Utility-Based Shortfall Risk & Optimized Certainty Equivalents

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Quantitative Risk Management

- Profit & loss distributions (P & L) are very complex. Appropriate summary statistics are needed:
  - Standardization facilitates communication
  - Simple tools for sensitivity analysis
  - Basis for capital regulation of financial firms
- Risk management of banks and insurance companies requires appropriate measures of the *downside risk*.
- Current industry standard Value at Risk is inappropriate in many cases.

Capital Regulation

"Capital regulation is the cornerstone of bank regulators' efforts to maintain a safe and sound banking system, a critical element of overall financial stability."

(Ben S. Bernanke, 2006)

- Major operational problems need, of course, to be fixed first:
  - myopic incentive schemes in financial institutions, lack of transparency, improper due diligence, limited accountability and limited liability of brokers and bankers, and other operational deficiencies
- Once banks work properly, careful risk management will be a key tool to ensure stable financial markets

Bailed out by the Tax Payer

Outline

(i) Background

- The industry standard: Value at risk
- Convex risk measures

(ii) Examples

- Utility-based Shortfall Risk
- Optimized Certainty Equivalents

Static Risk Measures

Value at Risk – A Good Risk Measure?

Criteria for good risk measures

- Economic properties:

  adequate assessment of diversification effects, large losses, additional information, etc.

- Implementation:

  efficient algorithms for the estimation of risk measures in realistic portfolio models

The Industry Standard – Value at Risk

Value at risk at level λ:

$$\mathrm{VaR}_\lambda(X) = \inf\{m \in \mathbb{R} : \ P[m + X < 0] \leq \lambda\}$$

"Smallest monetary amount to be added to a financial position such that the probability of a loss becomes smaller than λ."

Drawbacks of Value at Risk

- does not account for the size of extremely large losses
- does not encourage diversification

We will describe two distribution-based alternatives.

Value at Risk – Diversification

$$X_i = \begin{cases} 1 & \text{with probability } 50\% \\ -1 & \text{with probability } 50\% \end{cases}$$

The Value at Risk of X_i at level 50% is -1.

If X_1 and X_2 are independent, then

$$\frac{X_1 + X_2}{2} = \begin{cases} 1 & \text{with probability } 25\% \\ 0 & \text{with probability } 50\% \\ -1 & \text{with probability } 25\% \end{cases}$$

The VaR at level 50% of the diversified portfolio is 0.

Value at Risk – Large Losses

$$X_1 = \begin{cases} 1 & \text{with probability } 99\% \\ -1 & \text{with probability } 1\% \end{cases}$$

$$X_2 = \begin{cases} 1 & \text{with probability } 99\% \\ -10^{10} & \text{with probability } 1\% \end{cases}$$

The VaR at Level 1% of both positions is -1.

Value at Risk in the Media

"David Einhorn, who founded Greenlight Capital, a prominent hedge fund, wrote not long ago that VaR was

> '*relatively useless as a risk-management tool and potentially catastrophic when its use creates a false sense of security among senior managers and watchdogs. This is like an air bag that works all the time, except when you have a car accident.*' "

"Nicholas Taleb, the best-selling author of 'The Black Swan,' has crusaded against VaR for more than a decade. He calls it, flatly, '*a fraud.*' "

("Risk Mismanagement", New York Times, 2. Januar 2009)

Static Risk Measures

Risk measures

$\rho : \mathcal{X} \to \mathbb{R}$

- Monotonicity: If $X \leq Y$, then $\rho(X) \geq \rho(Y)$.
- Cash invariance: If $m \in \mathbb{R}$, then $\rho(X + m) = \rho(X) - m$.

