A Maximum Principle for Stochastic Differential Games with g–expectations and partial information

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Abstract

In this paper, we initiate a study on optimal control problem for stochastic differential games under generalized expectation via backward stochastic differential equations and partial information. We first prove a sufficient maximum principle for zero-sum stochastic differential game problem. And then extend our approach to general stochastic differential games (nonzero–sum games), and obtain an equilibrium point of such game. Finally we give some examples of applications.

Key words: Jump diffusion, stochastic control, stochastic differential game, forward-backward stochastic differential equations, g–expectation, sufficient maximum principle.

1 Introduction

Suppose the dynamics of a stochastic system is described by a stochastic differential equation on a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ of the form:

$$\begin{aligned} dX(t) &= b(t, X(t), u_0(t))dt + \sigma(t, X(t), u_0(t))dW(t) \\ &\quad + \int_{\mathbb{R}_0} \gamma(t, X(t^-), u_1(t^-, z), z)\widetilde{N}(\,dt,\,dz),\ t \in [0, T], \\ X(0) &= x \in \mathbb{R}^n. \end{aligned} \tag{1}$$

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Here $b : [0,T] \times \mathbb{R}^n \times K \to \mathbb{R}^n$; $\sigma : [0,T] \times \mathbb{R}^n \times K \to \mathbb{R}^{n\times n}$ and $\gamma : [0,T] \times \mathbb{R}^n \times K \times \mathbb{R}_0 \to \mathbb{R}^{n\times n}$ are given continuous functions, and $W(t)$ is n–dimensional Brownian motion, $\widetilde{N}(.,.)$ are n independent compensated Poisson random measures and K is a given closed subset of $\mathbb{R}^n$. The processes $u_0(t) = u_0(t,\omega)$ and $u_1(t) = u_1(t,z,\omega)$, $\omega \in \Omega$ are our *control processes*. We assume that $u_0(t), u_1(t,z)$ have values in a given set K for a.a. t, z and adapted to a given filtration $\{\mathcal{G}_t\}_{t\in[0,T]}$, where

$$\mathcal{G}_t \subseteq \mathcal{F}_t; \qquad t \in [0,T]. \tag{2}$$

For example, we could have a delayed information flow of the form

$$\mathcal{G}_t = \mathcal{F}_{(t-\delta)^+}; \qquad t \in [0,T],$$

where $(t-\delta)^+ = \max(0, t-\delta)$ and $\delta > 0$ is a given constant. We call $u = (u_0, u_1)$ an *admissible control* if (1) has a unique strong solution and

$$E^x\Big[\int_0^T \mid f(t, X(t), u_0(t)) \mid \, dt + \mid h(X(T)) \mid \Big] < \infty. \tag{3}$$

Let $f : [0,T] \times \mathbb{R}^n \times K \to \mathbb{R}$ be a continuous function, namely the *profit rate*, and $h : \mathbb{R}^n \to \mathbb{R}$ be a concave function, namely the *bequest function*. If u is an admissible control we define the *performance criterion* $J(u)$ by

$$J(u) = \mathbb{E}\Big[\int_0^T f(t, X(t), u(t))\, dt + h(X(T))\Big]. \tag{4}$$

Now suppose that the controls $u_0(t)$ and $u_1(t,z)$ have the form

$$\begin{aligned} u_0(t) &= (\theta_0(t), \pi_0(t)); \qquad t \geq 0, \\ u_1(t,z) &= (\theta_1(t,z), \pi_1(t,z)); \qquad (t,z) \in [0,\infty) \times \mathbb{R}^n. \end{aligned}$$

We let $\Theta_{\mathcal{G}}$ and $\Pi_{\mathcal{G}}$ be given families of admissible controls $\theta = (\theta_0, \theta_1)$ and $\pi = (\pi_0, \pi_1)$, respectively.

The *classical partial information zero–sum stochastic differential game problem* is to find $(\theta^*, \pi^*) \in \Theta_{\mathcal{G}} \times \Pi_{\mathcal{G}}$ such that

$$J(\theta^*, \pi^*) = \sup_{\pi\in\Pi_{\mathcal{G}}} \Big(\inf_{\theta\in\Theta_{\mathcal{G}}} J(\theta, \pi)\Big). \tag{5}$$

Such a control (θ^*, π^*) is called an *optimal control* (if it exists).

The intuitive idea is that there are two players, I and II. Player I controls $\theta := (\theta_0, \theta_1)$ and player II controls $\pi := (\pi_0, \pi_1)$. The actions of the players are antagonistic, which means that between I and II there is a payoff $J(\theta, \pi)$ which is a cost for I and a reward for II.

The problem (5) for jumps was studied recently by several authors, e.g. [2], [6], [9] and references therein. In this paper we study this game in the case when the performance criterion $J(u)$ in (4) is replaced by a criterion involving *risk*.

If we interpret *risk* in the sense of a *convex risk measure*, it can be represented as a nonlinear expectation called g–expectation. See [10] and [11] for more information about this. More precisely, let

$$g : \mathbb{R} \times \mathbb{R} \times L^2(\nu) \to \mathbb{R}$$

be a given convex function such that g is uniformly Lipschitz with respect to (y, k, l), i.e.,

$$| g(y,k,l) - g(y^{'},k^{'},l^{'}) | \leq K(| y - y^{'} | + | k - k^{'} | + | l - l^{'} |), \tag{6}$$

and such that, for each $T > 0$ and $(y, k, l) \in (\mathbb{R} \times \mathbb{R} \times L^2(\nu))$, $g(y, k, l)$ is progressively measurable.

Let $\mathbb{F}$ be a family of $\mathcal{F}_T$–measurable random variables $\xi : \Omega \to \mathbb{R}^n$, where $T > 0$ is a fixed constant. Consider the following backward stochastic differential equation (BSDE, for short):

$$\begin{aligned} dY(t) &= -g(K(t), L(t,\cdot)dt + K(t)dW(t) + \int_{\mathbb{R}_0} L(t,z)\widetilde{N}(dt,dz), \qquad (7)\\ Y(T) &= \xi. \end{aligned}$$

We then define

Definition 1.1. For each $\xi \in \mathbb{F}$, we call

$$\mathcal{E}_g(\xi) := Y(0) \tag{8}$$

the g*–expectation of* ξ related to g.

One can show that the map $\xi \to \mathcal{E}_g(\xi)$ keeps all the properties that $\mathbb{E}$ has, except possibly for the linearity. Further, it is clear that when $g(\cdot) = 0$, $\mathcal{E}_g$ is reduced to the original expectation $\mathbb{E}$.

