# Backward Stochastic Equations and Equilibrium Pricing in Incomplete Markets


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[1]This talk is based on joint work with many people.

# Outline

- Securitization.
- Equilibrium pricing and market completion.
  - Risk Processes, preferences, and payoffs.
  - The representative agent.
  - Existence and uniqueness of equilibrium.
- BSDE characterization of equilibrium pricing kernels.
- Numerical illustrations.
- The general theory in discrete time.

# Some Related Literature


- H & Müller: "On the spanning property of risk bond priced by equilibrium", Math of OR, 32, 784-807, 2007.
- H, Pirvu & Dos Reis: "On securitization, market completion and equilibrium risk transfer", Working paper, 2008.
- Cheridito et al.: "Equilibrium pricing in incomplete markets under translation invariant preferences".

# Securitization

# Securitization

- Securitization: Transformation of non-tradable risk factors into tradable financial securities.
- Economic problems related to securitization:
  - cross-hedging of financial securities;
  - optimization and pricing in incomplete markets;
  - ...
- Mathematical problems related to securitization:
  - utility optimization in incomplete markets;
  - backward stochastic differential equations;
  - ...
- Our focus is on equilibrium pricing in incomplete markets.

# The Model

# Risk Processes and Payoffs

- The agents are exposed financial and non-financial risk factors.
- The non-financial risk factor follows a BM with drift:

$$dR_t = a_t dt + b dW_t^R$$

- The agents can trade a stock with exogenous price process

$$\frac{dS_t}{S_t} = \theta^S(R_t)dt + dW_t^S$$

and a (weather) derivative written on the external risk factor:

$$H = h(S_T, R_T) + \int_0^T \varphi(u, S_u, R_u)du$$

The derivative is priced to match supply (exogenous) and demand.

# Preferences

- The agents' preferences are generated by a BSDE:

$$-dY_t^a = g^a(t, Z_t)dt + Z_t dW_t \quad \text{with} \quad Y_T^a = -\xi^a.$$

As such they are time-consistent and translation invariant:

$$Y_t^a(\xi^a) = Y_t^a \circ Y_{t+s}^a(\xi^a)$$

and

$$Y_t^a\left(\xi^a + \mathscr{F}_t\right) = Y_t^a(\xi^a) - \mathscr{F}_t.$$

- H. and Müller (2007) considered the entropic case

$$g^a(t, z) = \frac{1}{2\gamma_a}\|z\|^2 \quad \text{and} \quad \theta^S(R_t) \equiv \theta^S.$$

# Market Completion

# Market Completion

• With two sources of risk and two assets, our model is potentially complete.

• However: there is no a-priori reason to assume completeness in equilibrium.

• H. and Müller (2007) proved completeness for exponential utility functions using duality and PDE methods. We rely on probabilistic methods, in particular the Haussmann formula.

• The issue of completeness does not arise in discrete time models.

# Pricing Schemes

# Pricing Schemes

• We assume that the agents have no impact on stock prices so the restriction of any pricing measure on $\mathscr{F}^S$ is given by $\theta^S$.

• For any "admissible" pricing measure $\mathbb{Q}$ its density is a uniformly integrable martingale:

$$Z_t = \exp\left( - \int_0^t \begin{pmatrix} \theta^S \\ \theta_s^R \end{pmatrix} d \begin{pmatrix} W_s^S \\ W_s^R \end{pmatrix} - \frac{1}{2} \int_0^t |\theta_s|^2 ds \right)$$

• The set of pricing rules can be identified with the set of

market prices of external risk $\theta^R = (\theta_s^R)$

such that $(Z_t)$ defined by the equation above is a u.i. martingale.

# Pricing Schemes

- For any such pricing scheme the derivative price process is

$$B_t^\theta = \mathbb{E}^\theta[H|\mathscr{F}_t].$$

- Using the martingale representation property w.r.t. $\mathbb{P}^\theta$:

$$dB_t^\theta = \kappa_t^R(\theta_t^R dt + dW^R) + ...dt + ...dW_t^S.$$

- The market is complete in equilibrium iff the endogenous volatility process $\kappa^R$ is almost surely different from zero.

# Utility Optimization

# Utility Optimization

• For a candidate process $\theta^R$ we assume first that the market is complete and solve the agents' optimization problems under $\mathbb{P}^\theta$.