Capital requirement

- A position $X \in \mathcal{X}$ is acceptable, if $\rho(X) \leq 0$.
  The collection $\mathcal{A}$ of all acceptable positions is the *acceptance set*.
- ρ is a capital requirement, i.e.

$$\rho(X) = \inf \{m \in \mathbb{R} : \; X + m \in \mathcal{A}\}.$$

Diversification

Semiconvexity:

$$\rho(\alpha X + (1-\alpha)Y) \leq \max(\rho(X), \rho(Y)) \qquad (\alpha \in [0,1]).$$

$\Longrightarrow$

Convexity (Föllmer & Schied, 2002):

$$\rho(\alpha X + (1-\alpha)Y) \leq \alpha\rho(X) + (1-\alpha)\rho(Y) \qquad (\alpha \in [0,1]).$$

Geometric properties of the acceptance set

- ρ convex $\Leftrightarrow$ $\mathcal{A}$ convex.

VaR and AVaR

Value at Risk

$$\mathrm{VaR}_\lambda(X) = \inf\{m \in \mathbb{R} : \; P[m + X < 0] \leq \lambda\}.$$

- not convex,
- positively homogeneous.

Average Value at Risk

$$\mathrm{AVaR}_\lambda(X) = \frac{1}{\lambda}\int_0^\lambda \mathrm{VaR}_\gamma(X)d\gamma = \sup\left\{E(-X|A) : \; P(A) > \lambda\right\}.$$

- convex,
- positively homogeneous.

Robust Representation

Robust representations of convex risk measures are an immediate consequence of Fenchel's Theorem.

Theorem 1 *Let $\rho : L^\infty \to \mathbb{R}$ be a convex risk measure. Then the following conditions are equivalent:*

(i) If $X_n \searrow X$ P-a.s. then $\rho(X_n) \nearrow \rho(X)$.

(ii) ρ is lower semicontinous with respect to $\sigma(L^\infty, L^1)$.

(iii) For $X \in L^\infty$ the risk measure can be calculated as

$$\rho(X) = \sup_{Q \in \mathcal{M}_1(P)} \left(E_Q[-X] - \alpha(Q)\right)$$

with $\alpha(Q) = \sup_{Y \in L^\infty} (E_Q[-Y] - \rho(Y))$.

Average value at risk AVaR_λ

- Set $\mathcal{Q}_\lambda = \left\{ Q \in \mathcal{M}_1(P) : \frac{dQ}{dP} \le \frac{1}{\lambda} \right\}$.
- The minimal penalty function is given by

$$\alpha(Q) = \begin{cases} 0, & Q \in \mathcal{Q}_\lambda, \\ \infty, & else. \end{cases}$$

- A robust representation is given by

$$\mathrm{AVaR}_\lambda(X) = \sup_{Q \in \mathcal{Q}_\lambda} E_Q[-X]$$

Distribution-Based Risk Measures

A risk measure $\rho : L^\infty(\Omega, \mathcal{F}, P) \to \mathbb{R}$ is distribution-based, if

$$\mathcal{L}(X; P) = \mathcal{L}(Y; P) \Longrightarrow \rho(X) = \rho(Y).$$

Theorem 2 *A risk measure is convex, law-based and continuous from above if and only if*

$$\rho(X) = \sup_{\mu \in \mathcal{M}_1((0,1])} \left(\int_{(0,1]} AVaR_\lambda(X)\mu(d\lambda) - \beta_{\min}(\mu) \right),$$

where

$$\beta_{\min}(\mu) = \sup_{X \in \mathcal{A}_\rho} \int_{(0,1]} AVaR_\lambda(X)\mu(d\lambda).$$

Utility-based Shortfall Risk

Utility-based Shortfall Risk

$\ell : \mathbb{R} \to \mathbb{R}$ convex loss function, z interior point of the range of ℓ.

