Market Models for European Options: Dynamic Local Volatility and Tangent Lévy Models.

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(joint with Sergey Nadtochiy, OMI, Oxford)

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Talk Based on:

- Introductory Review:
  - R. Carmona, *HJM: A Unified Approach to Dynamic Models for Fixed Income, Credit and Equity Markets*, in "Paris - Princeton Lecutues in Mathematical Finance, 2005", *Lecture Notes in Mathematics*, **1919**, p. 3 – 45.
- ***Dynamic Local Volatility*** (DLV) Models:
  - R. Carmona and S. Nadtochiy, *Local volatility dynamic models*, Finance and Stochastics, 42(10) (2009).
  - R. Carmona and S. Nadtochiy, *An Infinite Dimensional Stochastic Analysis Approach to Local Volatility Dynamic Models*, Comm. on Stochastic Analysis, 2(1) (2008).
- Today's paper ***Tangent Lévy Market Models*** (TLM).

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HJM Philosophy

- Short Interest Rate Model:
  - $dr_t = \kappa(\overline{r} - r_t)dt + \sigma dW_t$ (Vasicek)
  - Explicit formulas for the forward (yield) curves
  - Too "rigid": can't always find the 3 parameters to match the observed forward curve $\tau \hookrightarrow f(\tau)$
- Replace $\overline{r}$ by $t \hookrightarrow \overline{r}(t)$ given by

$$\overline{r}(\tau = f'(\tau) + \kappa f(\tau 0 - \frac{\sigma^2}{2\kappa}(1 - e^{-\kappa\tau})(3e^{-\kappa\tau} - 1))$$

- **PERFECT** match, but **THIS IS NOT A MODEL**
- Tomorrow we have to **DO IT AGAIN**
  - $dr_t = \kappa(\overline{r}(t) - r_t)dt + \sigma dW_t$ **IS NOT A DYNAMIC MODEL**
  - Only used **ONE DAY** to reproduce observed prices!

Problem Formulation for Equity Markets

- Set Up:
  - underlying price $(S_t)_{t\geq 0}$
  - set of liguidly traded derivatives, e.g. European Call options for all strikes K and maturities T

$$\left(\{C_t(T,K)\}_{T,K>0}\right)_{t\geq 0}$$

- Goal: to describe a **large class** of time-consistent *market models*, i.e. stochastic models (say, SDE's) for S and $\{C(T,K)\}_{T,K>0}$, such that
  - each model is arbitrage-free
  - prices observed on the market serve (**only!**) as initial condition

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Motivation: Static Models

- Call Options have become liquid $\Rightarrow$ need for financial models consistent with the observed option prices.
- Stochastic volatility models (e.g. Hull-White, Heston, etc.) do not reproduce market prices for **all** strikes and maturities (fit the *implied volatility* surface).
- Solution: ***local volatility*** models.

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Dupire's Formula

Assume $r = 0$ &

$$dS_t = \sigma_t dW_t.$$

- *B.Dupire (1994)* showed that

$$d\tilde{S}_t = \tilde{S}_t \tilde{a}(t, \tilde{S}_t) dW_t,$$

with

$$\tilde{a}^2(T, K) := \frac{2 \frac{\partial}{\partial T} C(T, K)}{K^2 \frac{\partial^2}{\partial K^2} C(T, K)}$$

- gives the **same call prices** $C(T, K)$!

$$\mathbb{E}\{(\tilde{S}_T - K)^+\} = C(T, K)$$

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Main issue: frequent **recalibration**:

- *Stochastic volatility* models have different "optimal" parameters on each day.
- *Local volatility* surface changes as well...

