Portfolio Choice under Risk Constraints

Stefan Weber
Leibniz Universität Hannover
email: sweber@stochastik.uni-hannover.de
web: www.stochastik.uni-hannover.de/∼sweber

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Outline

(i) Motivation

(ii) Robust utility maximization under a joint budget and risk constraint in incomplete markets

- Reduction to a static problem
- Dual characterization and Föllmer measures

(iii) The approach of Cuoco, He & Isaenko

(iv) General equilibrium

(v) Further research

Portfolio Choice and Risk Measures

Purpose of risk measures

- provide appropriate measures for downside risk
- enforce appropriate rules for regulatory capital
- allocate risk capital to business units

Questions

- What are good risk measures?
- How do risk measures influence the actions of economic agents?

Both questions are related. The standard theory of risk measures neglects the impact of regulation on the actions of economic agents

Portfolio Choice and Risk Measures (cont.)

Approaches

- Investigate optimal portfolio choice under risk measure constraints
  - Single agent
  - Price processes exogenous
- Investigate market equilibrium under risk measure constraints
  - Multiple agents
  - Price processes endogenous

Robust Optimal Portfolio Choice

(Joint work with Anne Gundel)

Market Model

- Filtered probability space: $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, R)$ with $\mathcal{F} = \mathcal{F}_T$
  - R is a reference measure.
- Discounted price process: $\mathbb{R}^d$-valued semimartingale S
- Absolutely continuous martingale measures: $\mathcal{P}$ (with $\mathcal{P}_e \neq \emptyset$)

Problem

Find optimal financial position $X \in \mathcal{F}_T$ (plus integrability)
which is both affordable and acceptable:

$$X = V_T = x_2 + \int_0^T \xi_t dS_t \ge \bar{x}_u$$

Optimality

- Find maximal element with respect to some preference order $\preceq$.
- Typically, a preference order admits a numerical representation:

$$X \succeq \tilde{X} \quad \Leftrightarrow \quad U(X) \geq U(\tilde{X}).$$

Examples

Let $u : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ be a Bernoulli utility function.

- Expected utility: $U(X) = E_Q[u(X)]$
  - Axiomatic foundation by von Neumann & Morgenstern, Savage.
- Robust expected utility: $U(X) = \inf_{Q \in \mathcal{Q}_0} E_Q[u(X)]$
  - "Robust" extension of expected utility approach.
  - Axiomatic foundation by Gilboa & Schmeidler (1989).
  - Extensions by Maccheroni, Marinacci & Rustichini (2006).

Risk Constraint

Notation

- $\mathcal{A}_{Q_1} = \{X : E_{Q_1}[\ell(-X)] \leq x_1\}$
- $\rho_{Q_1}(X) := \inf\{m \in \mathbb{R} : X + m \in \mathcal{A}_{Q_1}\}$

Risk Constraint

- Shortfall risk constraint:
  - Downside risk $\rho_{Q_1}(X)$ is non positive, i.e., $E_{Q_1}[\ell(-X)] \leq x_1$.
- Robust risk constraint:
  - $\mathcal{Q}_1$ set of subjective probability measures
  - For all $Q_1 \in \mathcal{Q}_1$ downside risk $\rho_{Q_1}(X)$ is non positive, i.e.,

$$\sup_{Q_1 \in \mathcal{Q}_1} E_{Q_1}[\ell(-X)] \leq x_1.$$

Optimization Problems

Dynamic problem

Maximize $\inf_{Q_0 \in \mathcal{Q}_0} E_{Q_0}[u(V_T)]$ over all $V \in \mathcal{V}(x_2)$
that satisfy $\sup_{Q_1 \in \mathcal{Q}_1} E_{Q_1}[\ell(-V_T)] \leq x_1$.

Static problem

Maximize $\inf_{Q_0 \in \mathcal{Q}_0} E_{Q_0}[u(X)]$ over all X
under the constraints $\sup_{Q_1 \in \mathcal{Q}_1} E_{Q_1}[\ell(-X)] \leq x_1$ and $\sup_{P \in \mathcal{P}} E_P[X] \leq x_2$.