With the above defined generalized expectation, we now introduce the following cost function

$$J_g(\theta,\pi) = \mathcal{E}_g\Big[\int_0^T f(t, X(t), u(t))\, dt + h(X(T))\Big]. \tag{9}$$

We can formulate our problem with generalized expectation as follows

Problem 2 Find a $(\theta^*, \pi^*) \in \Theta_{\mathcal{G}} \times \Pi_{\mathcal{G}}$ such that

$$J_g(\theta^*,\pi^*) = \sup_{\pi\in\Pi_{\mathcal{G}}} \Big(\inf_{\theta\in\Theta_{\mathcal{G}}} J_g(\theta,\pi)\Big). \tag{10}$$

This problem can be expressed in a different way. Suppose the state $(X(t), Y(t))$ of our system is described by the following coupled forward-backward stochastic differential equation (FBSDE):

$$
\begin{cases}
dX(t) &= b(t,X(t),u_0(t))dt + \sigma(t,X(t),u_0(t))dW(t) \\
&\quad + \int_{\mathbb{R}_0} \gamma(t,X(t^-),u_1(t^-,z),z)\widetilde{N}(\,dt,\,dz), \\
dY(t) &= -[g(K(t),L(t,\cdot)) + f(t,X(t),u(t))]dt \\
&\quad + K(t)dW(t) + \int_{\mathbb{R}_0} L(t,z)\widetilde{N}(dt,dz), \\
X(0) &= X_0, \qquad Y(T) = h(X(T)),
\end{cases}
\tag{11}
$$

and the cost function is given of the form:

$$
\begin{aligned}
&J_g(\theta,\pi) = Y(0) \\
&= \mathbb{E}\Big[h(X(T)) + \int_0^T (g(K(t),L(t,\cdot)) + f(t,X(t),u(t)))dt\Big].
\end{aligned}
\tag{12}
$$

The problem is to find $u^* = (\theta^*,\pi^*) \in \Theta_{\mathcal{G}} \times \Pi_{\mathcal{G}}$ such that

$$
J_g(\theta^*,\pi^*) = \sup_{\pi\in\Pi_{\mathcal{G}}} \Big(\inf_{\theta\in\Theta_{\mathcal{G}}} J_g(\theta,\pi)\Big).
\tag{13}
$$

In Section 2 we study the partial optimal control problem for zero–sum stochastic differential games with g–expectations and we prove a partial information sufficient maximum principle for such problem. In Section 3 we generalize our approach to the general case, not necessarily of zero-sum type, and also give an equilibrium point for nonzero-sum games. Finally, in Section 4 we apply our results to finance market.

2 A maximum principle for zero–sum games with g–expectations

We now present a maximum principle for problem (13).

The *Hamiltonian*

$$
H : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times L^2(\nu) \times K_1 \times K_2 \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times L^2(\nu)
$$

is defined by

$$
\begin{aligned}
&H(t,x,y,k,l,\theta,\pi,\mu,\varphi,\psi,\phi) = g(k,l) + f(t,x,\theta,\pi) \\
&+ (g(k,l) + f(t,x,\theta,\pi))\mu + b(t,x,\theta,\pi)\varphi \\
&+ \sigma(t,x,\theta,\pi)\psi + \int_{\mathbb{R}_0} \gamma(t,x,\theta,\pi,z)\phi(t,z)\nu(dz).
\end{aligned}
\tag{14}
$$

We assume that H is differentiable in the variables x, y, k, l. The *adjoint* *equation* in the unknown adapted processes μ, φ, ψ, ϕ is following FBSDE:

$$\begin{cases} d\mu(t) &= \frac{\partial H}{\partial y}(t, X(t), Y(t), K(t), L(t), \theta(t), \pi(t), \mu(t), \varphi(t), \psi(t), \phi(,\cdot))dt \\ &+ \frac{\partial H}{\partial k}(t, X(t), Y(t), K(t), L(t), \theta(t), \pi(t), \mu(t), \varphi(t), \psi(t), \phi(,\cdot))dW(t) \\ &+ \int_{\mathbb{R}_0} \bigtriangledown_l H(t, X(t), Y(t), K(t), L(t), \theta(t), \pi(t), \mu(t), \varphi(t), \psi(t), \phi(,\cdot))\widetilde{N}(dt, dz), \\ d\varphi(t) &= -\frac{\partial H}{\partial x}(t, X(t), Y(t), K(t), L(t), \theta(t), \pi(t), \mu(t), \varphi(t), \psi(t), \phi(,\cdot))dt \\ &\quad +\psi(t)dW(t) + \int_{\mathbb{R}_0} \phi(t,z)\widetilde{N}(dt, dz), \\ \mu(0) &= 0, \qquad \varphi(T) = (1+\mu(T))h^{'}(X(T)), \end{cases} \tag{15}$$

where $\bigtriangledown_l H$ denotes the gradient (Frechet derivative) of H with respect to l.

With a slight abuse of notation we will let Θ and Π denote given sets of possible control values of $\theta(t)$, $t \in [0,T]$, and $\pi(t)$, $t \in [0,T]$, respectively.

Theorem 2.1. *Let* $(\hat{\theta}, \hat{\pi}) \in \Theta_{\mathcal{G}} \times \Pi_{\mathcal{G}}$ *with corresponding solutions* $\widehat{X}(t)$, $\widehat{Y}(t)$, $\widehat{K}(t)$, $\widehat{L}(t,z)$, $\widehat{\mu}(t)$, $\widehat{\varphi}(t)$, $\widehat{\psi}(t)$, $\widehat{\phi}(t,z)$ *of equations* (11) *and* (15). *Suppose that*

(The conditional minimum principle)

$$\begin{aligned} &\inf_{\theta\in\Theta} \mathbb{E}[H(t, \widehat{X}(t), \widehat{Y}(t), \widehat{K}(t), \widehat{L}(t,\cdot), \theta, \hat{\pi}(t), \hat{\mu}(t), \widehat{\varphi}(t), \widehat{\psi}(t), \widehat{\phi}(t,\cdot))|\mathcal{G}_t] \\ &= \mathbb{E}[H(t, \widehat{X}(t), \widehat{Y}(t), \widehat{K}(t), \widehat{L}(t,\cdot), \hat{\theta}(t), \hat{\pi}(t), \hat{\mu}(t), \widehat{\varphi}(t), \widehat{\psi}(t), \widehat{\phi}(t,\cdot))|\mathcal{G}_t] \\ &= \sup_{\pi\in\Pi} \mathbb{E}[H(t, \widehat{X}(t), \widehat{Y}(t), \widehat{K}(t), \widehat{L}(t,\cdot), \hat{\theta}(t), \pi(t), \hat{\mu}(t), \widehat{\varphi}(t), \widehat{\psi}(t), \widehat{\phi}(t,\cdot))|\mathcal{G}_t]. \end{aligned} \tag{16}$$

(i) Suppose that, for all $t \in [0,T]$, $h(x)$ *is concave and*

$$(x,y,k,l,\pi) \to H(t,x,y,k,l,\hat{\theta}(t), \pi, \hat{\mu}(t), \widehat{\varphi}(t), \widehat{\psi}(t), \widehat{\phi}(t,\cdot))$$

is concave. Then

$$J_g(\hat{\theta}, \hat{\pi}) \geq J_g(\hat{\theta}, \pi), \qquad \text{for all } \pi \in \Pi_{\mathcal{G}},$$

and

$$J_g(\hat{\theta}, \hat{\pi}) = \sup_{\pi\in\Pi_{\mathcal{G}}} J_g(\hat{\theta}, \pi).$$

