Liquidity Risk Measures

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Liquidity

- Illiquidity poses significant risk for investors
  - Short-term obligations can force asset holders to liquidate assets
  - In illiquid markets, prices of fire sales may be suboptimal
- Liquidity-Adjustments
  - The value of portfolios should be adjusted for these adverse scenarios
  - The liquidity-risk should be quantified
- Important model ingredients
  - Supply-demand curves
  - Short-term obligations

Outline

(i) Liquidity Constraints

- Supply-Demand Curves
- Liquidity and Portfolio Constraints

(ii) Liquidity Adjustments

- Portfolio Value
- Risk Measures

(iii) Numerical Case Studies

Liquidity Constraints

Supply-Demand Curves

- Marginal supply-demand curve (MSDC)

  $m : \mathbb{R}_* \to \mathbb{R}$ decreasing function

  - $y > 0$:

    Sell y shares for $\int_0^y m(x)dx$

  - $y < 0$:

    Buy $|y|$ shares for $\int_y^0 m(x)dx$, i.e. “receive” $\int_0^y m(x)dx$

  The convex cone of all MSDC’s is denoted by $\mathcal{M}$.

- The numbers $m^+ := m(0+)$ and $m^- := m(0-)$ are called the best bid and best ask, respectively.

- The difference $\Delta m : m^- - m^+ \geq 0$ is called the bid-ask spread.

Supply-Demand Curves (cont.)

- Proceeds of transaction: $P(s) = \int_0^s m(x)dx$
- Supply-demand curves:

$$S(x) := \frac{P(x)}{x}, \quad x \in \mathbb{R}_*$$

  - $x > 0$: average unit prices for sale
  - $x > 0$: average unit prices for purchase

Examples

Linear Supply-Demand Curve

- Supply-demand curve: $S(x) = a - b \cdot x$ for given $a, b > 0$
- Proceeds: $P(x) = x \cdot S(x) = x \cdot (a - b \cdot x)$
- Marginal supply-demand curve: $m(x) = a - 2b \cdot x$

Remark

Marginal prices become negative, if seller sells more than $a/2b$ units. This case will, however, not occur, since it is never optimal for the seller to sell at these prices.

Examples (cont.)

Exponential Supply-Demand Curve

- Marginal supply-demand curve: $m(x) = a \cdot e^{-bx}$ for $a, b > 0$
- Proceeds: $S(x) = \frac{a}{bx} \cdot \left(1 - e^{-bx}\right)$
- Supply-demand curve: $P(x) = \frac{a}{b} \cdot \left(1 - e^{-bx}\right)$

Polynomial Supply-Demand Curve

- Marginal supply-demand curve: $m(x) = \begin{cases} a(b-x)^\gamma, & x < b, \\ 0, & x \geq b. \end{cases}$

  for $a, b > 0$ and $\gamma > 1$
- Proceeds: $P(x) = \frac{a}{\gamma+1}(b^{\gamma+1} - (b-x)^{\gamma+1})$
- Supply-demand curve: $S(x) = \frac{a}{(\gamma+1)x} \cdot (b^{\gamma+1} - (b-x)^{\gamma+1})$

Markets and Portfolios

- Spot market of assets:

  $\bar{m} = (m_0, m_1, \ldots, m_N) \in \mathcal{M}^{N+1}$.

  We will always assume that asset 0 corresponds to cash and set $m_0 = 1$.

- Portfolio in a spot market of N risk assets:

  $\bar{\xi} = (\xi_0, \xi_1, \ldots, \xi_N) = (\xi_0, \xi) \in \mathbb{R}^{N+1}$

  "Number of assets at time 0"

Liquidation versus Mark-to-Market Value

- Liquidation value

$$L(\bar{\xi}, \bar{m}) \;=\; \sum_{i=0}^{N} \int_0^{\xi_i} m_i(x)dx \;=\; \xi_0 + \sum_{i=1}^{N} \int_0^{\xi_i} m_i(x)dx.$$

- Maximal mark-to market value

$$U(\bar{\xi}, \bar{m}) \;=\; \xi_0 + \sum_{i=1}^{N} \tilde{m}_i(\xi_i) \cdot \xi_i \;=\; L(\bar{\xi}, \bar{m}) - \sum_{i=1}^{N} \int_0^{\xi_i} \hat{m}_i(x)dx,$$

where $\tilde{m}_i(\xi_i) = \begin{cases} m_i^+, & \text{if } \xi_i \geq 0, \\ m_i^-, & \text{if } \xi_i < 0. \end{cases}$

