# Monte Carlo Simulation

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# Quantifying and Hedging the Downside Risk

Risk management and financial regulation rely on the proper assessment of the downside risk of financial positions.

**Macroeconomic perspective – regulators**

- design and enforce appropriate rules for regulatory capital
- VaR current industry standard for downside risk – not optimal
- New Basel Capital Accord (Basel II) only intermediate step

**Business perspective – financial institutions**

- improve portfolio value respecting regulatory capital rules
- hedge financial risks, in particular downside risk

# Implementing Risk Measures

- Estimation of risk measures essential in practice
- Simulation algorithms are very tractable and applicable in many models
- Large losses are rare: risk measure simulation can be slow
- This requires variance reduction techniques

# Outline

(i) Toy example: Normal Copula Model (NCM)

- Credit portfolios
- Credit metrics
- Risk measurement

(ii) Monte Carlo Simulation

- Importance sampling
- Stochastic approximation

# Normal Copula Model

# Better Manage Your Risks!

# Credit Portfolios

- One-period model with time periods $t = 0, 1$
- Financial positions at $t = 1$ are modeled as random variables

## Credit Portfolio Losses

- Portfolio with $m$ positions (obligors)
- The random loss at time $1$ due to a default of obligor $i = 1, 2, \ldots, m$ is denoted by $l_i$
- The total losses are given by

$$L = \sum_{i=1}^{m} l_i$$

- Typical decomposition: $l_i = v_i D_i$ with exposure $v_i$ and default indicator $D_i \in \{0, 1\}$

# Credit Portfolios (2)

- Framework above is completely general (if we focus on one-period credit loss models)
- Risk assessment in practice requires specific models that need to be estimated/calibrated and evaluated
- **Examples**
  - Credit Metrics
    - JP Morgan; based on Normal Copula
  - Credit Risk+
    - Credit Suisse; Poisson mixture model
  - Copula models like the t-copula model
    - general family; Gaussian mixture like t-copula particularly tractable

# Credit Metrics

- Model of overall losses of credit portfolio over fixed time horizon
- Losses $L = -X \geq 0$ are given by:

$$L = \sum_{i=1}^{m} v_i D_i.$$

  - Default indicators:

$$D_i = \mathbf{1}_{\{Y_i > y_i\}}$$

  - Marginal default probabilities:

$$p_i = P\{D_i = 1\}$$

  - $m$-dimensional normal factor with standardized marginals:

$$Y = (Y_1, Y_2, \ldots, Y_m)$$

  - Threshold levels:

$$y_i = \Phi^{-1}(1 - p_i)$$

# Credit Metrics (continued)

In industry applications the covariance matrix of the Gaussian vector $Y$ is often specified through a factor model:

$$\begin{aligned} Y_i &= A_{i0}\varepsilon_i + \sum_{j=1}^{d} A_{ij} Z_j \qquad i = 1, \ldots, m,\ d < m; \\ 1 &= A_{i0}^2 + A_{i1}^2 + \ldots + A_{id}^2 \qquad A_{i0} > 0,\ A_{ij} \geq 0, \end{aligned}$$

where

- $Z_1, \ldots, Z_d$ are $d$ independent standard normal random variables (systematic risks), and
- $\varepsilon_1, \ldots, \varepsilon_m$ are $m$ independent standard normal random variables which are independent of $Z_1, \ldots, Z_d$ (idiosyncratic risks).

# Credit Metrics (continued)

- JP Morgan's Credit Metrics is a simplistic toy model.
  - The dependence structure is based on a Gaussian copula and ad hoc.
  - The model exhibits no tail-dependence.
  - Credit Metrics model is "like Black-Scholes".
- Credit Metrics is also called Normal Copula Model (NCM).
- The NCM can be used as a basis for Gaussian mixture models like the t-copula model.
- Risk estimation techniques that work in the NCM can often be extended to Gaussian mixture models and other models.

# Risk Measurement

## Credit Portfolio Losses

Losses $L = -X \geq 0$ are given by:

$$L = \sum_{i=1}^{m} v_i D_i.$$

## Downside Risk

For a given portfolio model, the downside risk $\rho(X)$ needs to be computed for a given risk measure $\rho$.

## Example: Shortfall risk

If $\rho$ is utility-based shortfall risk, then $\rho(X)$ is given by the unique root $s_*$ of the function

$$f(s) := E[\ell(-X - s)] - z.$$

# Risk Measurement and Monte Carlo

## Shortfall risk

Shortfall risk $\rho(X)$ is given by the unique root $s_*$ of the function

$$f(s) := E[\ell(-X - s)] - z.$$

## Computational Problems

- Downside risk focuses on the tail. Rare events are hard to simulate.
- Stochastic root finding problem needs to be solved.