• For a given admissible trading strategy $\pi$ the risk follows:

$$-dY_t^a(\pi) = g^a(t, Z_t^a)dt - Z_t^a dW_t$$

with terminal condition

$$Y_T^a(\pi) = -H^a - V_T^{a,\theta}(\pi).$$

• The optimization problem is thus given by:

$$\min_{\pi^a \in \mathbb{S}^\theta} Y_0^a(\pi^a)$$

where the set of admissible strategies $\mathbb{S}^\theta$ depends on the driver, ...

# Utility Optimization

- The optimization problem is equivalent to minimizing the residual risk which follows a BSDE of the form

$$-d\hat{Y}_t^a(\pi) = G^a(t, \pi_t^a, X_t, \hat{Z}_t^a)dt - \hat{Z}_t^a dW_t$$

with terminal condition

$$\hat{Y}_T^a(\pi) = -H^a.$$

Theorem (Existence of optimal trading strategies)

*If some form of comparison principle holds, then*

$$\Pi^a(z) = \mathrm{argmin}_\pi G^a(t, \pi, x, z)$$

*yields an optimal strategy* $\hat{\pi}_t^a = \Pi^a(\tilde{Z}_t^a)$ *where* $(\tilde{Y}^a, \tilde{Z}^a)$ *solves*

$$-d\tilde{Y}_t^a = G^a(t, \Pi^a(\tilde{Z}^a), X_t, \tilde{Z}_t^a)dt - \tilde{Z}_t^a dW_t, \quad \tilde{Y}_T^a = -H^a$$

# The Representative Agent

# The representative agent

- We characterize an equilibrium in terms of a BSDE associated with the preferences of a representative agent.

- The risk preferences are generated by the BSDE

$$-dY_t^{ab} = g^{ab}(t, Z_t^{ab})dt - Z_t^{ab}dW_t$$

with the terminal condition

$$Y_T^{ab}(\pi) = -H^a - H^b - H - V_T^{ab,\theta}(\pi)$$

where the driver $g^{a,b}$ is defined by the inf-convolution:

$$g^{ab}(t,z) = \inf_x \{g^a(t, z-x) + g^b(t,x)\}.$$

# The first order conditions

- The first order conditions for optimality are:

$$g^{ab}_{z_1}\left(t, Z^{ab}_t - \pi^{ab,1}_t \begin{pmatrix} S_t \\ 0 \end{pmatrix} - \pi^{ab,2}_t \kappa^\theta_t\right) = -\theta^S_t,$$

and

$$g^{ab}_{z_2}\left(t, Z^{ab}_t - \pi^{ab,1}_t \begin{pmatrix} S_t \\ 0 \end{pmatrix} - \pi^{ab,2}_t \kappa^\theta_t\right) = -\theta^R_t.$$

- Equilibrium condition is $\pi^{ab,2}_t \equiv 0$; equations independent of $\kappa^\theta$.
- The second equation yields equilibrium condition for $\theta^R$.

# The representative agent

## Theorem (Existence and Characterization of Equilibrium)

*Let us assume that (in addition to other technical conditions) the following conditions are satisfied:*

- *There exists a solution* $(Y, Z)$ *of the backward equation*

$$Y_t = -H^a - H^b - H + \int_t^T G(s, Z_s)\, ds - \int_t^T Z_s\, dW_s$$

*with driver*

$$G(t, Z_t) \triangleq g^{ab}\left(t, Z_t - \tilde{\pi}_t^1(Z_t)\begin{pmatrix} S_t \\ 0 \end{pmatrix}\right) - \tilde{\pi}_t^1(Z_t) S_t \theta_t^S,$$

*where* $\tilde{\pi}_t^1 = \tilde{\pi}_t^1(z)$ *is a solution (in* $x$*) of the equation*

$$g_{z_1}^{ab}\left(t, z - x\begin{pmatrix} S_t \\ 0 \end{pmatrix}\right) = -\theta_t^S.$$

# The representative agent

## Theorem (continued)

- *The process* $\theta^R = \theta^R(z)$ *defined implicitly by*

$$-\theta_t^R(z) = g_{z_2}^{ab}\left(t, z - \tilde{\pi}_t^{ab,1}(z)\begin{pmatrix} S_t \\ 0 \end{pmatrix}\right) \quad (1)$$

  *can be represented in terms of a Lipschitz continuous function of the forward process:*

$$\theta_t^R = u(t, S_t, R_t).$$

*Then, the process* $\theta^R$ *along with the market price of financial risk* $\theta^S$ *defines an equilibrium pricing measure.*