Attainable Portfolios

- Portfolio at time 0: $\bar{\xi} \in \mathbb{R}^{N+1}$
- Any portfolio which is attainable from $\bar{\xi}$ has the form:

$$\left(\xi_0 + \sum_{i=1}^{N} \int_0^{\eta_i} m_i(x)dx, \xi - \eta\right) \quad (\eta \in \mathbb{R}^N)$$

We denote the set of attainable portfolios by $\mathcal{A}(\bar{\xi}, \bar{m})$.

Remark

The maximal mark-to-market values of the attainable portfolios will typically be an interval that is bounded above by the mark-to-market of the original portfolio.

Liquidity Adjustments

Liquidity Constraints

- Short-term cash flows

  Continuous function $\phi : \mathbb{R}^N \to \mathbb{R} \cup \{-\infty\}$ such that $\phi(0_N) = 0$.

  We will usually assume that ϕ is concave and non positive.

  Notation: $\phi \in \Phi$

- Set of liquid portfolios

  Letting $\phi \in \Phi$ and $a \in \mathbb{R}$, the set of liquid portfolios that are attainable from $\bar{\xi}$ are defined as

$$\mathcal{L}(\bar{\xi}) = \mathcal{L}(\bar{\xi}, \bar{m}, \phi, c) = \left\{\bar{\eta} \in \mathcal{A}(\bar{\xi}, \bar{m}) : \eta_0 + \phi(\eta) \geq a\right\}.$$

Liquidity Constraints (cont.)

Examples

- Proportional margin constraints:

  the obligations are proportional to the number of assets on which the investor is short, i.e.

$$\phi(\xi) = -\sum_{i=1}^{N} \alpha_i \cdot \xi_i^-, \quad \alpha_i \geq 0, \quad i = 1, \dots, N$$

- Quadratic margin constraints: the obligations are quadratic in the number of assets on which the investor is short, i.e.

$$\phi(\xi) = -\sum_{i=1}^{N} \alpha_i \cdot (\xi_i^-)^2, \quad \alpha_i \geq 0, \quad i = 1, \dots, N$$

Portfolio Constraints

Let $\mathcal{K} \subseteq \mathbb{R}^N$ be non-empty, closed, convex set.

Requirement

- $\eta \in \mathcal{K}$ for any admissible portfolio $\bar{\eta} = (\eta_0, \eta)$ at the end of the time horizon, $t = 1$
- We suppose that $0_N \in \mathcal{K}$, i.e. holding cash only is acceptable, as long as the borrowing constraint $\eta_0 \geq a$ is satisfied.

We will always assume that the portfolio constraint can be expressed in terms of r convex functions $\psi_1, \ldots, \psi_r : \mathbb{R}^N \to \mathbb{R}$, i.e.

$$\eta \in \mathcal{K} \quad \Longleftrightarrow \quad \psi_1(\eta) \leq 0, \ldots, \psi_r(\eta) \leq 0.$$

Portfolio Value

Attainable portfolios

$$\mathcal{L}(\bar{\xi}) = \mathcal{L}(\bar{\xi}, \bar{m}, \phi, c) = \left\{\bar{\eta} \in \mathcal{A}(\bar{\xi}, \bar{m}) : \eta_0 + \phi(\eta) \geq a\right\}$$

Value of a portfolio

$$V(\bar{\xi}, \bar{m}, \phi, a, \mathcal{K}) = \sup\{U(\bar{\eta}, \bar{m}) : \bar{\eta} \in \mathcal{L}(\bar{\xi}, \bar{m}, \phi, a) \cap \mathcal{K}\}.$$

(Acerbi & Scandolo, 2008; Anderson, Liese & W., 2009)

Liquidity Risk Measures

- Assume that the MSDC m and the ICF ϕ are random.
- Let $\mathcal{A}$ be the acceptance set of a risk measure ρ.