Existing Results

- *E.Derman, I.Kani (1997)*: idea of ***dynamic local volatility*** for continuum of options
- *P.Schönbucher, M.Schweizer, J.Wissel (1998-2008)*: consider fixed maturity and all strikes, fixed strike and all maturities, finitely many strikes and maturities (using mixture of Implied and local volatility).
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- We **define local volatility** $\tilde{a}^2(T, K)$ **by Dupire's formula** (now it depends upon t and ω),

$$\tilde{a}_t^2(T, K) := \frac{2\frac{\partial}{\partial T} C_t(T, K)}{K^2 \frac{\partial^2}{\partial K^2} C_t(T, K)}$$

- change variables

$$a_t(\tau, x) = \tilde{a}_t(t + \tau, S_t e^x),$$

- and work with

$$h_t(\tau, x) := \log a_t^2(\tau, x)$$

Pricing PDE

$\left[D_x := \frac{1}{2}(\partial^2_{x^2} - \partial_x)\right]$

- Normalized call prices

$$c(\tau, x) = \frac{1}{S_t} C_t(t+\tau, S_t e^x)$$

satisfy Dupire's forward PDE

$$\begin{cases} \partial_\tau c(\tau, x) = e^{h(\tau,x)} D_x c(\tau, x), & \tau > 0, x \in \mathbb{R} \\ c(0, x) = (1 - e^x)^+. \end{cases}$$

- Introduce operator $\mathbf{F} : h \mapsto c$, for $h \in \mathcal{B} \subset C^{1,5}([0, \bar{\tau}] \times \mathbb{R})$.

local vol surface $\hookrightarrow$ call price surface

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DLV Model

- Risk-neutral drift of underlying and interest rate are zero.
- Pricing is done with expectation under some pricing measure $\mathbb{Q}$ (doesn't have to be unique).
- $B = (B^1, \dots, B^m)$ m-dimensional Brownian motion (m could be ∞).
- Under $\mathbb{Q}$, price process S is a martingale, and we have the following dynamics

$$\begin{cases} dh_t = \alpha_t dt + \sum_{n=1}^{m} \beta_t^n dB_t^n \\ dS_t = S_t \sigma_t dB_t^1, \end{cases}$$

for some regular enough $\mathcal{B}$-valued processes α, $\{\beta^n\}_{n=1}^m$, and scalar random process σ.

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Questions to answer

1. When is such a model consistent? Equivalently, find necessary and sufficient conditions for all call prices to be martingales.
2. What are the free input parameters of consistent models? (among σ, α, β) and how to specify them?

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”Dual” Fundamental Solution

- Denote by $p(h)$ the fundamental solution of the forward PDE

$$\partial_\tau w(\tau, x) = e^{h(\tau,x)} D_x w(\tau, x)$$

- Introduce $q(h)$ as fundamental solution of ”dual” backward PDE

$$\partial_\tau w(\tau, x) = -e^{h(\tau,x)} D_x w(\tau, x)$$

Operators I, J, K

- $I[\Gamma_2, f, \Gamma_1](\tau_2, x_2; \tau_1, x_1)$

$$:= \int_{\tau_1}^{\tau_2} \int_{\mathbb{R}} \Gamma_2(\tau_2, x_2; u, y) f(u, y) D_y \Gamma_1(u, y; \tau_1, x_1) dy du,$$

- $J[\Gamma_2, f, \Gamma_1](\tau_2, x_2; \tau_1, x_1)$

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Operator **F** is twice continuously Frechét-differentiable. And, for any $h, h', h'' \in \mathcal{B}$, we have

$$\mathbf{F}'(h)[h'] = \frac{1}{2}K[p(h), h'e^h, q(h)],$$

and

$$\mathbf{F}''(h)[h', h''] = \frac{1}{2}\Big(K\Big[I\Big[p(h), h''e^h, p(h)\Big], h'e^h, q(h)\Big] + K\Big[p(h), h'e^h, J\Big[q(h), h''e^h, q(h)\Big]\Big]\Big)$$

Consistency Conditions

$$\left[\tilde{h}_t(T,x) := h_t(T-t, x-\log S_t), \quad L(h) := \log q(h)\right]$$

1. *Drift restriction:*

$$\tilde{\alpha}_t = \sigma_t \tilde{\beta}_t^1 \left(\partial_x L(\tilde{h}_t) - L'(\tilde{h}_t)[\partial_x \tilde{h}_t]\right) - \frac{1}{2}\sum_{n=1}^{\infty} \tilde{\beta}_t^{n\,2} - \sum_{n=1}^{\infty} \tilde{\beta}_t^n L'(\tilde{h}_t)[\tilde{\beta}_t^n]$$

2. *Spot volatility specification:*

$$\tilde{h}_t(t, S_t) = 2\log\sigma_t$$

Progress so far

Consistency conditions answer the first question. Let us consider the second one: what are the free parameters and how to choose them?