The acceptance set is defined as

$$\mathcal{A} = \{X \in L^\infty : \ E_P\left[\ell(-X)\right] \leq z\}$$

$\mathcal{A}$ induces the shortfall risk measure ρ:

$$\rho(X) = \inf\{m \in \mathbb{R} : \ X + m \in \mathcal{A}\}$$

- convex,
- positively homogeneous only under conditions on ℓ:

$$\ell(x) = z - \alpha x^- + \beta x^+, \quad \beta \geq \alpha \geq 0.$$

Utility-based Shortfall Risk (2)

Robust Representation

Utility-based shortfall risk $\rho : L^\infty \to \mathbb{R}$ is a convex risk measure that continuous from below with robust representation

$$\rho(X) = \max_{Q \in \mathcal{M}_1} \left(E_Q(-X) - \alpha_{\min}(Q) \right), \quad X \in L^\infty$$

and penalty function

$$\alpha(Q) = \inf_{\lambda > 0} \frac{1}{\lambda} \left(z + E \left[\ell^* \left(\lambda \frac{dQ}{dP} \right) \right] \right), \quad Q \in \mathcal{M}_1(P).$$

Utility-based Shortfall Risk (3)

Examples

- Entropic risk measure: $\ell(x) = e^{\beta x}$, $\beta > 0$

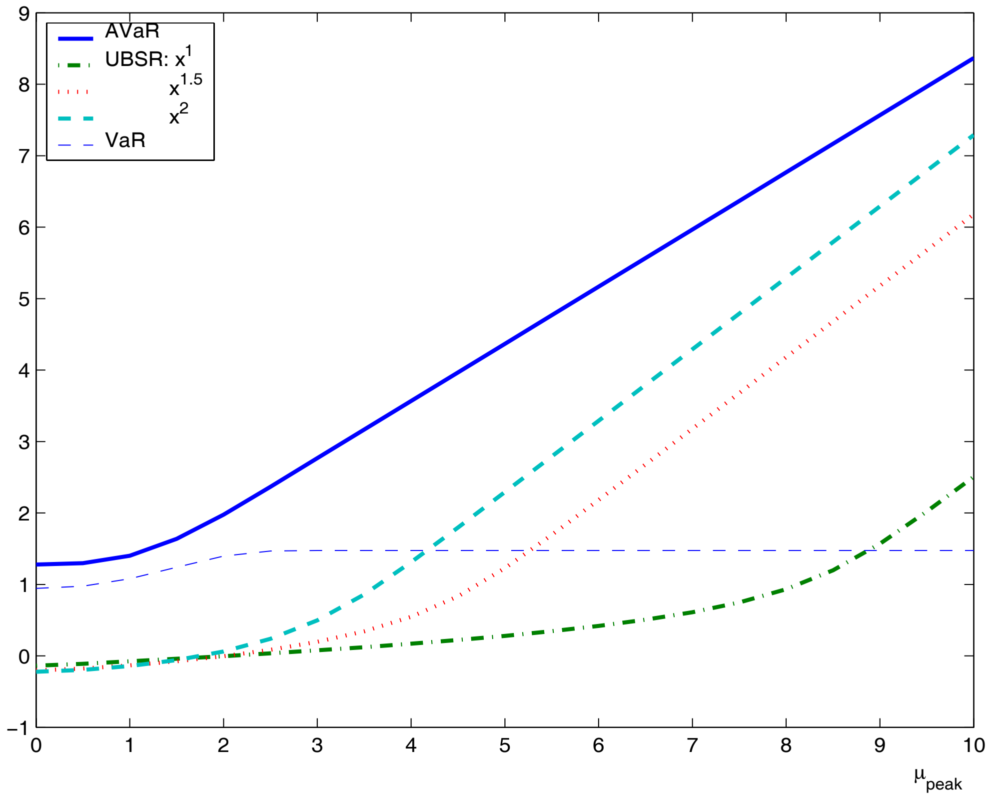
$$\alpha(Q) = \frac{1}{\beta}\left(H(Q|P) - \log z\right)$$

- Polynomial loss function: $\ell(x) = \frac{1}{p} x^p \cdot \mathbf{1}_{[0,\infty)}(x)$, $p > 1$

  Denoting by $q = p/(p-1)$ the dual coefficient, the minimal penalty function is given by

$$\alpha(Q) = (pz)^{1/p} \cdot E\left[\left(\frac{dQ}{dP}\right)^q\right]^{1/q}.$$

Utility-based Shortfall Risk (4)



$\mathrm{VaR}_{0.05}$, $\mathrm{AVaR}_{0.05}$ and utility-based shortfall risk with parameters $p \in \{1, \frac{3}{2}, 2\}$ and $z = 0.3$ as a function of μ for a mixture of a t (weight 0.96) and Gaussian with mean μ (weight 0.04).