Liquidity-based risk measure (Anderson, Liese & W., 2009)

- The risk of a portfolio $\bar{\xi}$ is defined as

$$\rho^V(\bar{\xi}) = \inf\{k : V(k + \bar{\xi}) \in \mathcal{A}\}$$

- The mapping $\rho^V : \mathbb{R}^{N+1} \to \mathbb{R}$ is a convex risk measure.
- $\rho^V(\bar{\xi})$ is equal to the unique solution $k \in \mathbb{R}$ of the equation

$$0 = \rho(V(\bar{\xi} + k))$$

- In the case of UBSR this amounts to solving the equation

$$0 = E\left[\ell(-V(\bar{\xi} + k)\right] - z$$

Liquidity Risk Measures (cont.)

Robust Representation

Let Δ^N be the N-dimensional simplex in $\mathbb{R}^{N+1}$.

Then ρ^V admits a robust representation

$$\rho^V(\bar{\xi}) = -\min_{\bar{v}\in\Delta^N}\left(\bar{v}\cdot\bar{\xi} - \beta(\bar{v})\right), \quad \bar{\xi}\in\mathbb{R}^{N+1},$$

with penalty function

$$\beta(\bar{v}) = -\inf_{V(\bar{\xi})\in\mathcal{A}} \bar{v}\cdot\bar{\xi}, \quad \bar{v}\in\Delta^N.$$