## Efficient Computation

- Variance reduction techniques increase the accuracy/rate of convergence, e.g. importance sampling (Dunkel & W., 2007)
- Stochastic approximation (Dunkel & W., 2008a, 2008b)

# Monte Carlo Simulation

# Importance Sampling

## Shortfall risk

Shortfall risk $\rho(X)$ is given by the unique root $s_*$ of the function

$$f(s) := E[\ell(-X - s)] - z.$$

**First task**:

Estimate

$$E_P[\ell(L - s)] = E_P[h(L)]$$

with $h(L) = \ell(L - s)$.

**Problem**:

(i) Tail events are rare; if we compute the sample average of iid replications under $h(L)$, we will converge very slowly to $E_P[h(L)]$.

(ii) Solution: simulate the important tail part more frequently!

# Importance Sampling (2)

**First task**:

Estimate

$$E_P[\ell(L-s)] = E_P[h(L)]$$

with $h(L) = \ell(L-s)$.

**Two-step variance reduction**:

(i) Importance sampling for $L$ conditional on factor $Z$.

(ii) Variance reduction for factor $Z$.

# Special case: independent default events

In the factor model, independence corresponds to

$$A_{i0} = 1, \quad A_{ij} = 0 \qquad i = 1, \ldots, m, \quad j = 1, \ldots, d.$$

**Importance Sampling:**

- If $Q$ is an equivalent probability measure with a density of the form

$$\frac{dQ}{dP} = g(L),$$

  then $E_P[h(L)] = E_Q\left[\frac{h(L)}{g(L)}\right]$.

- Sampling $L_k$ independently from the distribution of $L$ under $Q$, we get an unbiased, consistent estimator of $E_P[h(L)]$:

$$J_n^g = \frac{1}{n} \sum_{k=1}^{n} \frac{h(L_k)}{g(L_k)}.$$

# Exponential twisting

**SR loss function**:

Suppose that $\ell(x) = \gamma^{-1} x^{\gamma} \mathbf{1}_{[0,\infty)}(x)$ is polynomial.

**Measure change**:

Consider class of probability measures $Q_\theta$, $\theta \geq 0$, with

$$\frac{dQ_\theta}{dP} = \frac{\exp(\theta L)}{\psi(\theta)},$$

where $\psi(\theta) = \log E[\exp(\theta L)] = \sum_{i=1}^{m} \log[1 + p_i(e^{\theta v_i} - 1)]$.

**'Optimal' measure change**:

Minimize an upper bound for the $L^2$-error of the estimator of $E_P[\ell(L - s)]$. This suggests an 'optimal' $\theta_s$.

# Exponential twisting (continued)

(i) Calculate

$$q_i(\theta_s) := \frac{p_i e^{v_i \theta_s}}{1 + p_i \left(e^{v_i \theta_s} - 1\right)}.$$

(ii) Generate $m$ Bernoulli-random numbers $D_i \in \{0, 1\}$, such that $D_i = 1$ with probability $q_i(\theta_s)$.