Theorem 1 *The static optimization problem admits a solution, if and only if the dynamic optimization problem admits a solution. In this case, let $X^* \in \mathcal{X}(x_1, x_2)$ be a solution to the static problem, and let $V^* \in \mathcal{V}(x_2)$ be a solution to the dynamic problem. Then $V_T^* = X^*$.*

Auxiliary Problem without Model Uncertainty

Optimization Problem

Let $P \in \mathcal{P}$, $Q_0 \in \mathcal{Q}_0$, and $Q_1 \in \mathcal{Q}_1$ be fixed.

Problem

Maximize $E_{Q_0}[u(X)]$ over all X
under the constraints $E_{Q_1}[\ell(-X)] \leq x_1$ and $E_P[X] \leq x_2$.

Solution

The optimal claim has the form

$$x^* \left(\lambda_1^* \frac{dQ_1}{dQ_0}, \lambda_2^* \frac{dP}{dQ_0} \right),$$

where

- $x^* : [0, \infty) \times (0, \infty) \to (\bar{x}_u, \infty)$ continuous deterministic function,
- λ_1^*, λ_2^* suitable real parameters.

The Dual Problem

Define

$$v_{\lambda_1,\lambda_2}(P|Q_1|Q_0) = E_R\left[v\left(\lambda_2\frac{dP}{dR}, \lambda_1\frac{dQ_1}{dR}, \frac{dQ_0}{dR}\right)\right]$$

with $v(p, q_1, q_0) = \sup_{x\in\mathbb{R}}(q_0 u(x) - q_1\ell(-x) - xp)$.

Original Problem

Maximize $E_{Q_0}[u(X)]$ over all X
under the constraints $E_{Q_1}[\ell(-X)] \leq x_1$ and $E_P[X] \leq x_2$.

Dual Problem

Minimize $v_{\lambda_1,\lambda_2}(P|Q_1|Q_0) + \lambda_1 x_1 + \lambda_2 x_2$ over all λ_1, λ_2

Solutions of Problems

$$x^*\left(\lambda_1^*\frac{dQ_1}{dQ_0}, \lambda_2^*\frac{dP}{dQ_0}\right) \qquad \longleftrightarrow \qquad \lambda_1^*, \lambda_2^*$$

Incomplete Market with Model Uncertainty

Optimization Problem

Problem

Maximize $\inf_{Q_0 \in \mathcal{Q}_0} E_{Q_0}[u(X)]$ over all X
under the constraints $\sup_{Q_1 \in \mathcal{Q}_1} E_{Q_1}[\ell(-X)] \leq x_1$ and $\sup_{P \in \mathcal{P}} E_P[X] \leq x_2$.

Assumptions

- $\bar{x}_u > -\infty$, say w.l.o.g $\bar{x}_u = 0$,
- Weak compactness of $\mathcal{Q}_0$ and $\mathcal{Q}_1$

Solution

Given suitable integrability conditions, the optimal claim has the form

$$x^*\left(\lambda_1^*\frac{dQ_1^*}{dQ_0^*},\lambda_2^*\frac{dP^*}{dQ_0^*}\right),$$

where

- $x^* : [0,\infty)\times(0,\infty)\to(\bar{x}_u,\infty)$ deterministic function as before,
- $Q_0^*\in\mathcal{Q}_0$, $Q_1^*\in\mathcal{Q}_1$,
- $P^*\approx R$ suitable measure with $P^*(\Omega)\leq 1$,
- λ_1^*,λ_2^* suitable real parameters.

Question: How can we find Q_0^*, Q_1^*, P^*, and λ_1^*, λ_2^*?