Numerical Case Studies

Market and Portfolio

- Portfolio vector

$$\bar{\xi} = (\xi_0, \xi_1, \xi_2) = (0, -3, 4)$$

- Exponential marginal demand-supply curves

$$m(x) = a \cdot e^{-bx}$$

with $a - 20 \sim \mathcal{LN}(1/2, 1/2)$

- Liquidity constraints

$$\phi(\xi) = \alpha \cdot (\xi_1^- + \xi_2^-)$$

- Portfolio constraints

$$\mathcal{K} = [-4, \infty)^2$$

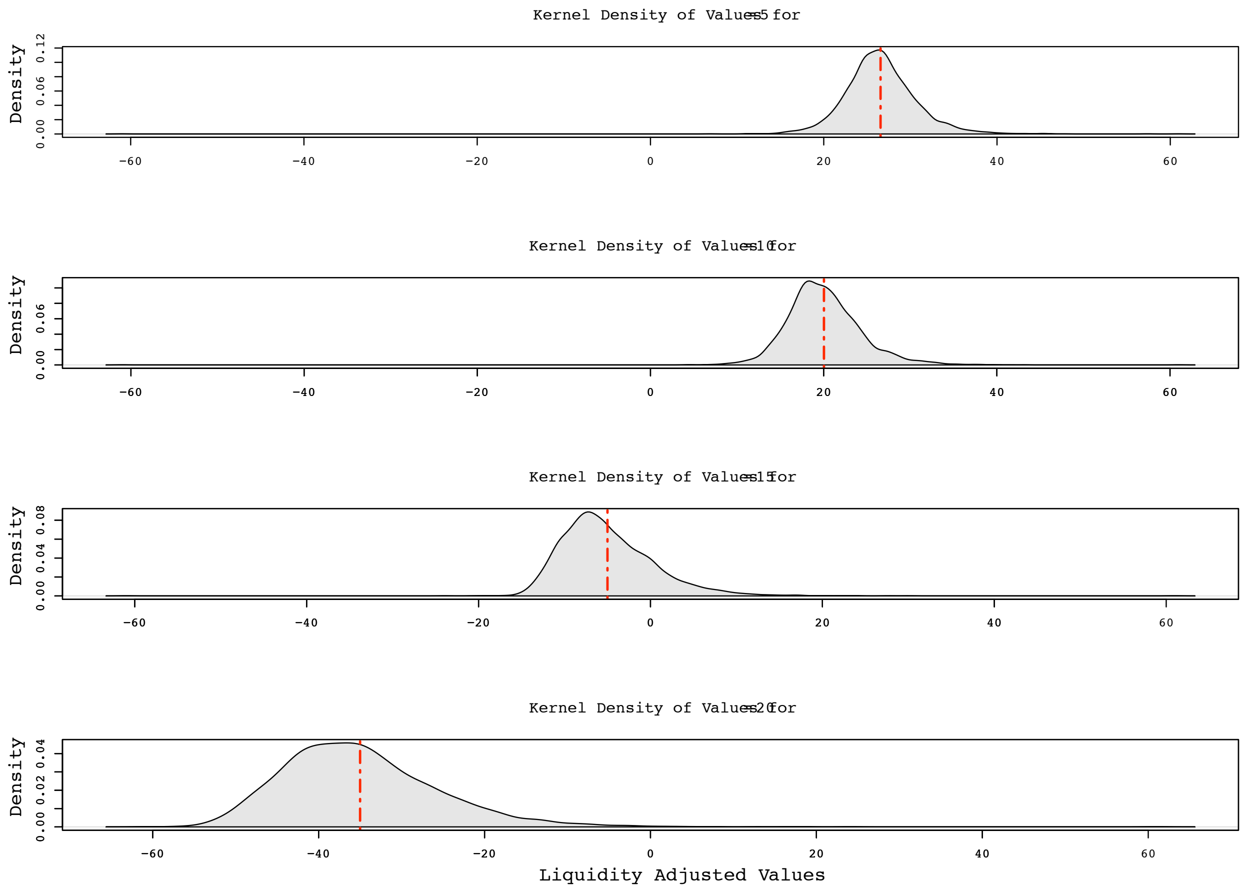
Liquidity-Adjusted Value

	b=0.005			
α	Mean	Variance	Skewness	Kurtosis
5	27.8587	15.4756	0.0802	29.5952
10	27.8102	16.6191	0.1181	29.6297
15	27.8545	16.3723	0.0946	29.3827
20	27.7346	16.5052	0.0939	29.3998

	b=0.5			
α	Mean	Variance	Skewness	Kurtosis
5	26.5572	16.3905	0.0823	29.7312
10	20.0344	17.5126	0.1516	28.7874
15	-5.0026	28.285	0.2567	27.3533
20	-35.0029	85.1734	0.1806	26.8235

	b=1			
α	Mean	Variance	Skewness	Kurtosis
5	24.969	16.0884	0.0796	29.1924
10	-4.2628	34.8094	0.2768	27.1233
15	-18.0512	78.4985	0.1125	27.3698
20	—	—	—	—

Liquidity-Adjusted Value (cont.)



Liquidity Adjusted Portfolio Values with $b = 0.5$, $\alpha = 5, 10, 15, 20$

Liquidity-Adjusted Entropic Risk Measures

	b=0.0005			
	α=5	α=10	α=15	α=20
	k_{RM}/k_{PR}	k_{RM}/k_{PR}	k_{RM}/k_{PR}	k_{RM}/k_{PR}
$\beta = 0.02$	-2.5275/-2.3939	-2.2368/-2.3088	-2.4667/-2.3942	-2.2287/-2.3851
$\beta = 0.08$	-15.8750/-16.5501	-15.5920/-15.5051	-15.0820/-15.6940	-15.6400/-15.7803
$\beta = 0.15$	-21.1540/-21.4882	-21.4530/-21.8234	-21.5020/-21.4335	-21.9300/-21.6615
VaR(0.05)	-21.8780	-21.7000	-21.6150	-21.5810

	b=0.5			
	α=5	α=10	α=15	α=20
	k_{RM}/k_{PR}	k_{RM}/k_{PR}	k_{RM}/k_{PR}	k_{RM}/k_{PR}
$\beta = 0.02$	-0.8572/-2.3939	3.6768/3.8298	11.7150/11.7200	22.3320/22.3854
$\beta = 0.08$	-12.6150/-12.9923	-5.5968/-5.3172	5.0683/5.1998	17.8920/17.9012
$\beta = 0.15$	-14.7170/-14.9323	-7.1308/-6.9363	4.0099/4.3038	17.2890/17.3566
VaR(0.05)	-20.4360	-13.9420	12.0350	47.9940

	b=1			
	α=5	α=10	α=15	α=20
	k_{RM}/k_{PR}	k_{RM}/k_{PR}	k_{RM}/k_{PR}	k_{RM}/k_{PR}
$\beta = 0.02$	0.7344/0.6064	8.6733/8.6553	20.8170/20.9401	–
$\beta = 0.08$	-8.5896/-8.4589	3.3786/3.5234	18.7970/18.4807	–
$\beta = 0.15$	-9.7309/-9.7902	2.0792/2.8749	18.3900/18.3592	–
VaR(0.05)	-18.7540	11.8280	31.4780	–

Conclusion

Conclusion

(i) Portfolio values and static risk measures can be adjusted for liquidity risk

(ii) Liquidity-adjusted risk measures **are risk measures**!

(iii) Numerical case studies clearly demonstrate liquidity effects

(iv) Further research: dynamic liquidity-adjusted risk measures