(iii) Calculate

$$\psi(\theta_s) = \sum_{i=1}^{m} \log[1 + p_i(e^{\theta_s v_i} - 1)]$$

and $L = \sum_{i=1}^{m} v_i D_i$, and return the estimator

$$\ell(L - s) \exp\left[-L\theta_s + \psi\left(\theta_s\right)\right].$$

# Exponential twisting (continued)

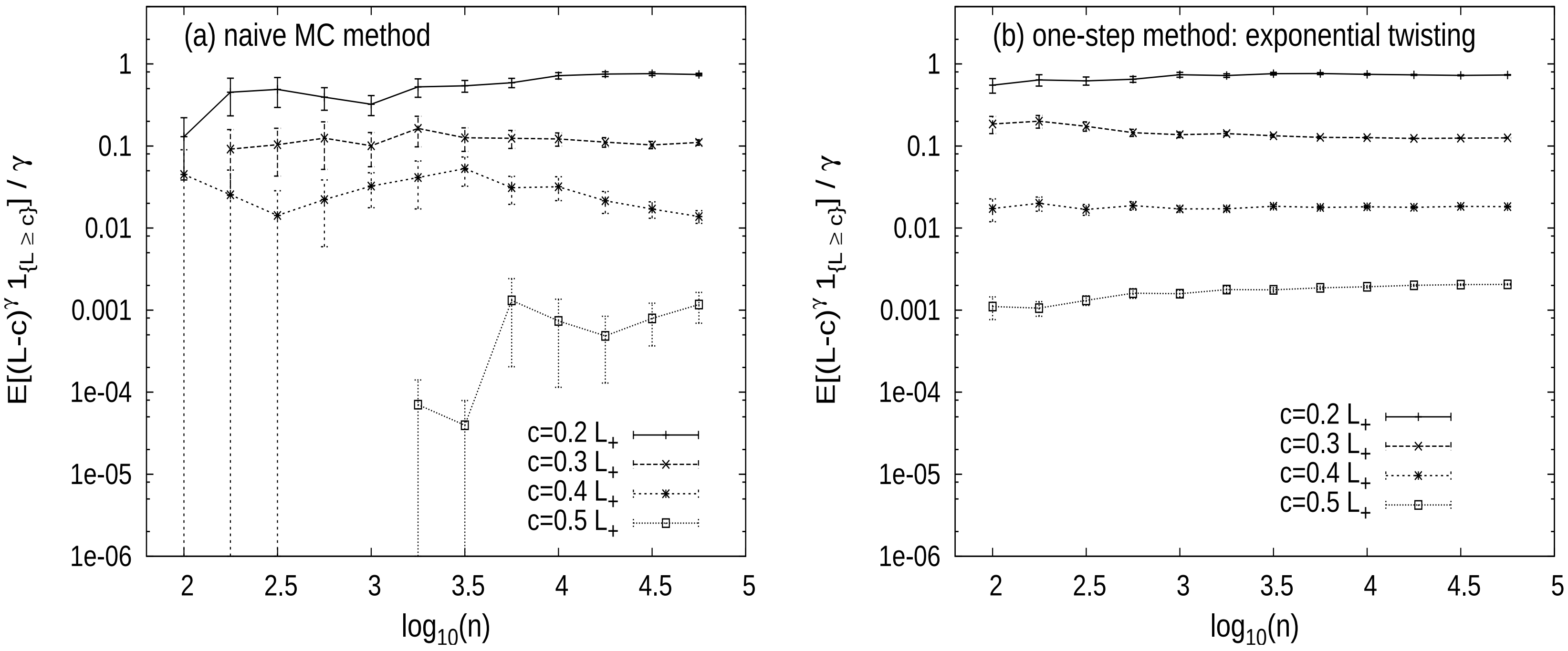



Figure 1: MC results for estimating SR with piecewise polynomial loss function in the NCM. Length of error bars is sample standard deviation of estimator.

# Stochastic Approximation

Stochastic approximation methods provide more efficient root-finding techniques for shortfall risk (Dunkel & W., 2008).

**Robbins-Monro Algorithm**

- Let $\hat{Y}_s : [0,1] \to \mathbb{R}$ such that $E[\hat{Y}_s(U)] = f(s)$ for $U$ ∼unif[0,1].
- Choose a constant $\gamma \in \left(\frac{1}{2}, 1\right]$, $c > 0$, and a starting value $s_1 \in [a,b] \ni s^*$.
- For $n \in \mathbb{N}$ we define recursively:

$$s_{n+1} \;=\; \Pi_{[a,b]}\left[s_n + \frac{c}{n^\gamma} \cdot Y_n\right] \tag{1}$$

with

$$Y_n = \hat{Y}_{s_n}(U_n) \tag{2}$$

for a sequence $(U_n)$ of independent, unif[0,1]-distributed random variables.