(ii) Suppose that, for all $t \in [0,T]$, $h(x)$ *is convex and*

$$(x,y,k,l,\theta) \to H(t,x,y,k,l,\theta, \hat{\pi}(t), \hat{\mu}(t), \widehat{\varphi}(t), \widehat{\psi}(t), \widehat{\phi}(t,\cdot))$$

is convex. Then

$$J_g(\hat{\theta}, \hat{\pi}) \leq J_g(\theta, \hat{\pi}), \qquad \text{for all } \theta \in \Theta_{\mathcal{G}},$$

and

$$J_g(\hat{\theta}, \hat{\pi}) = \inf_{\theta\in\Theta_{\mathcal{G}}} J_g(\theta, \hat{\pi}).$$

(iii) If both cases (i) and (ii) hold (which implies, in particular, that h *is an affine function), then* $(\theta^*, \pi^*) := (\hat{\theta}, \hat{\pi})$ *is an optimal control and*

$$J_g(\hat{\theta}, \hat{\pi}) = \sup_{\pi\in\Pi_{\mathcal{G}}} \Big(\inf_{\theta\in\Theta_{\mathcal{G}}} J_g(\theta,\pi) \Big) = \inf_{\theta\in\Theta_{\mathcal{G}}} \Big(\sup_{\pi\in\Pi_{\mathcal{G}}} J_g(\theta,\pi) \Big). \tag{17}$$

Proof. i) Suppose (i) holds. Choose $(\theta, \pi) \in \Theta_{\mathcal{G}} \times \Pi_{\mathcal{G}}$ with corresponding solutions $X(t)$, $Y(t)$, $K(t)$, $L(t,z)$, $\mu(t)$, $\varphi(t)$, $\psi(t)$ and $\phi(t,z)$. In the following we write

$$\begin{aligned}
\widehat{H}(t) &= H(t, \widehat{X}(t), \widehat{Y}(t), \widehat{K}(t), \widehat{L}(t,\cdot), \hat{\theta}(t), \hat{\pi}(t), \hat{\mu}(t), \widehat{\varphi}(t), \widehat{\psi}(t), \widehat{\phi}(t,\cdot)), \\
H^{\hat{\theta}}(t) &= H(t, X^{\hat{\theta}}(t), Y^{\hat{\theta}}(t), K^{\hat{\theta}}(t), L^{\hat{\theta}}(t), \hat{\theta}(t), \pi(t), \hat{\mu}(t), \widehat{\varphi}(t), \widehat{\psi}(t), \widehat{\phi}(t,\cdot)), \\
H^{\hat{\pi}}(t) &= H(t, X^{\hat{\pi}}(t), Y^{\hat{\pi}}(t), K^{\hat{\pi}}(t), L^{\hat{\pi}}(t), \theta(t), \hat{\pi}(t), \hat{\mu}(t), \widehat{\varphi}(t), \widehat{\psi}(t), \widehat{\phi}(t,\cdot))
\end{aligned}$$

and similarly with $\hat{f}(t)$, $f^{\hat{\theta}}(t)$, $f^{\hat{\pi}}(t)$ ... etc. Then

$$J_g(\hat{\theta}, \hat{\pi}) - J_g(\hat{\theta}, \pi) = I_1 + I_2, \tag{18}$$

where

$$I_1 = \mathbb{E}\Big[\int_0^T (\widehat{g}(t) - g^{\hat{\theta}}(t) + \widehat{f}(t) - f^{\hat{\theta}}(t))dt\Big]$$

and

$$I_2 = \mathbb{E}[h(\widehat{X}(T)) - h(X^{\hat{\theta}}(T))].$$

By the definition of H we have

$$\begin{aligned}
I_1 = \mathbb{E}\Big[\int_0^T \Big\{ & \widehat{H}(t) - H^{\hat{\theta}}(t) - (\hat{g}(t) - g^{\hat{\theta}}(t) + \hat{f}(t) - f^{\hat{\theta}}(t))\mu(t) \\
& - (\hat{b}(t) - b^{\hat{\theta}}(t))\widehat{\varphi}(t) - (\widehat{\sigma}(t) - \sigma^{\hat{\theta}}(t))\widehat{\psi}(t) \\
& - \int_{\mathbb{R}_0} (\widehat{\gamma}(t) - \gamma^{\hat{\theta}}(t))\widehat{\phi}(t,z)\nu(dz)\Big\}dt\Big].
\end{aligned} \tag{19}$$

Since $\hat{\mu}(0) = 0$, we can rewrite I_2 as following:

$$I_2 = \mathbb{E}[h(\widehat{X}(T)) - h(X^{\hat{\theta}}(T)) + (\widehat{Y}(0) - Y^{\hat{\theta}}(0))\hat{\mu}(0)]. \tag{20}$$

By the Itô formula, we have

$$\begin{aligned}
\mathbb{E}[(\widehat{Y}(0) - Y^{\hat{\theta}}(0))\hat{\mu}(0)] &= \mathbb{E}[(\widehat{Y}(T) - Y^{\hat{\theta}}(T))\hat{\mu}(T)] \\
&- \mathbb{E}\Big[\int_0^T \Big\{(\widehat{Y}(t) - Y^{\hat{\theta}}(t))d\hat{\mu}(t) + \hat{\mu}(t)d(\widehat{Y}(t) - Y^{\hat{\theta}}(t)) \\
&+ \frac{\partial \widehat{H}}{\partial k}(t)(\widehat{K}(t) - K^{\hat{\theta}}(t)) + \int_{\mathbb{R}_0} \nabla_l \widehat{H}(t)(\widehat{L}(t) - L^{\hat{\theta}}(t))\nu(dz)\Big\}dt\Big] \\
&= \mathbb{E}[(h(\widehat{X}(T)) - h(X^{\hat{\theta}}(T)))\hat{\mu}(T)] + I_3,
\end{aligned}$$

where

$$\begin{aligned}
I_3 = -\,\mathbb{E}\Big[\int_0^T & \Big\{(\hat{g}(t) - g^{\hat{\theta}}(t) + \hat{f}(t) - f^{\hat{\theta}}(t))\mu(t) + \frac{\partial \widehat{H}}{\partial y}(t)(\widehat{Y}(t) - Y^{\hat{\theta}}(t)) \\
& + \frac{\partial \widehat{H}}{\partial k}(t)(\widehat{K}(t) - K^{\hat{\theta}}(t)) + \int_{\mathbb{R}_0} \nabla_l \widehat{H}(t)(\widehat{L}(t) - L^{\hat{\theta}}(t))\nu(dz)\Big\}dt\Big].
\end{aligned}$$