Monte Carlo Simulation

Shortfall risk $\rho(X)$ is given by the unique root s_* of the function

$$f(s) := E[\ell(-X - s)] - z.$$

Efficient Computation

- Variance reduction techniques increase the accuracy/rate of convergence, e.g. importance sampling (Dunkel & W., 2007)
- Stochastic approximation (Dunkel & W., 2008)

Compound Lotteries and Risk

(i) A distribution μ is called acceptable, if $\rho(\mu) \leq 0$.

(ii) The acceptance set on the level of distributions $\mathcal{N}$ consists of all acceptable distributions.

Convexity:

- If the lotteries μ and ν are acceptable (resp. not acceptable),
- then for $\alpha \in [0, 1]$ the compound lottery

$$\alpha\mu + (1 - \alpha)\nu$$

is also acceptable (resp. not acceptable).

Compound Lotteries and Risk (cont.)

Examples

The following risk measures have convex acceptance and rejections sets on the level of distributions:

- Negative expected value: $E(-X)$
- Worst-case measure: $\|X^-\|_\infty$
- Value at risk: $VaR_\lambda(X) = -q_\lambda^+(X)$
- Shortfall risk

Theorem (W., 2006):

Shortfall risk is the only convex risk measure with convex acceptance and rejection sets on the level of distributions.

Characterization Theorem

ψ-Weak Topology

- C_ψ denotes for a fixed continuous function $\psi : \mathbb{R} \to [1, \infty)$ the vector space of all continuous functions $f : \mathbb{R} \to \mathbb{R}$ for which we can find a constant $c \in \mathbb{R}$ such that for all $x \in \mathbb{R}$, $|f(x)| \leq c\psi(x)$.

  ψ is called a gauge function.

- $\mathcal{M}_c^+(\mathbb{R})$ designates the space of finite measures with compact support.

The ψ-weak topology on the set $\mathcal{M}_c^+(\mathbb{R})$ is the initial topology of the family $\mu \mapsto \int f(x)\mu(dx)$, $(\mu \in \mathcal{M}_c^+(\mathbb{R}), f \in C_\psi)$.

The ψ-weak topology is finer than the weak topology.

Characterization Theorem (cont.)

Theorem 3 (W., 2006) *Let ρ be a distribution-based risk measure.*

Assume there exists $x \in \mathbb{R}$ with $\delta_x \in \mathcal{N}$ such that for $y \in \mathbb{R}$, $\delta_y \in \mathcal{N}^c$,

$$(1-\alpha)\delta_x + \alpha\delta_y \in \mathcal{N}$$

for sufficiently small $\alpha > 0$.

Then the following statements are equivalent:

(i) *$\mathcal{N}$ is ψ-weakly closed for some gauge function $\psi : \mathbb{R} \to [1, \infty)$, $\mathcal{N}$ and $\mathcal{N}^c$ are both convex.*

(ii) *For some left-continuous loss function $\ell : \mathbb{R} \to \mathbb{R}$ and a scalar $z \in \mathbb{R}$ in the interior of the convex hull of the range of ℓ:*

$$\mathcal{N} = \left\{ \mu \in \mathcal{D} : \int \ell(-x)\mu(dx) \leq z \right\}.$$

Dynamic Risk Measurement

Dynamic Risk Measures

- $(\Omega, \mathcal{F}, P)$ standard Borel probability space
- Time steps $t = 0, 1, 2, \ldots, T$ with terminal time T
- $(\mathcal{F}_t)_{t=0,1,\ldots,T}$ filtration with
  - $\mathcal{F}_0 = \{\emptyset, \Omega\}$
  - $\mathcal{F}_T = \mathcal{F}$

General assumption

We will always suppose that the underlying filtered probability space is sufficiently rich.