# Stochastic Approximation (2)

## Averaging procedure

**Theorem 1** *Suppose that* $\gamma \in \left(\frac{1}{2}, 1\right)$*. For arbitrary* $\rho \in (0,1)$ *we define*

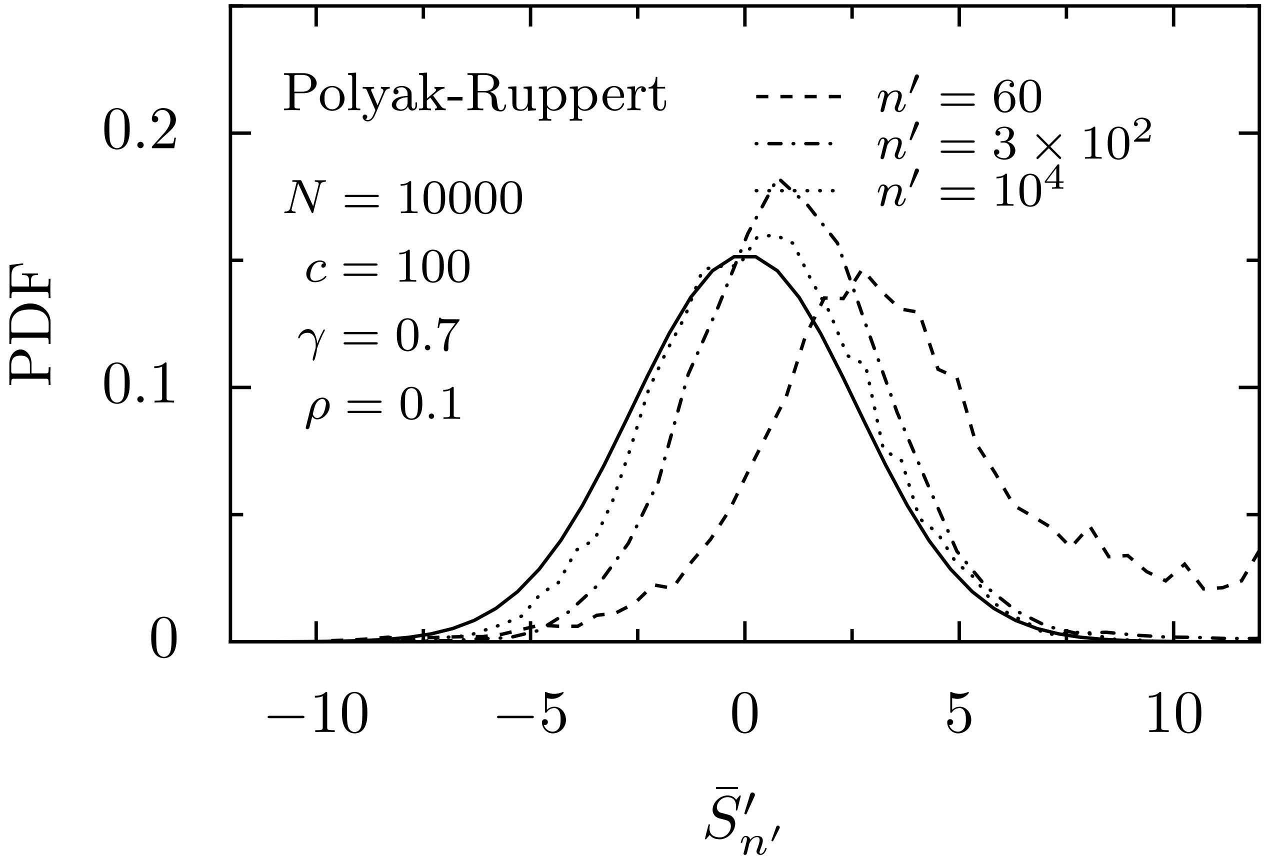
$$\bar{s}_n = \frac{1}{\rho \cdot n} \sum_{i=(1-\rho)n}^{n} s_i.$$

*Then* $\bar{s}_n \to s^*$ $P$*-almost surely. For every* $\varepsilon > 0$ *there exists another process* $\hat{s}$ *such that* $P(\bar{s}_n = \hat{s}_n \ \forall n) \ \geq \ 1 - \varepsilon$ *and*

$$\sqrt{\rho n} \cdot (\hat{s}_n - s^*) \to \mathcal{N}\left(0, \frac{\sigma^2(s^*)}{\left(f'(s^*)\right)^2}\right).$$

- Optimal rate and asymptotic variance guaranteed
- Finite sample properties usually good

# Stochastic approximation (3)




Averaging algorithmus: $\sqrt{\rho n}(\tilde{s}_n - s^*)$ is asymptotically normal (simulation of UBSR with polynomial loss function in the NCM with IS).

# Conclusion

# Conclusion

(i) Axiomatic theory of risk measures

- VaR is not a good risk measure
- Better risk measures have been designed, e.g. Utility-based Shortfall Risk

(ii) Implementation in credit portfolio models

- Importance sampling
- Stochastic approximation

# Further research

- Comparison of stochastic average approximation with stochastic approximation
- Extension of the proposed techniques to a larger class of risk measures, e.g. optimized certainty equivalents
- Adjustments for liquidity risk
- Dynamic risk measurement procedures, and their numerical implementation

# Thank you for your attention!