- Looks like β is the free parameter: determines both α and σ.
- However, $L(h)$ has singularity at $\tau = 0$... Some work is needed to make sure that "drift restriction" produces $\alpha_t \in \mathcal{B}$.

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Short-term L

- The following is a modification of results by *S.Molchanov, S.Varadhan, I.Chavel* and others:

$$L(h)(\tau, x) = -\frac{1}{2}\log\tau - \frac{\left(\int_0^x e^{-\frac{1}{2}h(0,y)}dy\right)^2}{2\tau} + \hat{L}(h)(\tau, x),$$

where $\hat{L}(h)(.,.)$ is a smooth function.

- Notice that $\hat{L}(h)(.,.)$ satisfies an initial-value problem without singularities!

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Short-term β

- Denote $\delta_h(x) := \int_0^x e^{-\frac{1}{2}h(0,y)}dy$.
- In order for $\alpha(.,.)$ to be smooth at $\tau = 0$, we need

$$\frac{1}{2}\left(\int_0^x \left[\beta^1(0,y) - \sigma\partial_y h(0,y)\right] d\delta_h(y) + 2\right)^2 + \sum_{n=2}^{m}\left(\int_0^x \beta^n(0,y)d\delta_h(y)\right)^2 = 2$$

Example of feasible β

- Consider smooth $f : \mathbb{R} \mapsto \mathbb{R}_+$ with

$$0 < \int_0^\infty yf(y)dy = -\int_{-\infty}^0 yf(y)dy \leq 1$$

- Then the following specification, for example, takes care of singularities in "drift restriction"

$$\beta^1(h; 0, x) = \sigma \partial_x h(0, x) - 4xf(x)e^{\frac{1}{2}h(0,x)}$$

$$\beta^2(h; 0, x) = e^{\frac{1}{2}h(0,x)} \frac{\sqrt{2}xf(x)\left(1 - 2\int_0^x yf(y)dy\right)}{\sqrt{\int_0^x yf(y)dy(1 - \int_0^x yf(y)dy)}}$$

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DLV: summary of results

- Consistent dynamics of "Log-Local-Vol" ($\frac{1}{2}h_t$) take form of equation
$$dh_t = \alpha(h_t, \beta_t)dt + \beta_t dB_t,$$
where β satisfies "short-term condition" and $\alpha(h, \beta)$ is given by "drift restriction".
- Can, for example, choose $\beta_t := \beta(h_t)$, satisfying "short-term condition" and implement explicit Euler scheme for resulting SDE
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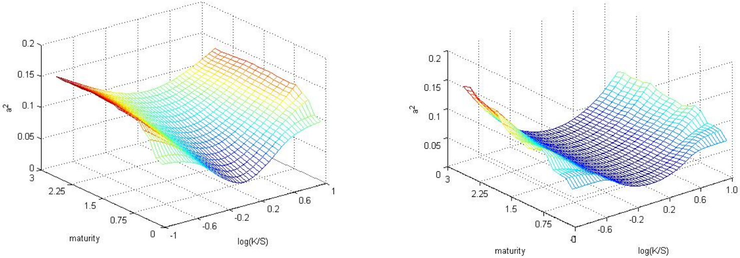

Local Volatility in Heston (left) and Hull-White (right) models.

- When can we use local vol as a (static) code for Option Prices?

  I. Gyongyi: it is possible if underlying follows regular enough Ito process.

- Natural relaxation of this assumption on underlying?

  Introduce jumps.

- What is the right substitute for local volatility?