By the concavity of h and using the Itô formula again, we get

$$\begin{aligned}
I_2 &= \mathbb{E}[(h(\widehat{X}(T)) - h(X^{\hat{\theta}}(T)))(1+\hat{\mu}(T))] + I_3 \\
&\geq \mathbb{E}[(\widehat{X}(T) - X^{\hat{\theta}}(T))h^{'}(\widehat{X}(T)(1+\hat{\mu}(T))] + I_3 \\
&= \mathbb{E}[(\widehat{X}(T) - X^{\hat{\theta}}(T))\widehat{\varphi}(T)] + I_3 \\
&= \mathbb{E}\Big[\int_0^T \Big\{\widehat{\varphi}(t)d(\widehat{X}(t) - X^{\hat{\theta}}(t)) + (\widehat{X}(t) - X^{\hat{\theta}}(t))d\widehat{\varphi}(t) \\
&\quad + (\widehat{\sigma}(t) - \sigma^{\hat{\theta}}(t))\widehat{\psi}(t) + \int_{\mathbb{R}_0} (\widehat{\gamma}(t) - \gamma^{\hat{\theta}}(t))\widehat{\phi}(t,z)\nu(dz)\Big\}dt\Big] + I_3 \\
&= \mathbb{E}\Big[\int_0^T \Big\{-\frac{\partial \widehat{H}}{\partial x}(t)(\widehat{X}(t) - X^{\hat{\theta}}(t)) + (\hat{b}(t) - b^{\hat{\theta}}(t))\widehat{\varphi}(t) \\
&\quad + (\widehat{\sigma}(t) - \sigma^{\hat{\theta}}(t))\widehat{\psi}(t) + \int_{\mathbb{R}_0} (\widehat{\gamma}(t) - \gamma^{\hat{\theta}}(t))\widehat{\phi}(t,z)\nu(dz)\Big\}dt\Big] + I_3. \qquad (21)
\end{aligned}$$

Hence

$$\begin{aligned}
I_1 + I_2 = \mathbb{E}\Big[\int_0^T \Big\{\hat{H}(t) - H^{\hat{\theta}}(t) - \Big(&\frac{\partial \widehat{H}}{\partial x}(t)(\widehat{X}(t) - X^{\hat{\theta}}(t)) \\
&+ \frac{\partial \widehat{H}}{\partial y}(t)(\widehat{Y}(t) - Y^{\hat{\theta}}(t)) + \frac{\partial \widehat{H}}{\partial k}(t)(\widehat{K}(t) - K^{\hat{\theta}}(t)) \\
&+ \int_{\mathbb{R}_0} \nabla_l \widehat{H}(t)(\widehat{L}(t) - L^{\hat{\theta}}(t))\nu(dz)\Big)\Big\}dt\Big]. \qquad (22)
\end{aligned}$$

On the other hand, the function

$$(x,y,k,l,\pi) \to H(t,x,y,k,l,\hat{\theta}(t),\pi,\hat{\mu}(t),\widehat{\varphi}(t),\widehat{\psi}(t),\widehat{\phi}(t,\cdot))$$

is concave, we have

$$\begin{aligned}
\widehat{H}(t) - H^{\hat{\theta}}(t) \geq\ &\frac{\partial \widehat{H}}{\partial x}(t)(\widehat{X}(t) - X^{\hat{\theta}}(t)) + \frac{\partial \widehat{H}}{\partial y}(t)(\widehat{Y}(t) - Y^{\hat{\theta}}(t)) \\
&+ \frac{\partial \widehat{H}}{\partial k}(t)(\widehat{K}(t) - K^{\hat{\theta}}(t)) + \int_{\mathbb{R}_0} \nabla_l \widehat{H}(t)(\widehat{L}(t) - L^{\hat{\theta}}(t))\nu(dz) \\
&+ \frac{\partial \widehat{H}}{\partial \pi}(t)(\hat{\pi}(t) - \pi(t)). \qquad (23)
\end{aligned}$$

Combining (22), (23) and the condition (16), we conclude that

$$\begin{aligned} J_g(\hat{\theta}, \hat{\pi}) - J_g(\hat{\theta}, \pi) &\geq \mathbb{E}\Big[\int_0^T \frac{\partial \widehat{H}}{\partial \pi}(t)(\hat{\pi}(t) - \pi(t))dt\Big] \\ &= \mathbb{E}\Big[\int_0^T \mathbb{E}\Big[\frac{\partial \widehat{H}}{\partial \pi}(t)(\hat{\pi}(t) - \pi(t))\Big|\mathcal{G}_t\Big]dt\Big] \\ &= \mathbb{E}\Big[\int_0^T \mathbb{E}\Big[\frac{\partial \widehat{H}}{\partial \pi}(t)\Big|\mathcal{G}_t\Big](\hat{\pi}(t) - \pi(t))dt\Big] \\ &= \mathbb{E}\Big[\int_0^T \frac{\partial}{\partial \pi}\mathbb{E}[\widehat{H}(t)|\mathcal{G}_t](\hat{\pi}(t) - \pi(t))dt\Big] \geq 0. \end{aligned} \tag{24}$$

Since this holds for all $\pi \in \Pi_{\mathcal{G}}$, $\hat{\pi}$ is optimal.

ii) Proceeding in the same way as in (i) we can show that, if (ii) holds,

$$J_g(\hat{\theta}, \hat{\pi}) \leq J_g(\theta, \hat{\pi}),$$

for all $\theta \in \Theta_{\mathcal{G}}$. Then $\hat{\theta}$ is optimal.

iii) If both (i) and (ii) hold then

$$J_g(\hat{\theta}, \pi) \leq J_g(\hat{\theta}, \hat{\pi}) \leq J_g(\theta, \hat{\pi}),$$

for any $(\theta, \pi) \in \Theta_{\mathcal{G}} \times \Pi_{\mathcal{G}}$. Thereby

$$J_g(\hat{\theta}, \hat{\pi}) \leq \inf_{\theta \in \Theta_{\mathcal{G}}} J_g(\theta, \hat{\pi}) \leq \sup_{\pi \in \Pi_{\mathcal{G}}} \big(\inf_{\theta \in \Theta_{\mathcal{G}}} J_g(\theta, \pi)\big).$$

On the other hand,

$$J_g(\hat{\theta}, \hat{\pi}) \geq \sup_{\pi \in \Pi_{\mathcal{G}}} J_g(\hat{\theta}, \pi) \geq \inf_{\theta \in \Theta_{\mathcal{G}}} \big(\sup_{\pi \in \Pi_{\mathcal{G}}} J_g(\theta, \pi)\big).$$

Now due to the inequality

$$\inf_{\theta \in \Theta_{\mathcal{G}}} \big(\sup_{\pi \in \Pi_{\mathcal{G}}} J_g(\theta, \pi)\big) \geq \sup_{\pi \in \Pi_{\mathcal{G}}} \big(\inf_{\theta \in \Theta_{\mathcal{G}}} J_g(\theta, \pi)\big)$$

we have

$$J_g(\hat{\theta}, \hat{\pi}) = \sup_{\pi \in \Pi_{\mathcal{G}}} \big(\inf_{\theta \in \Theta_{\mathcal{G}}} J_g(\theta, \pi)\big) = \inf_{\theta \in \Theta_{\mathcal{G}}} \big(\sup_{\pi \in \Pi_{\mathcal{G}}} J_g(\theta, \pi)\big).$$