# Example: Entropic Utilities

# (Semi-) Entropic utilities

## Theorem (Equilibrium for (semi-)entropic utilities)

- *The preferences come from:*

$$g^a(z) = \frac{1}{2\gamma_a}\|z\|^2.$$

- *The market price of financial risk is of the form:*

$$(\theta_t^S)^2 = \Gamma(R_t)$$

  *for a bounded function* $\Gamma$ *with bounded* $1^{st}$ *and* $2^{nd}$ *derivative.*

- *The derivative's payoff is increasing in the external risk factor and strictly increasing on a set of positive measure.*

*Then an equilibrium exists. More precisely, the following holds:*

# (Semi-) Entropic utilities

## Theorem (continued)

- *There exists a unique equilibrium market price of external risk. It is given by the second component of the integrand part of the solution* $(Y, Z)$ *of the BSDE*

  $$Y_t = H^{rep} - \int_t^T z_s dW_s + \frac{1}{2}\int_t^T [-(z_s^2)^2 + (\theta_s^S)^2 - 2\theta_s^S z_s^1]\, ds$$

  *with terminal condition*

  $$H^{rep} \triangleq \frac{H^a + H^b + nH}{\gamma_R} \quad \textit{where} \quad \gamma^R = \gamma^a + \gamma^b.$$

- *The equilibrium market price of external risk is differentiable with respect to* $\gamma_R$ *and the number of available derivatives.*

# Some Numerical Illustrations

# Numerical illustrations for entropic utilities

There are two sources of randomness described by two diffusions

$$R_t = 4t + 2W_t^R; \quad \frac{dS_t}{S_t} = \theta^S(R_t)dt + dW_t^S$$

and two agent with dynamic entrpoic utility functions and payoffs

$$H^i = \frac{1}{2}S_T + \int_0^T \exp\left\{-0.5(R_t - R_i)^2 dt\right\} \quad (i = a, b)$$