Dynamic Risk Measures (continued)

- The space of financial positions is given by $\mathcal{D} := L^\infty(\Omega, \mathcal{F}, P)$, i.e., by "terminal positions"
- Evaluate the risk of a terminal financial position at different dates $t = 1, 2, \ldots, T$, as new information becomes available:

$$\Psi_t(D) = \Theta_t(\mathcal{L}(D|\mathcal{F}_t)). \tag{1}$$

  - Θ_t is a static distribution-based risk measure on $\mathcal{M}_{1,c}(\mathbb{R})$.
  - The corresponding acceptance sets are given by

$$\mathcal{N}_t = \{\mu \in \mathcal{M}_{1,c}(\mathbb{R}) : \ \Theta_t(\mu) \leq 0\}.$$

- An axiomatic characterization of dynamic risk measures $(\Psi_t)_t$ with representation (1) can easily be derived.

Dynamic Consistency

Risk measurement

$$\Psi_t(D) = \Theta_t(\mathcal{L}(D|\mathcal{F}_t))$$

Acceptability

$D \in \mathcal{D}$ acceptable in scenario $\omega \in \Omega$ at time t, if $\Psi_t(D)(\omega) \leq 0$.

Definition 1 *A risk measure $\Psi = (\Psi_t)_t$ is*

- *weakly* acceptance consistent, *if for financial positions $D \in \mathcal{D}$,*

$$\Psi_{t+1}(D) \leq 0 \quad P-\text{a.s.} \qquad \Longrightarrow \qquad \Psi_t(D) \leq 0 \quad P-\text{a.s.}$$

- *weakly* rejection consistent, *if for financial positions $D \in \mathcal{D}$,*

$$\Psi_{t+1}(D) > 0 \quad P-\text{a.s.} \qquad \Longrightarrow \qquad \Psi_t(D) > 0 \quad P-\text{a.s.}$$

Representation of Consistent Risk Measures

Proposition 1 *If $\Psi = (\Psi_t)_t$ is both acceptance and rejection consistent, then it can be represented by*

$$\Psi_t(D) = \Theta(\mathcal{L}(D|\mathcal{F}_t)) \tag{2}$$

for a unique static risk measure Θ.

Question

If a dynamic risk measure has the form (2):
Under which conditions on Θ do we obtain dynamic consistency?

Consistency and Measure Convexity

Measure convexity

A subset M of a locally convex space is called measure convex, if for every $\gamma \in \mathcal{M}_1(M)$

- the barycenter $s_\gamma = \int_M m\gamma(dm)$ exists, and
- s_γ is contained in M.

Locally measure convex subsets of $\mathcal{M}_{1,c}(\mathbb{R})$

A measurable subset $\mathcal{C}$ of $\mathcal{M}_{1,c}(\mathbb{R})$ is locally measure convex, if for all $c \in \mathbb{R}$ the set

$$\mathcal{C} \cap \mathcal{M}_1([-c, c])$$

is measure convex.

Consistency and Measure Convexity (cont.)

Theorem 4 Θ *static risk measure,* $\mathcal{N} \subseteq \mathcal{M}_{1,c}(\mathbb{R})$ *its acceptance set.*

We define dynamic risk measurements by

$$\Psi_t(D) = \Theta\left(\mathcal{L}(D|\mathcal{F}_t)\right).$$

(i) $\Psi = (\Psi_t)_t$ *is* weakly acceptance consistent*, if and only if* $\mathcal{N}$ *is a* locally measure convex *set of probability measures.*

(ii) $\Psi = (\Psi_t)_t$ *is* weakly rejection consistent*, if and only if* $\mathcal{N}^c$ *is a* locally measure convex *set of probability measures.*

Examples

Negative expected value, worst-case measure, value at risk, shortfall risk

Further Implications

Suppose that the dynamic risk measure $\Psi = (\Psi_t)_t$ is

(i) acceptance and rejection consistent,

(ii) convex (in the sense of Föllmer and Schied, i.e., on the space of random variables),

(iii) additional technical conditions.