□

3 A maximum principle for nonzero-sum games with g–expectations

In this section, we study a nonzero sum stochastic differential games problem with g–expectation. For notational simplification, we consider only two players;

it is similar for n players. The control system is given as before, which is

$$\begin{aligned} dX(t) &= b(t, X(t), u_0(t))dt + \sigma(t, X(t), u_0(t))dW(t) \\ &\quad + \int_{\mathbb{R}_0} \gamma(t, X(t^-), u_1(t^-, z), z)\widetilde{N}(\,dt,\,dz),\ t \in [0, T], \\ X(0) &= x \in \mathbb{R}^n. \end{aligned} \tag{25}$$

Let $u = (u_0, u_1) = (\theta, \pi)$, where $\theta = (\theta_0, \theta_1)$ and $\pi = (\pi_0, \pi_1)$ are controls for player 1 and 2, respectively. Let $\mathcal{G}_t^1 \subseteq \mathcal{F}_t$ and $\mathcal{G}_t^2 \subseteq \mathcal{F}_t$ be two sub–filtrations, representing the information available to player 1 and player 2, respectively, and let $\Theta_{\mathcal{G}^1}$, $\Pi_{\mathcal{G}^2}$ be the corresponding families of admissible control processes $\theta(t)$, $\pi(t)$; $t \in [0, T]$. We denote by $J^i_{g_i}(\theta, \pi)$, $i = 1, 2$, the cost functions corresponding to the two players 1 and 2:

$$J^i_{g_i}(\theta, \pi) = \mathcal{E}_{g_i}\Big[\int_0^T f_i(t, X(t), u(t))\,dt + h_i(X(T))\Big], \qquad i = 1, 2, \tag{26}$$

where $g_i : \mathbb{R}\times\mathbb{R}\times L^2(\nu) \to \mathbb{R}$ are given convex functions satisfying (6). Thus $\mathcal{E}_{g_i}$ represents the preference of player i, $i = 1, 2$. The problem is to find a control $(\theta^*, \pi^*) \in \Theta_{\mathcal{G}^1} \times \Pi_{\mathcal{G}^2}$ such that

$$\begin{cases} J^1_{g_1}(\theta, \pi^*) & \leq J^1_{g_1}(\theta^*, \pi^*), \qquad \text{for all} \quad \theta \in \Theta_{\mathcal{G}^1}; \\ J^2_{g_2}(\theta^*, \pi) & \leq J^2_{g_2}(\theta^*, \pi^*), \qquad \text{for all} \quad \pi \in \Pi_{\mathcal{G}^2}. \end{cases} \tag{27}$$

The pair of controls (θ^*, π^*) is called a *Nash equilibrium* for the game. Note that when player 1 (resp. 2) acts with the strategy θ^* (resp. π^*), the best that 2 (resp. 1) can do is to act with π^* (resp. θ^*).

We use the same method as in the previous section, but adapted to the new situation. We now consider the following forward-backward SDEs:

$$\begin{cases} dX(t) &= b(t, X(t), u_0(t))dt + \sigma(t, X(t), u_0(t))dW(t) \\ &\quad + \int_{\mathbb{R}_0} \gamma(t, X(t^-), u_1(t^-, z), z)\widetilde{N}(\,dt,\,dz), \\ dY_1(t) &= -[g_1(K_1(t), L_1(t,\cdot)) + f_1(t, X(t), u(t))]dt \\ &\quad + K_1(t)dW(t) + \int_{\mathbb{R}_0} L_1(t,z)\widetilde{N}(dt,dz), \\ dY_2(t) &= -[g_2(K_2(t), L_2(t,\cdot)) + f_2(t, X(t), u(t))]dt \\ &\quad + K_2(t)dW(t) + \int_{\mathbb{R}_0} L_2(t,z)\widetilde{N}(dt,dz), \\ X(0) &= X_0, \qquad Y_1(T) = h_1(X(T)), \qquad Y_2(T) = h_2(X(T)). \end{cases} \tag{28}$$

The cost functions $J^i_{g_i}(\theta, \pi)$, $i = 1, 2$, now take the form:

$$\begin{aligned} &J^i_{g_i}(\theta, \pi) = Y_i(0) \\ &= \mathbb{E}\Big[h_i(X(T)) + \int_0^T (g_i(K_i(t), L_i(t,\cdot)) + f_i(t, X(t), u(t)))dt\Big],\ i = 1, 2. \end{aligned} \tag{29}$$

We want to find a Nash equilibrium for the game, i.e. a pair (θ^*, π^*), such that the inequalities (27) are satisfied.

Let us introduce two Hamiltonian functions associated with this game, namely H_1 and H_2, as follows:

$$H_i: \ [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times L^2(\nu) \times K \times K \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times L^2(\nu) \to \mathbb{R}$$

are defined by

$$\begin{aligned} & H_i(t,x,y_i,k_i,l_i,\theta,\pi,\mu_i,\varphi_i,\psi_i,\phi_i) = g_i(k_i,l_i) + f_i(t,x,\theta,\pi) \\ & + (g_i(k_i,l_i) + f_i(t,x,\theta,\pi))\mu_i + b(t,x,\theta,\pi)\varphi_i \\ & + \sigma(t,x,\theta,\pi)\psi_i + \int_{\mathbb{R}_0} \gamma(t,x,\theta,\pi,z)\phi_i(t,z)\nu(dz), \qquad i = 1,2. \end{aligned} \tag{30}$$

We assume that H_i is differentiable with respect to the variables x, y_i, k_i, l_i, respectively. The *adjoint equations* in the unknown adapted processes μ_i, φ_i, ψ_i and ϕ_i, $i = 1,2$, is following FBSDE:

$$\begin{cases} d\mu_i(t) &= \frac{\partial H_i}{\partial y}(t,X(t),Y_i(t),K_i(t),L_i(t),\theta(t),\pi(t),\mu_i(t),\varphi_i(t),\psi_i(t),\phi_i(,\cdot))dt \\ &+ \frac{\partial H_i}{\partial k_i}(t,X(t),Y_i(t),K_i(t),L_i(t),\theta(t),\pi(t),\mu_i(t),\varphi_i(t),\psi_i(t),\phi_i(,\cdot))dW(t) \\ &+ \int_{\mathbb{R}_0} \nabla_{l_i} H_i(t,X(t),Y_i(t),K_i(t),L_i(t),\theta(t),\pi(t),\mu_i(t),\varphi_i(t),\psi_i(t),\phi_i(,\cdot))\widetilde{N}(dt,dz), \\ d\varphi_i(t) &= -\frac{\partial H_i}{\partial x}(t,X(t),Y_i(t),K_i(t),L_i(t),\theta(t),\pi(t),\mu_i(t),\varphi_i(t),\psi_i(t),\phi_i(,\cdot))dt \\ &+ \psi_i(t)dW(t) + \int_{\mathbb{R}_0} \phi_i(t,z)\widetilde{N}(dt,dz), \\ \mu_i(0) &= 0, \qquad \varphi_i(T) = (1+\mu_i(T))h_i'(X(T)). \end{cases} \tag{31}$$

The following result is a generalization of Theorem 2.1: (As in Section 2 we let Θ and Π denote the set of possible control values of $\theta(t)$, $t \in [0,T]$ and $\pi(t)$, $t \in [0,T]$, respectively.)