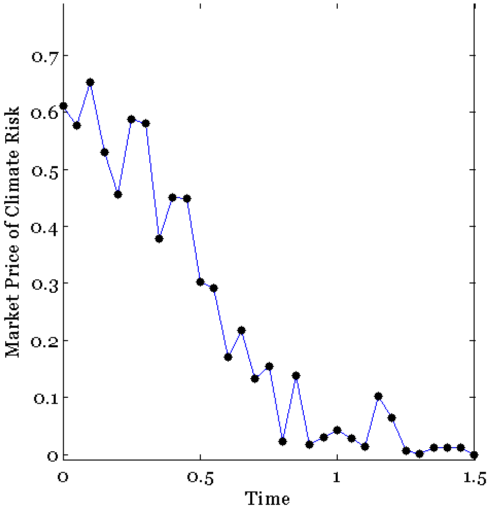
where

$$R^a = 4; \quad R^b = -1.$$

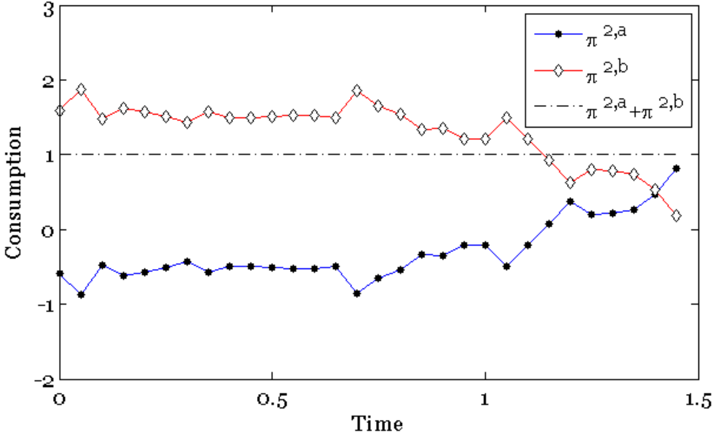
The dividend pays interest at a rate

$$\varphi_t = \exp\left\{-\left(4t - R_t\right)^+\right\}.$$

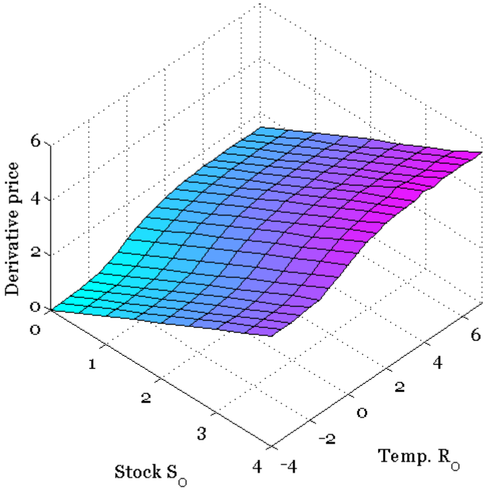
# Sample Market Price of Risk



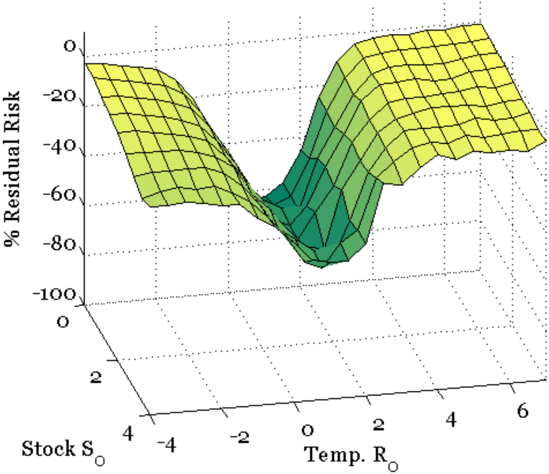
# Sample Trading Strategies



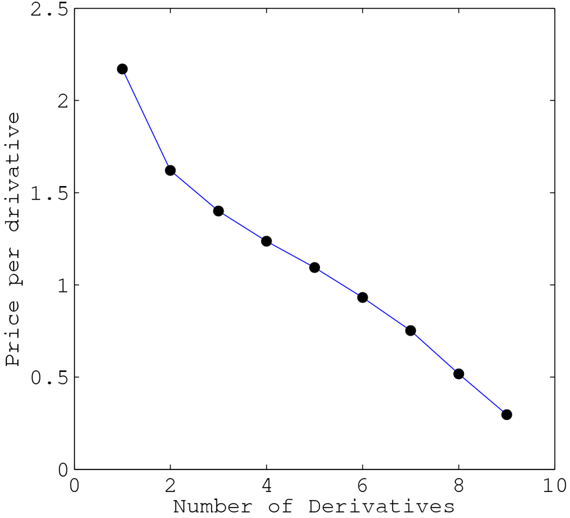
# Equilibrium Prices



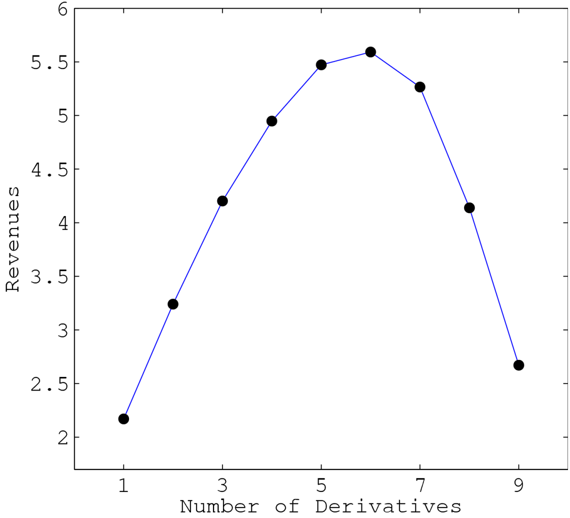
# Benefits of Financial Innovation



# Equilibrium as a Non-Linear Pricing Scheme



# Equilibrium as a Non-Linear Pricing Scheme



# Discrete Time
# - The General Structure -

# Equilibrium Pricing in Discrete Time

The agents are endowed with random payoffs $H^a$; at any time $t \in \{1, 2, ..., T\}$ they maximize a preference functional

$$U_t^a : L(\mathscr{F}_T) \to L(\mathscr{F}_t)$$

which is normalized, monotone, $\mathscr{F}_t$-translation invariant

$$U_t^a(X + Z) = U_t^a(X) + Z \quad \text{for all} \quad Z \in \mathscr{F}_t$$

convex and strongly time consistent, i.e.,

$$U_t^a(X) = U_t^a \circ U_{t+1}^a(X).$$

AGENTS MAXIMIZE UTILITY FROM TRADING IN A FINANCIAL MARKET.