Dual Problem

- Convex conjugate: $v(p, q_1, q_0) = \sup_{x\in\mathbb{R}} (q_0 u(x) - q_1 \ell(-x) - xp)$

Define

$$v_{\lambda_1,\lambda_2}(P|Q_1|Q_0) = E_R\left[v\left(\lambda_2 \frac{dP}{dR}, \lambda_1 \frac{dQ_1}{dR}, \frac{dQ_0}{dR}\right)\right].$$

Then there exists a minimizer
$(\lambda_1^*, \lambda_2^*, Q_0^*, Q_1^*, P^*) \in (\mathbb{R}_+)^2 \times \mathcal{Q}_0 \times \mathcal{Q}_1 \times \mathcal{P}^T$ of

$$v_{\lambda_1,\lambda_2}(P|Q_1|Q_0) + \lambda_1 x_1 + \lambda_2 x_2.$$

Remark: The set of martingale measures $\mathcal{P}$ has to be replaced by appropriate projections $\mathcal{P}^T$ of Föllmer martingale measures.

Worst Case Measures

- Q_0^*, Q_1^*, P^* are worst case measures, i.e.,

$$\begin{aligned} E_{P^*}[X^*] &= \sup_{P \in \mathcal{P}^T} E_P[X^*] \\ E_{Q_1^*}[\ell(-X^*)] &= \sup_{Q_1 \in \mathcal{Q}_1} E_{Q_1}[\ell(-X^*)] \\ E_{Q_0^*}[u(X^*)] &= \inf_{Q_0 \in \mathcal{Q}_0} E_{Q_0}[u(X^*)] \end{aligned}$$

with $X^* = x^* \left(\lambda_1^* \frac{dQ_1^*}{dQ_0^*}, \lambda_2^* \frac{dP^*}{dQ_0^*} \right)$

- The robust solution is the classical solution under the worst case measures

- The robust utility of the optimal claim is given by

$$\inf_{Q_0 \in \mathcal{Q}_0} E_{Q_0}[u(X^*)] = v_{\lambda_1^*, \lambda_2^*}(P^* | Q_1^* | Q_0^*) + \lambda_1^* x_1 + \lambda_2^* x_2$$

Föllmer Measures

- Extended measurable space:

  Consider the product space $\bar{\Omega} := \Omega \times (0, \infty]$.

  Set $\mathcal{F}_t := \mathcal{F}_T$ for $t > T$ with the predictable σ-algebra

$$\bar{\mathcal{F}} := \sigma(\{A \times (t, \infty] : A \in \mathcal{F}_t, t \geq 0\}).$$

  The predictable filtration $(\bar{\mathcal{F}}_t)_{t \geq 0}$ is defined in the same manner.

- Extended trading strategies:

  $\bar{\mathcal{V}}(x)$ denotes the class of value processes $\bar{V} = (\bar{V}_t)_{t \geq 0}$ on $(\bar{\Omega}, \bar{\mathcal{F}}, (\bar{\mathcal{F}}_t)_{t \geq 0})$ with

$$\bar{V}_t(\omega, s) = V_t(\omega) \mathbf{1}_{\{s > t\}} \qquad (V \in \mathcal{V}(x)).$$

Föllmer Measures (continued)

- Föllmer martingale measures:

  A probability measure $\bar{P}$ on $(\bar{\Omega}, \bar{\mathcal{F}})$ will be called a *Föllmer martingale measure* if

  (i) $P^t \ll R$ on $\mathcal{F}_t$ $(t \geq 0)$,

  (ii) Any $\bar{V} \in \bar{\mathcal{V}}(1)$ is a supermartingale under $\bar{P}$.