Theorem 3.1. *Let $(\hat{\theta},\hat{\pi}) \in \Theta_{\mathcal{G}^1} \times \Pi_{\mathcal{G}^2}$ with corresponding state processes $\widehat{X}(t)$, $\widehat{Y}_1(t)$ and $\widehat{Y}_2(t)$. Suppose there exists a solution $(\hat{\varphi}_i(t),\hat{\psi}_i(t),\hat{\phi}_i(t,z))$, $i = 1,2$, of the corresponding adjoint equation* (31) *such that*

$$\begin{aligned} & \max_{\pi\in\Pi} \mathbb{E}[H_1(t,\widehat{X}(t),\widehat{Y}_1(t),\widehat{K}_1(t),\widehat{L}_1(t),\hat{\theta}(t),\pi,\widehat{\mu}_1(t),\widehat{\varphi}_1(t),\widehat{\psi}_1(t),\widehat{\phi}_1(,\cdot))|\mathcal{G}_t^2] \\ & = \mathbb{E}[H_1(t,\widehat{X}(t),\widehat{Y}_1(t),\widehat{K}_1(t),\widehat{L}_1(t),\hat{\theta}(t),\hat{\pi}(t),\widehat{\mu}_1(t),\widehat{\varphi}_1(t),\widehat{\psi}_1(t),\widehat{\phi}_1(,\cdot))|\mathcal{G}_t^2], \end{aligned} \tag{32}$$

and

$$\begin{aligned} & \max_{\theta\in\Theta} \mathbb{E}[H_2(t,\widehat{X}(t),\widehat{Y}_2(t),\widehat{K}_2(t),\widehat{L}_2(t),\theta,\hat{\pi}(t),\widehat{\mu}_2(t),\widehat{\varphi}_2(t),\widehat{\psi}_2(t),\widehat{\phi}_2(,\cdot))|\mathcal{G}_t^1] \\ & = \mathbb{E}[H_2(t,\widehat{X}(t),\widehat{Y}_2(t),\widehat{K}_2(t),\widehat{L}_2(t),\hat{\theta}(t),\hat{\pi}(t),\widehat{\mu}_2(t),\widehat{\varphi}_2(t),\widehat{\psi}_2(t),\widehat{\phi}_2(,\cdot))|\mathcal{G}_t^1]. \end{aligned} \tag{33}$$

Moreover, suppose that, for all $t \in [0,T]$, $H_i(t,x,y_i,k_i,l_i,\theta,\pi,\hat{\mu}_i,\hat{\varphi}_i,\hat{\psi}_i,\hat{\phi}_i)$ is concave in x, y_i, k_i, l_i, θ, π and $g_i(x)$ is concave in x, $i = 1,2$. Then $(\hat{\theta}(t),\hat{\pi}(t))$ is a Nash equilibrium for the game.

Proof. Proceeding as in proof of Theorem 2.1 we have

$$
\begin{aligned}
J^1_{g_1}(\hat\theta,\hat\pi) - J^1_{g_1}(\hat\theta,\pi) &= \mathbb{E}\Big[h_1(\hat X(T)) - h_1(X^{\hat\theta}(T)) \\
&\quad + \int_0^T \big\{\hat g_1(t) - g_1^{\hat\theta}(t) + \hat f_1(t) - f_1^{\hat\theta}(t)\big\}dt\Big] \\
&= \mathbb{E}[h_1(\widehat X(T)) - h_1(X^{\hat\theta}(T)) + (\widehat Y_1(0) - Y_1^{\hat\theta}(0))\hat\mu_1(0)] \\
&+ \mathbb{E}\Big[\int_0^T \Big\{\widehat H_1(t) - H_1^{\hat\theta}(t) - \Big((\hat g_1(t) - g_1^{\hat\theta}(t) + \hat f_1(t) - f_1^{\hat\theta}(t))\hat\mu_1(t) \\
&\quad - (\hat b(t) - b^{\hat\theta}(t))\widehat\varphi_1(t) - (\widehat\sigma(t) - \sigma^{\hat\theta}(t))\widehat\psi_1(t) \\
&\quad - \int_{\mathbb{R}_0} (\widehat\gamma(t) - \gamma^{\hat\theta}(t))\widehat\phi_1(t,z)\nu(dz)\Big)\Big\}dt\Big] \\
&= \mathbb{E}\Big[\int_0^T -\Big(\bigtriangledown_x \hat H(t)(\widehat X(t) - X^{\hat\theta}(t)) + \bigtriangledown_{y_1}\widehat H(t)(\widehat Y_1(t) - Y_1^{\hat\theta}(t)) \\
&\quad + \bigtriangledown_{k_1}\widehat H(t)(\widehat K_1(t) - K_1^{\hat\theta}(t)) + \int_{\mathbb{R}_0} \bigtriangledown_{l_1}\widehat H(t)(\widehat L_1(t) - L_1^{\hat\theta}(t))\nu(dz)\Big)dt \\
&+ \mathbb{E}\Big[\int_0^T \Big((\hat g_1(t) - g_1^{\hat\theta}(t) + \hat f_1(t) - f_1^{\hat\theta}(t))\hat\mu_1(t) + (\hat b(t) - b^{\hat\theta}(t))\widehat\varphi_1(t) \\
&\quad + (\widehat\sigma(t) - \sigma^{\hat\theta}(t))\widehat\psi_1(t) + \int_{\mathbb{R}_0} (\widehat\gamma(t) - \gamma^{\hat\theta}(t))\widehat\phi_1(t,z)\nu(dz)\Big)dt\Big] \\
&+ \mathbb{E}\Big[\int_0^T \Big\{\widehat H_1(t) - H_1^{\hat\theta}(t) - \Big((\hat g_1(t) - g_1^{\hat\theta}(t) + \hat f_1(t) - f_1^{\hat\theta}(t))\hat\mu_1(t) \\
&\quad + (\hat b(t) - b^{\hat\theta}(t))\widehat\varphi_1(t) + (\widehat\sigma(t) - \sigma^{\hat\theta}(t))\widehat\psi_1(t) \\
&\quad + \int_{\mathbb{R}_0} (\widehat\gamma(t) - \gamma^{\hat\theta}(t))\widehat\phi_1(t,z)\nu(dz)\Big)\Big\}dt\Big] \\
&= \mathbb{E}\Big[\int_0^T \Big\{\widehat H_1(t) - H_1^{\hat\theta}(t) - \Big(\bigtriangledown_x \widehat H_1(t)(\widehat X(t) - X^{\hat\theta}(t)) \\
&\quad + \bigtriangledown_{y_1}\widehat H_1(t)(\widehat Y_1(t) - Y_1^{\hat\theta}(t)) + \bigtriangledown_{k_1}\widehat H_1(t)(\widehat K_1(t) - K_1^{\hat\theta}(t)) \\
&\quad + \int_{\mathbb{R}_0} \bigtriangledown_{l_1}\widehat H_1(t)(\widehat L_1(t) - L_1^{\hat\theta}(t))\nu(dz)\Big)\Big\}dt\Big]. \qquad (34)
\end{aligned}
$$