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Extensions of Local Volatility

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Fitting option prices with Lévy-based models

- *J.Cox, S.Ross, R.Merton (1976)*: introduced jumps in the price of underlying.
- *S.Kou (2002)*: *Double Exponential Jump Diffusion* model.
- *P.Carr, H.Geman, D.Madan, M.Yor, E.Seneta (1990-2005)*: infinite activity jumps, *Variance Gamma* and *CGMY* models.
- *P.Carr, H.Geman, D.Madan, M.Yor (2004)*: use Markovian time change of Lévy process – *Local Lévy* models.

Definition

- Consider a pure jump martingale

$$\tilde{S}_t = \tilde{S}_0 + \int_0^t \int_{\mathbb{R}} \tilde{S}_{u-}(e^x - 1)\left[N(dx, du) - \eta(dx, du)\right],$$

where $N(dx, du)$ is a *Poisson random measure* associated with jumps of $\log(\tilde{S})$, given by its compensator

$$\eta(dx, du) = \kappa(u, x) dx du,$$

- If the model above reproduces market prices of call options, we call κ the ***Tangent Lévy Density***.

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Pricing PIDE

$$\left[\tilde{C}(T,x) := C(T,e^x),\ D_x := \partial^2_{x^2} - \partial_x\right]$$

Call Prices at time t, produced by κ, satisfy the following PIDE

$$\begin{cases} \partial_T \tilde{C}_t(T,x) = \int_{\mathbb{R}} \psi(T,x-y) D_y \tilde{C}_t(T,y)dy \\ \tilde{C}_t(T,x)\Big|_{T=t} = (\tilde{S}_t - e^x)^+, \end{cases}$$

where

$$\psi(T,x) = \begin{cases} \int_{-\infty}^{x} (e^x - e^z)\kappa(T,z)dz & x < 0 \\ \int_{x}^{\infty} (e^z - e^x)\kappa(T,z)dz & x > 0 \end{cases}$$

Fourier Transform

- Introduce $\Delta(T,x) = -\partial_x \tilde{C}(T,x)$.
- Take Fourier transform in x (use "hat" for values in Fourier space), and obtain

$$\begin{cases} \partial_T \hat{\Delta}(T,x) = -\left(4\pi^2 x^2 + 2\pi i x\right) \hat{\psi}(T,x)\hat{\Delta}(T,x) \\ \hat{\Delta}(T,x)\Big|_{T=t} = \frac{\exp\left\{\log(\tilde{S}_t)(1-2\pi i x)\right\}}{1-2\pi i x} \end{cases} \quad (1)$$

- Obtain mapping: $\tilde{C} \to \hat{\Delta} \to \hat{\psi} \to \kappa$.

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- Obtain mapping: $\tilde{C} \to \hat{\Delta} \to \hat{\psi} \to \kappa$.

From κ to Call Prices

- Solve (1) for $\hat{\Delta}$ in closed form.
- Invert Fourier transform and integrate to obtain

$$\tilde{C}_t(T,x) = \tilde{S}_t \lim_{\lambda\to+\infty} \int_{\mathbb{R}} \frac{e^{2\pi i y\lambda} - e^{2\pi i y(x-\log \tilde{S}_t)}}{2\pi i y(1-2\pi i y)} \cdot \exp\left(-2\pi(2\pi y^2 + iy)\int_{t\wedge T}^{T} \hat{\psi}(u,y)du\right) dy$$

- Obtain mapping: $\kappa \to \hat{\psi} \to \tilde{C}$.

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Tangent Lévy Model

- $B = (B^1, \ldots, B^m)$ is the m-dimensional Brownian motion (m can be ∞).
- Under pricing measure $\mathbb{Q}$, process S is a martingale, and we have the following dynamics

$$\begin{cases} S_t = S_0 + \int_0^t \int_{\mathbb{R}} S_{u-}(e^x - 1)(M(dx, du) - K_u(x)dxdu) \\ \kappa_t = \kappa_0 + \int_0^t \alpha_u du + \sum_{n=1}^m \int_0^t \beta_u^n dB_u^n, \end{cases}$$

  for some *integer valued random measure* M with compensator $K_u(\omega, x)dxdu$.