$\Longrightarrow$

$\exists$ continuous, convex loss function $\ell : \mathbb{R} \to \mathbb{R}$ with associated shortfall risk Θ such that Ψ can be represented as

$$\Psi_t(D) = \Theta(\mathcal{L}(D|\mathcal{F}_t)).$$

Strong Dynamic Consistency

Definition 2 *A risk measure $\Psi = (\Psi_t)_t$ is dynamically consistent, if*

$$\Psi_t(D) = \Psi_t[-\Psi_{t+1}(D)] \quad P - \textit{a.s.}$$

- Weak dynamic consistency is necessary for this Bellman principle.
- Shortfall risk is i.g. not dynamically consistent.

Optimized Certainty Equivalents

OCE – Definition

- Convex family of utility functions

$$\mathbb{U}_0 = \{u : \mathbb{R} \to [-\infty, \infty) \text{ concave utility}, u(0) = 0, 1 \in \partial u(0)\}$$

- Financial positions are modeled as bounded random variables on an atomless probability space $(\Omega, \mathcal{F}, P)$

The optimized certainty equivalent (OCE) of $X \in L^\infty$ is defined by

$$S_u(X) := \sup_{\eta \in \mathbb{R}} \{\eta + Eu(X - \eta)\}.$$

Remark

- If u is continuously differentiable and strictly concave, then η^* is the unique solution of $Eu'(X - \eta^*) = 1$.
- This property can be exploited for the numerical estimation of OCE (stochastic root finding).

OCE – Risk Measure

$$\rho_u(X) = -S_u(X) \qquad\qquad (X \in L^\infty)$$

defines a convex risk measure.

Risk aversion

- As δ increases from 0 to ∞ the degree of risk aversion corresponding to $u_\delta(t) := \frac{u(\delta t)}{\delta}$ increases:

$$\rho_{u_\delta}(X) = \frac{1}{\delta}\rho_u(\delta X) \quad \geq \quad \rho_u(X) \quad \forall \delta \geq 1,$$
$$\rho_{u_\delta}(X) = \frac{1}{\delta}\rho_u(\delta X) \quad \leq \quad \rho_u(X) \quad \forall \delta \in [0,1]$$

- $\lim_{\delta\to\infty} \rho_{u_\delta}(X) = \rho_{\max}(X)$
- $\lim_{\delta\to 0} \rho_{u_\delta}(X) = -E(X)$

OCE – Exponential and Quadratic Utility

Exponential Utility

If $u(t) = 1 - e^{-t}$,

$$\rho_u(X) = \log Ee^{-X}$$

coincides with the entropic risk measure, a special case of utility-based shortfall risk.

Quadratic Utility

If $u(t) = \begin{cases} t - 1/2t^2, & t < 1, \\ 1/2 & t \geq 1 \end{cases}$, then

$$\rho_u(X) = -E(X) + 1/2var(x),$$

if $\|X^+\|_\infty \leq 1 + E(X)$.

OCE – Piecewise Linear Utility

Setting $[z]_+ = 0 \vee z$, the piecewise linear utility $u(t) = \gamma_1 [t]_+ - \gamma_2 [-t]_+$ with $0 \leq \gamma_1 < 1 < \gamma_2$ defines the risk measure

$$\rho_u(X) = \inf_{\eta \in \mathbb{R}} \left\{ \eta + \gamma_2 E(-\eta - X)^+ - \gamma_1 E(X + \eta)^+ \right\}.$$

In the special case $\gamma_1 = 0$ we obtain AVaR_α with $\alpha = 1/\gamma_2$:

$$\mathrm{AVaR}_\alpha(X) = \inf_{\eta \in \mathbb{R}} \left\{ \eta + \frac{1}{\alpha} E(X + \eta)^- \right\}.$$

OCE – Piecewise Linear Utility (2)

Define $\alpha = \frac{1-\gamma_1}{\gamma_2-\gamma_1}$.