Since H_1 is concave in x, y_1, k_1, l_1 and π, we get,

$$
\begin{aligned}
\mathbb{E}\Big[\int_0^T (\widehat H_1(t) - H_1^{\hat\theta}(t))dt\Big] &\geq \mathbb{E}\Big[\int_0^T \Big(\bigtriangledown_x \widehat H_1(t)(\widehat X(t) - X^{\hat\theta}(t)) \\
&+ \bigtriangledown_{y_1}\widehat H_1(t)(\widehat Y_1(t) - Y_1^{\hat\theta}(t)) + \bigtriangledown_{k_1}\widehat H_1(t)(\widehat K_1(t) - K_1^{\hat\theta}(t)) \\
&+ \int_{\mathbb{R}_0} \bigtriangledown_{l_1}\widehat H_1(t)(\widehat L_1(t) - L_1^{\hat\theta}(t))\nu(dz) + \bigtriangledown_\pi \widehat H_1(t)(\hat\pi(t) - \pi(t))\Big)dt\Big]. \qquad (35)
\end{aligned}
$$

Combining the above we get

$$\begin{aligned} J^1_{g_1}(\hat{\theta},\hat{\pi}) - J^1_{g_1}(\hat{\theta},\pi) &\geq \mathbb{E}\Big[\int_0^T \bigtriangledown_\pi \widehat{H}_1(t)(\hat{\pi}(t)-\pi(t))dt\Big] \\ &= \mathbb{E}\Big[\int_0^T \mathbb{E}[\bigtriangledown_\pi \widehat{H}_1(t)(\hat{\pi}(t)-\pi(t))|\mathcal{G}^2_t]dt\Big]. \qquad (36) \end{aligned}$$

On the other hand, the condition (32) gives,

$$\mathbb{E}\Big[\bigtriangledown_\pi \widehat{H}_1(t)(\hat{\pi}(t)-\pi(t))|\mathcal{G}^2_t\Big] = (\hat{\pi}(t)-\pi(t)) \bigtriangledown_\pi \mathbb{E}[\widehat{H}_1(t)|\mathcal{G}^2_t]_{\pi=\hat{\pi}(t)} \geq 0. \quad (37)$$

Hence

$$J^1_{g_1}(\hat{\theta},\hat{\pi}) - J^1_{g_1}(\hat{\theta},\pi) \geq 0. \quad (38)$$

In the same way we show that

$$J^2_{g_2}(\hat{\theta},\hat{\pi}) - J^2_{g_2}(\theta,\hat{\pi}) \geq 0, \quad (39)$$

whence the desired result. □

4 Application in finance

We now apply our result in the previous section to study the worst case model risk management problem. Firstly, we recall the definition of the convex risk measure and its relation to g–expectation.

Definition 4.1. Let $\mathbb{F} = L^p(\mathbb{P})$ for some $p \in [1,\infty]$. A *convex risk measure* is a functional $\rho : \mathbb{F} \to \mathbb{R}$ that satisfies the following properties:

(i) (convexity) $\rho(\lambda X + (1-\lambda)Y) \leq \lambda\rho(X) + (1-\lambda)\rho(Y)$; $X, Y \in \mathbb{F}, \lambda \in (0,1)$,

(ii) (monotonicity) If $X \leq Y$ then $\rho(X) \geq \rho(Y)$, $\qquad X, Y \in \mathbb{F}$,

(iii) (translation invariance)

$$\rho(X+m) = \rho(X) - m, \qquad X \in \mathbb{F}, m \in \mathbb{R}.$$

To connect to the above theory we give another representation of the convex risk measure in term of g–expectation:

Definition 4.2. The *risk* $\rho(\xi)$ of random variable ξ (ξ can be seen as a *financial position* of a trader in a a financial market) is defined by

$$\rho(\xi) := \mathcal{E}_g[-\xi] := Y(0) \quad (40)$$

where $Y(0)$ is the value at $t = 0$ of the solution BSDE (7).

Suppose that the finance market consists of one risky finance asset, whose unit price is denoted by $S_1(t)$, and one risk–free asset, whose price at time t is denoted by $S_0(t)$. We use the following stochastic differential equation to describe this financial market.

$$\begin{cases} dS_0(t) &= r(t)S_0(t)dt; \qquad S_0(0)=1, \\ dS_1(t) &= S_1(t^-)\Big[\alpha(t)dt+\beta(t)dW(t)+\int_{\mathbb{R}}\gamma(t,z)\widetilde{N}(dt,dz)\Big], \\ S_1(0) &> 0, \end{cases} \tag{41}$$

where $r(t)$ is a deterministic function, $\alpha(t),\beta(t)$ and $\gamma(t,z)$ are given $\mathcal{F}_t$-predictable functions satisfying the following integrability condition:

$$E\Big[\int_0^T \Big\{ \mid r(s)\mid + \mid \alpha(s)\mid + \frac{1}{2}\beta(s)^2 + \int_{\mathbb{R}} \mid \log(1+\gamma(s,z))-\gamma(s,z)\mid \nu(dz)\Big\}ds\Big] < \infty, \tag{42}$$

where T is fixed. We assume that

$$\gamma(t,z)\geq -1 \qquad \text{for a.a. } t,z\in[0,T]\times\mathbb{R}_0, \tag{43}$$

where $\mathbb{R}_0=\mathbb{R}\backslash\{0\}$. This model represents a natural generalization of the classical Black-Scholes market model to the case where the coefficients are not necessarily constants, but allowed to be (predictable) stochastic processes. Moreover, we have added a jump component. See e.g. [3] or [7] for discussions of such markets.

Let $\mathcal{G}_t\subseteq\mathcal{F}_t$ be a given sub-filtration and $\pi(t)$ be a portfolio, representing the *fraction* of the total wealth invested in the risky asset at time t. Then the dynamics of the corresponding wealth process $X^{(\pi)}(t)$ is

$$\begin{cases} dX^{(\pi)}(t) &= X^{(\pi)}(t^-)\Big[\{r(t)+(\alpha(t)-r(t))\pi(t)\}dt \\ &+ \ \pi(t)\beta(t)dW(t)+\pi(t^-)\int_{\mathbb{R}}\gamma(t,z)\widetilde{N}(dt,dz)\Big], \\ X^{(\pi)}(0) &= x>0. \end{cases} \tag{44}$$

A portfolio π is called *admissible* if it is a measurable càdlàg stochastic process adapted to filtration $\mathcal{G}_t$ and satisfies

$$\pi(t^-)\gamma(t,z)>-1 \qquad \text{a.s.}$$

and

$$\int_0^T \Big\{ \mid r(t)+(\alpha(t)-r(t))\pi(t)\mid + \pi^2(t)\beta^2(t) + \pi^2(t)\int_{\mathbb{R}}\gamma^2(t,z)\nu(dz)\Big\}dt<\infty \qquad \text{a.s.} \tag{45}$$