- Projections:

  Letting $Q^T(A) = \bar{Q}(A \times (T, \infty])$, we set

$$\mathcal{P}^T := \{P^T : \bar{P} \in \bar{\mathcal{P}}\}.$$

Föllmer Measures (continued)

Föllmer martingale measures correspond to the dual set of nonnegative supermartingales in Kramkov & Schachermayer (1999):

$$\mathcal{Y} = \{Y \geq 0 : \; Y_0 = 1, \; VY \; R-\text{supermartingale} \quad \forall \, V \in \mathcal{V}(1)\}$$

- The supermartingale $Y \in \mathcal{Y}$ corresponds to the Föllmer martingale measure $\bar{P}^Y$ defined as

$$\bar{P}^Y[A \times (t, \infty]] = E_R[Y_t; A] \qquad (A \in \mathcal{F}_t, \; t \geq 0)$$

- The Föllmer martingale measure corresponds to the supermartingale $Y \in \mathcal{Y}$ defined as

$$Y_t = \frac{dP^t}{dR} \qquad (t \geq 0)$$

Critical Remarks

- Modeling approach for risk constraint inspired by the (heuristic) article of Basak & Shapiro (2001)
  - Risk constraint is static and never reevaluated after the initial date
  - Optimal trading strategy is a commitment solution, strategy specified for all contingencies at time 0
- Better approach
  - Dynamic risk measurement
  - Model which is consistent with current practice

The approach of Cuoco, He & Isaenko (2008)

Dynamic risk constraints

- Modeling approach suggested by Cuoco, He & Isaenko (2008)
- Extensions by Pirvu & Zitkovic (2009)

Financial Market

Stocks are modeled by n-dimensional Itô-process:

$$\begin{aligned} dS_t^0 &= S_t^0 r dt \\ dS_t^i &= S_t^i \left(\alpha_t^i dt + \sum_{j=1}^{m} \sigma_t^{ij} dW_t^i \right), \quad i = 1, 2, \ldots, n \end{aligned}$$

with

- money market account S^0 and stocks $S^1, \ldots, S^n$,
- mean rate of return process α and variance-covariance process σ

Trading Strategies and Wealth Process

- Portfolio-proportion process ξ:

  The n coordinates of $\xi = (\xi_t)_{t\in[0,\infty)}$ signify the fractions of current wealth V_t^ξ invested in each of the n stocks, $t \in [0,\infty)$

- Wealth process V^ξ:

$$\begin{aligned} dV_t^\xi &= V_t^\xi \left((r + \xi_t^T \mu_t)dt + \xi_t^T \sigma_t dW_t\right), \\ V_t^\xi &= V_0 \exp\left(\int_0^t (r + \xi_u^T \mu_u - \frac{1}{2}\|\xi_u^T \sigma_u\| du) + \int_0^t \xi_u^T \sigma_u dW_u\right) \end{aligned}$$

Risk Constraint

- Measurement horizon τ
- Distribution-invariant risk measure ρ and threshold level α
- Projected change in wealth

(i) Fixed coefficients:

$$P_t^\xi = V_t^\xi \cdot \exp\left((r + \xi_t^T \mu_t - \frac{1}{2} \|\xi_t^T \sigma_t\|)\tau + \xi_t^T \sigma_t (W_{t+\tau} - W_t) \right) - V_t^\xi$$

(ii) Dynamic coefficients:

$$P_t^\xi = V_t^\xi \cdot \left(\int_t^{t+\tau} (r + \xi_t^T \mu_u - \frac{1}{2} \|\xi_t^T \sigma_u\|) du + \int_t^{t+\tau} \xi_t^T \sigma_u dW_u \right) - V_t^\xi$$

- Risk constraint at time t:

$$\rho\left(\mathcal{L}(P_t^\xi | \mathcal{F}_t) \right) \leq \alpha$$

Optimization Problem

Admissible trading strategies ξ and wealth processes V^ξ:

(i) Budget constraint:

$$\begin{aligned} V_0^\xi &= x_2 \\ dV_t^\xi &= V_t^\xi \left((r + \xi_t^T \mu_t)dt + \xi_t^T \sigma_t dW_t\right) \end{aligned}$$

(ii) Risk constraint:

$$\rho\left(\mathcal{L}(P_t^\xi | \mathcal{F}_t)\right) \leq \alpha \quad \forall t \in [0, \infty)$$

(iii) Technical conditions

Problem:

Find “optimal” ξ!