- The associated OCE risk measure can be computed as

$$\rho_u(X) = -\gamma_2 \int_{-\infty}^{\mathrm{VaR}_\alpha(X)} t dF_X(t) - \gamma_1 \int_{\mathrm{VaR}_\alpha(X)}^{\infty} t dF_X(t).$$

- For $\gamma_1 = 0$ one obtains the classical formula

$$\mathrm{AVaR}_\alpha(X) = \frac{1}{\alpha} \int_0^{\mathrm{VaR}_\alpha(X)} t dF_X(t) = \frac{1}{\alpha} \int_0^{\alpha} \mathrm{VaR}_z(X) dz.$$

OCE – Coherence

If $u \in \mathbb{U}_0$, then $u(t) \le t \; \forall t$. The subfamily $\mathbb{U}_0^< \subsetneq \mathbb{U}_0$ consists of utilities with strongly risk-averse utilities, i.e. $u \in\subsetneq \mathbb{U}_0^< \Leftrightarrow \forall t \neq 0 : u(t) < t$.

Theorem 5 *The following statements are equivalent:*

(i) $u \in \mathbb{U}_0^<$*, and* ρ_u *is coherent.*

(ii) $u(t) = \gamma_1 [t]_+ - \gamma_2 [-t]_+$ *with* $0 \le \gamma_1 < 1 < \gamma_2$*.*

The coherent OCE risk measures defined by strongly risk averse utilities correspond to the class of piecewise linear utilities.

OCE – Coherence (cont.)

$u \in \mathbb{U}_0^<$ is essential hypothesis (for $(i) \Rightarrow (ii)$)!

Define

$$u(t) = \begin{cases} 1 - e^{-t}, & t \geq 0, \\ t, & t \leq 0 \end{cases}$$

Then $u \in \mathbb{U}_0 \setminus \mathbb{U}_0^<$, and

$$\rho_u(X) = -E(X),$$

thus ρ_u is actually coherent.

OCE – Robust Representation

g-divergence

$g : [0, \infty) \to \mathbb{R} \cup \{\infty\}$ lower semicontinuous convex function, $g(1) < \infty$, $g(x)/x \to \infty$ as $x \to \infty$

The g-divergence is defined by

$$I_g(Q|P) = E_P\left[g\left(\frac{dQ}{dP}\right)\right], \qquad Q \ll P$$

"Statistical distance between Q and P"

- Level set $\{dQ/dP : I_g(Q|P) \leq c\}$ convex, weakly compact in $L^1(P)$

OCE – Robust Representation (cont.)

For $\phi : \mathbb{R} \to (-\infty, \infty]$ we define the convex conjugate by

$$\phi^*(s) = \sup_{t \in \mathbb{R}} \{st - \phi(t)\}.$$

Setting $u(t) = -g^*(-t)$, we obtain

$$\rho_u(X) \quad = \quad \sup_{Q \ll P} (E_Q[-X] - I_g(Q|P)), \qquad (X \in L^\infty)$$

Conversely,

$$I_g(Q|P) \quad = \quad -\inf_{X \in L^\infty} (\rho_u(X) + E_Q(X)), \qquad (Q \ll P)$$

UBSR & OCE

Utility-Based Shortfall Risk can be recovered from a family of OCE-risk measures.

Denote Shortfall Risk by

$$\mathrm{SR}(X) = \inf\{\eta : E(\ell(-X-\eta)) \leq z\},$$

and define

$$u(t) = -\ell(-t) + z.$$

Then

$$\mathrm{SR}(X) = \sup_{\lambda>0} \rho_{\lambda u}(X).$$

As a consequence, robust representations theorems for SR can be deduced from the corresponding results for OCE risk measures.

Conclusion

- UBSR and OCE provide interesting distribution-based families of risk measures with reasonable properties.
- These risk measures are convex and do not neglect large losses.
- They can be characterized as solutions to stochastic root finding problems that can be exploited for their computation.