The requirement that π should be adapted to the filtration $\mathcal{G}_t$ is a mathematical way of requiring that the choice of the portfolio value $\pi(t)$ at time t is only allowed to depend on the information (σ-algebra) $\mathcal{G}_t$ only. The wealth process corresponding to an admissible portfolio π is the solution of (44):

$$
\begin{aligned}
X^{(\pi)}(t) = x \exp\Big[\int_0^t &\{r(t) + (\alpha(t) - r(t))\pi(t) - \frac{1}{2}\pi^2(t)\beta^2(t) \\
&+ \int_{\mathbb{R}} (\ln(1 + \pi(s)\gamma(s,z)) - \pi(z)\gamma(s,z))\nu(dz)\}ds \\
&+ \int_0^t \pi(s)\beta(s)dW(s) + \int_0^t \int_{\mathbb{R}} \ln(1 + \pi(s)\gamma(s,z))\widetilde{N}(ds,dz)\Big]. \quad (46)
\end{aligned}
$$

Suppose that the market chooses the mean relative growth rate $\alpha(t)$ of the risky asset. Let $u(\cdot) = \alpha(\cdot)$ be the control process for the market. We denote by $\mathcal{A}$ and Π be the families of admissible controls for u and π, respectively. The cost function is now defined as follows:

$$J_g(u,\pi) := \rho(X^{(u,\pi)}(T)). \quad (47)$$

By the performance of (40) this cost function becomes

$$J_g(u,\pi) = \mathcal{E}_g[-X^{(u,\pi)}(T)]\big). \quad (48)$$

We then introduce our problem is to find $(u^*,\pi^*) \in \mathcal{A} \times \Pi$ such that

$$J_g(u^*,\pi^*) = \mathcal{E}_g[-X^{(u^*,\pi^*)}(T)] = \sup_{u\in\mathcal{A}} \big(\inf_{\pi\in\Pi} \mathcal{E}_g[-X^{(u,\pi)}(T)]\big). \quad (49)$$

Similarly as in the previous section the corresponding state process for $X(t) = X^{(u,\pi)}(t)$, $Y(t) = Y^{(\pi)}(t)$, $K(t) = K^{(\pi)}(t)$, $L(t,z) = L^{(\pi)}(t,z)$ in (11) becomes

$$
\begin{cases}
dX(t) &= X(t^-)\Big[\{r(t) + (u(t) - r(t))\pi(t)\}dt \\
&\quad + \pi(t)\beta(t)dW(t) + \pi(t^-)\int_{\mathbb{R}} \gamma(t,z)\widetilde{N}(dt,dz)\Big], \\
dY(t) &= -g(t,K(t),L(t))dt + K(t)dW(t) + \int_{\mathbb{R}_0} L(t,z)\widetilde{N}(dt,dz), \\
X(0) &= x, \qquad Y(T) = -X^{\pi}(T)
\end{cases} \quad (50)
$$

Then the problem (49) now is written in the following form

$$
\begin{aligned}
J_g(u^*,\pi^*) &= \sup_{u\in\mathcal{A}} \big(\inf_{\pi\in\Pi} J_g(u,\pi)\big) \\
&= \sup_{u\in\mathcal{A}} \Big(\inf_{\pi\in\Pi} \mathbb{E}\Big[-X^{(u^*,\pi^*)}(T) + \int_0^T g(t,K(t),L(t))dt\Big]\Big) \quad (51)
\end{aligned}
$$

By (14) the modification Hamiltonian becomes

$$
\begin{aligned}
H(t,x,y,k,l,u,\pi,\varphi,\psi,\phi) &= g(t,k,l) + g(t,k,l)\mu + x(r(t) + (u - r(t))\pi)\varphi \\
&\quad + x\beta(t)\pi\psi + \int_{\mathbb{R}_0} x\pi\gamma(t,z)\phi(\cdot,z)\nu(dt,dz). \quad (52)
\end{aligned}
$$

And FBSDE of adjoint equation is in following form

$$\begin{cases} d\mu(t) &= (1+\mu(t))\Big[\frac{\partial g}{\partial k}(t,k,l)dW(t) + \int_{\mathbb{R}_0} \bigtriangledown_l g(t,k,l)\widetilde{N}(dt,dz)\Big], \\ d\varphi(t) &= -\Big((r(t)+(\alpha(t)-r(t))\pi(t))\varphi(t) + \beta(t)\pi(t)\psi(t) \\ &+ \int_{\mathbb{R}_0} \pi(t)\gamma(t,z)\phi(t,z)\nu(dt,dz)\Big)dt + \psi(t)dW(t) + \int_{\mathbb{R}} \phi(t,z)\widetilde{N}(dt,dz), \\ \mu(0) &= 0, \qquad \varphi(T) = -(1+\mu(T)) \end{cases} \tag{53}$$

Let $(\hat{u},\hat{\pi})$ be a candidate for an optimal control and let $\hat{X}(t)$, $\hat{Y}(t)$ be the corresponding optimal processes with corresponding solution $\hat{\mu}(t)$, $\hat{\varphi}(t)$, $\hat{\psi}(t)$ and $\hat{\phi}(t)$ of the adjoint equation (53). Then the condition (16) becomes:

$$\begin{aligned} &\mathbb{E}\Big[\frac{\partial H}{\partial \pi}(t,\hat{X}(t),\hat{Y}(t),\hat{K}(t),\hat{L}(t),\hat{u}(t),\pi,\hat{\varphi}(t),\hat{\psi}(t),\hat{\phi}(t))\Big|\mathcal{G}_t\Big]_{\pi=\hat{\pi}(t)} \\ &= \mathbb{E}\Big[(\hat{u}(t)-r(t))\hat{\varphi}(t) + \beta(t)\hat{\psi}(t) + \int_{\mathbb{R}_0} \gamma(t,z)\hat{\phi}(t,z)\nu(dt,dz)\Big|\mathcal{G}_t\Big]_{\pi=\hat{\pi}(t)} = 0 \end{aligned} \tag{54}$$

and

$$\begin{aligned} &\mathbb{E}\Big[\frac{\partial H}{\partial u}(t,\hat{X}(t),\hat{Y}(t),\hat{K}(t),\hat{L}(t),u,\hat{\pi}(t),\hat{\varphi}(t),\hat{\psi}(t),\hat{\phi}(t))\Big|\mathcal{G}_t\Big]_{u=\hat{u}(t)} \\ &= \mathbb{E}[\hat{\pi}(t)\hat{\varphi}(t)|\mathcal{G}_t]_{u=\hat{u}(t)} = 0 \end{aligned} \tag{55}$$

For (55) to exist, we must have $\hat{\pi}(t) = 0$. Hence the optimal control for the worst case scenario problem (49) is $(u^*,\pi^*) = (u(t),0)$, for all $u(t)$ which satisfy (54).

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