Recent Results

Cuoco, He & Isaenko (2008)

- Model setup:
  - Complete multi-dimensional Black-Scholes market
  - Finite time-horizon
  - Expected utility from terminal wealth
  - VaR & AVaR
- Results:
  - Characterization of optimal trading strategy (multiple of Merton proportion, factor ≤ 1) and terminal wealth in terms of HJB-equation
  - Numerical case studies for CRRA utility
  - Equivalence of VaR & AVaR

Recent Results (continued)

Pirvu & Zitković (2009)

- Model setup:
  - Market driven by Itô-process, ergodicity assumption on the market parameter processes α and σ
  - Growth optimality: $\liminf_{t\to\infty} \frac{\log(V_t^\xi)}{t}$
  - Projected wealth with fixed coefficients
  - Relatively general class of risk measures
- Results:
  - Characterization of growth-optimal strategies (multiple of Merton proportion, factor ≤ 1) and optimal growth rate of wealth

General Equilibrium

A Simple Model

- Modeling approach suggested by Berkelaar, Cumperayot, & Kouwenberg (2002)
- Builds on Basak & Shapiro (2001)
- Needs to be extended to incorporate dynamic risk measurement

Market Model

Aggregate endowment is modeled by a geometric Brownian motion:

$$d\delta_t = \mu_\delta \delta_t dt + \sigma_\delta \delta_t dB_t$$

with constant drift μ_δ and volatility σ_δ, and BM B.

The single consumption good is perishable.

A Simple Model (cont.)

- Money market β in zero supply
- Stock S in constant net supply of 1
- Price Dynamic

$$\begin{aligned} d\beta_t &= r_t \beta_t dt, \\ d(S_t + \delta_t) &= S_t \left(\mu_t dt + \sigma_t dB_t \right), \end{aligned}$$

- Equilibrium

  Processes r, μ, σ are not exogenously given, but determined in equilibrium.

A Simple Model (cont.)

- Agents
  - Unregulated agents: type $i = 1$
  - Regulated agents: type $i = 2$
- Utilities
  - Consumption: U^i
  - Terminal wealth: H^i
  - Relative importance: ρ^i

A Simple Model (cont.)

- Unregulated agents solve the standard optimization problem under a budget constraint for $i = 1$, i.e.,

$$\begin{aligned} \max_{c^i,\ \xi^i,\ \psi^i} \quad & E\left[\int_0^T U^i(c_s^i)ds + \rho^i H^i(W_T^i)\right] \\ \text{s.t.} \quad & W_0^i = w^i \\ & dW_t^i = \xi_t^i d(S_t + \delta_t) + \psi_t^i d\beta_t - c_t^i dt \\ & W_t^i \geq 0, \quad \text{for } \forall t \in [0, T] \end{aligned}$$

- Regulated agents solve the same problem for $i = 2$ under an additional VaR-constraint at level α with threshold q, i.e.,

$$P[W_T^2 \geq q] \geq 1 - \alpha.$$

A Simple Model (cont.)

Market Clearing Condition

(i) Clearing of the commodity market:

$$\hat{c}_t^1 + \hat{c}_t^2 = \delta_t, \qquad 0 \le t \le T.$$

(ii) Clearing of the stock market:

$$\hat{\xi}_t^1 + \hat{\xi}_t^2 = 1, \qquad 0 \le t \le T.$$

(iii) Clearing of the money market:

$$\hat{\psi}_t^1 + \hat{\psi}_t^2 = 0, \qquad 0 \le t \le T.$$

A Simple Model (cont.)

Berkelaar, Cumperayot, & Kouwenberg (2002)

- Model assumptions
  - CRRA/HARA utility functions
- Results
  - Results of Basak & Shapiro (2001) regarding terminal wealth distribution qualitatively still valid
  - Volatility typically decreased in equilibrium by VaR-constraints, but maybe increased in bad states
- Further questions
  - (Semi-)Dynamic risk measurement
  - Comparison of different models

Conclusion

Further Research

(i) Models with more general
- preferences
- market structure
- risk measures

(ii) Utility from consumption

(iii) Equilibrium models