# Equilibrium Pricing in Discrete Time

The agents can trade stocks and securities. The holdings in stocks and securities at time $t$are denoted

$$\eta_t^a \quad \text{and} \quad \vartheta_t^a.$$

- Stock prices follow an exogenous stochastic process $\{S_t\}_{t=1}^T$
- Securities are in fixed supply and priced to match supply and demand. Security prices $\{R_t\}_{t=1}^T$ are endogenous.
- Securities pay a dividend $R$ at maturity so

$$R_T = R.$$

We derive an existence and uniqueness of equilibrium result using a representative agent approach.

# Equilibrium

An equilibrium consists of a trading strategy $\{(\hat{\eta}_t^a, \hat{\vartheta}_t^a)\}$ for every agent $a \in \mathbb{A}$ and a price process $\{R_t\}$ with $R_T = R$ s.t.:

- Individual optimality:

$$\begin{aligned} & U_t^a\left(H^a + \sum_{s=t}^{T-1}\{\hat{\eta}_s^a \cdot \Delta S_{s+1} + \hat{\vartheta}_s^a \cdot \Delta R_{s+1}\}\right) \\ \geq \quad & U_t^a\left(H^a + \sum_{s=t}^{T-1}\{\eta_s^a \cdot \Delta S_{s+1} + \vartheta_s^a \cdot \Delta R_{s+1}\}\right) \end{aligned}$$

  for all $a \in \mathbb{A}$, $t = 1, ..., T$, and $(\eta_{t+1}^a, \vartheta_{t+1}^a), ..., (\eta_T^a, \vartheta_T^a)$.

- Market clearing in the securities market:

$$\sum_{a \in \mathbb{A}} \hat{\vartheta}_t^a = n \quad \text{for all} \quad t = 1, ..., T-1.$$

The problem of dynamic equilibrium pricing can be reduced to a sequence of one period models.

# Existence of Equilibrium

### Theorem (Equilibrium and the Representative Agent)

*An equilibrium exists if and only if some representative agent has an optimal trading strategy.*

### Theorem (Existence of Equilibrium)

*If the agents are sensitive to large losses in the sense that*

$$\lim_{\lambda \to \infty} U_1^a(\lambda X) = -\infty \quad \textit{if} \quad \mathbb{P}[X < 0] > 0,$$

*then the optimal utility of the representative agent is attained. In particular, an equilibrium exists.*

# Computing Equilibria
# - Discrete BSDEs -

# Event Trees

Let the uncertainty be generated by independent random walks

$$b_t^i = \sum_{s=1}^{t} \Delta b_s^i \quad (i = 1, ..., d).$$

Then any random variable $X \in L(\mathscr{F}_{t+1})$ can be represented as

$$X = \mathbb{E}[X|\mathscr{F}_t] + \sum_{i=1}^{M} \xi_t^i(X) \Delta b_{t+1}^i \quad (X \in L(\mathscr{F}_{t+1}))$$

where the random coefficients $\xi_t^i(X)$ are given by

$$\xi_t^i(X) = \mathbb{E}[X \Delta b_{t+1}^i | \mathscr{F}_t].$$

THE UTILITY FUNCTION FOLLOWS A BACKWARD EQUATION.

# Equilibrium Dynamics

## Theorem (Equilibrium Dynamics)

*There exist random functions $g_t^R$ and $g_t^a$ such that the equilibrium price process $\{R_t\}$ and the equilibrium utility processes $\{H_t^a\}$ satisfy the coupled system of discrete BSDEs*

$$\begin{aligned} R_t &= R_{t+1} - g^R(Z^a_{t+1}, Z^R_{t+1}) + Z^R_{t+1} \cdot \Delta b_{t+1} \\ H^a_t &= H^a_{t+1} - g^a(Z^a_{t+1}, Z^R_{t+1}) + Z^a_{t+1} \cdot \Delta b_{t+1} \end{aligned}$$

*with terminal conditions*

$$R_T = R \quad \textit{and} \quad H^a_T = H^a.$$

An extension to continuous time is an open problem.

# Conclusion

- Dynamic GE model; preferences induced by BSDEs.
- Existence and characterization of equilibrium result.
- Equilibrium market price of risk characterized in terms of a BSDE.
- Model is amenable to efficient numerical illustrations.
- Continuous time: market completeness is equilibrium is key.
- Discrete time: anything goes.

Thank You!