- $\kappa_t(T, .) \in \mathbb{L}^1\left(\mathbb{R}, |x|\left(|x| \wedge 1\right)\left(1 + e^x\right)\right)$.

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Simplifying notation

- Introduce

$$\psi^n(T,x) := \begin{cases} \int_{-\infty}^{x} (e^x - e^z)\beta^n(T,z)dz & x < 0 \\ \int_x^{\infty} (e^z - e^x)\beta^n(T,z)dz & x > 0, \end{cases}$$

- and

$$\bar{\psi}^n(T,x) := \begin{cases} \int_{-\infty}^{x} (e^x - e^z) \int_0^T \beta^n(u,z)dudz & x < 0 \\ \int_x^{\infty} (e^z - e^x) \int_0^T \beta^n(u,z)dudz & x > 0, \end{cases}$$

Consistency Conditions

Assuming $\kappa \geq 0$, consistency of the model is equivalent to

1. *Drift restriction:*

$$\alpha_t(T,x) = -\frac{1}{2}\sum_{n=1}^{m}\int_{\mathbb{R}} \partial^4_{y^4}\bar{\psi}^n(T,y)\Big[\psi^n(T,x-y) - \Big(1 - y\partial_x + \frac{y^2}{2}\partial^2_{x^2} - \frac{y^3}{6}\partial^3_{x^3}\Big)\psi^n(T,x)\Big]$$
$$+ 2\partial^3_{y^3}\bar{\psi}^n(T,y)\Big[\psi^n(T,x-y) - \Big(1 - y\partial_x + \frac{y^2}{2}\partial^2_{x^2}\Big)\psi^n(T,x)\Big]$$
$$+ \partial^2_{y^2}\bar{\psi}^n(T,y)\left[\psi^n(T,x-y) - (1 - y\partial_x)\psi^n(T,x)\right]dy$$

2. *Compensator specification:* $\kappa_t(t,x) = K_t(x)$.

Model Specification

- Change variables from κ to $\tilde{\kappa}$:

$$\kappa(T,x) = e^{-\lambda|x|}\left(|x|^{-1-\delta} \vee 1\right)\tilde{\kappa}(T,x),$$

  where $\lambda > 1$ and $\delta \in (0,1)$.

- $\tilde{\kappa}$ takes values in a separable Banach space $\tilde{\mathcal{B}}$ consisting of continuous functions.

- Introduce $\tilde{\alpha}$ as integrable process in $\tilde{\mathcal{B}}$,

- and $\left\{\tilde{\beta}^n\right\}_{n=1}^m$ as square integrable processes with values in $\tilde{\mathcal{H}} \subset \tilde{\mathcal{B}}$.

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Model Specification

- Introduce stopping time

$$\tau_0 = \inf\left\{t \geq 0 : \inf_{T \in [t,\bar{T}], x \in \mathbb{R}} \tilde{\kappa}_t(T,x) \leq 0\right\},$$

- Clearly, τ_0 is predictable and $\tilde{\kappa}_{t\wedge\tau_0}$ is nonnegative.
- Denote $\rho(x) := e^{-\lambda|x|}\left(|x|^{-1-\delta} \vee 1\right)$.
- Due to "compensator specification", without loss of generality, we will substitute $K_t(x)$ by $\rho(x)\tilde{\kappa}_t(t,x)$.

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Model Specification

- Assume that market filtration is generated by $\{B^n\}_{n=1}^m$ and independent Poisson random measure N with compensator $\rho(x)dxdt$.

- Denote by $\{(t_n, x_n)\}_{n=1}^\infty$ the atoms of N. Then M can be chosen such that its atoms are

$$\{(t_n, W[\tilde{\kappa}_{t_n}(t_n, .)](x_n))\}_{n=1}^\infty ,$$

  for some deterministic mapping $f(.) \mapsto W[f](.)$

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For the following class of TL models

$$\begin{cases} S_t = S_0 + \int_0^t \int_{\mathbb{R}} S_{u-}(e^{W[\tilde{\kappa}_u(u,.)](x)} - 1)(N(dx, du) - \rho(x)dxdu) \\ \tilde{\kappa}_t = \tilde{\kappa}_0 + \int_0^{t\wedge\tau_0} \tilde{\alpha}_u du + \sum_{n=1}^m \int_0^{t\wedge\tau_0} \tilde{\beta}_u^n dB_u^n, \end{cases}$$

we have:

If $\left\{\tilde{\beta}^n\right\}_{n=1}^m$ are square integrable $\tilde{\mathcal{H}}$-valued random processes, then $\tilde{\alpha}$, given by the "drift restriction", is an integrable random process with values in $\tilde{\mathcal{B}}$, and the above system defines a consistent Tangent Lévy model.

Example of TL Model

- Choose $m = 1$, and $\tilde{\beta}_t(T, x) = \xi_t C(x)$,
- where $\xi_t = \frac{\sigma}{\epsilon}\left(\inf_{T \in [t, \bar{T}], x \in \mathbb{R}} \tilde{\kappa}_t(T, x) \wedge \epsilon\right)$
- and $C(x) = \text{sign}(x) e^{-\lambda' |x|} \left(|x| \wedge 1\right)^{1+2\delta} \left(\lambda + \lambda' + \delta |x|^{-1-\delta}\right)$.
- Then

$$\tilde{\kappa}_t(T, x) = \tilde{\kappa}_0(T, x) + \frac{T - t}{2} A(x) \int_0^t \xi_u^2 du + C(x) \int_0^t \xi_u dB_u,$$

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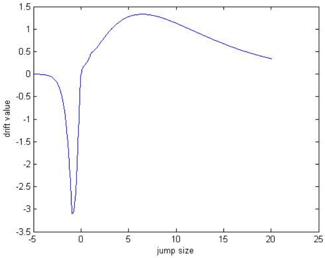
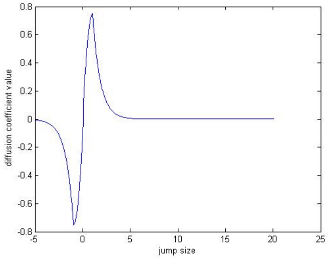
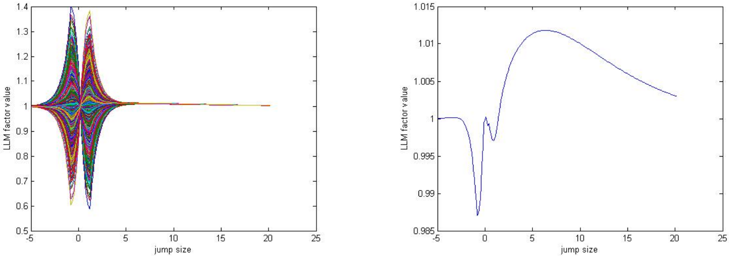




Drift (left) and diffusion coefficient (right) of $\tilde{\kappa}$, as functions of jump size x.

Example of TL Model



Simulated values of $\tilde{\kappa}_T(T, .)$ (left) and their average (right).

Conclusions

- We have described two classes of **consistent** stochastic dynamic models for the call price surface when it can be coded by:
  - ***Local Volatility***
  - ***Tangent Lévy density***.
- Each class corresponds to a different type of **dynamics of the underlying**:
  - ***continuous***
  - ***pure jump***.

  while keeping the semimartingale property.
- The description of **Tangent Lévy models** is complete: *for any admissible value of the free parameter in a given linear space, we can construct a unique arbitrage-free market model*.

Connection with Gyongyi's Theorem

- **Dupire local volatility** can be understood in terms of the original Gyongyi's theorem.
- **Pure Jump Martingales** can be treated the same way via *conditional expectations of the local characteristics* to give the Itô-Lévy system of a pure jump Markov process with the same marginals (Cont BenTata)
- As side product of our analysis, we exhibited a large class of **Infinitely Divisible** pure jump martingales for which the "conditional expectation treatment" gives an **additive** or **non-homogeneous Lévy** process with the same marginals.
- We would like to have simple conditions on the original characteristics of the underlier for the existence of **Tangent Lévy** structure.

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