LAGRANGIAN AND HAMILTONIAN METHODS IN GEOPHYSICAL FLUID DYNAMICS

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This note is an introduction to the variational formulation of fluid dynamics and the geometrical structures thus made apparent. A central theme is the role of continuous symmetries and the associated conservation laws. These are used to reduce more complex to simpler ones, and to study the stability of such systems. Many of the illustrations are taken from models arising from geophysical fluid dynamics.

Contents

Our space is three-dimensional and euclidean, and time is one-dimensional

Arnold [4]

1. Introduction: Why Hamiltonian?

The dynamics of inviscid fluids has a rich geometrical structure that appears most clearly when one considers its variational and/or Hamiltonian formulation. In these lectures, I will attempt to describe the most basic of this structure and how they may be usefully applied (e.g., to prove nonlinear stability). The examples are chosen to illustrate the mathematics as clearly as possible. References therefore are given to point the readers to more physically relevant models, which are often more messy mathematically.

We start by reviewing the basics of Lagrangian and Hamiltonian mechanics for finite-dimensional system, followed by a study of the free rigid body as an explicit example in section 3. We then give the Hamiltonian formulation of a number of models used in geophysical fluid dynamics, all in the Eulerian description, in section 4. In the following section, we consider more closely the role of continuous symmetries; this is then used to obtain the Eulerian description of the compressible Euler equations in two dimensions from its Lagrangian description. In section 6, we describe how Hamiltonian techniques can be used to prove nonlinear stability and more.

We point out that the term "Lagrangian" is used in two different senses in these notes: one to describe mechanics using the Euler–Lagrange equation (2.4) as opposed to Hamilton's equation (2.7); the other is to describe fluid motion using the particle labels $\boldsymbol{a}$ as opposed to (the Eulerian picture) using a fixed physical position $\boldsymbol{x}$ in space (see section 5.2). Some readers will note that the name "Euler" has also been used in two different senses, but let's get going ...

2. Review of Lagrangian and Hamiltonian Mechanics

Let us consider a mechanical system having $\boldsymbol{q} = (q_i)$, $i = 1, \cdots, n$, as (generalised) coordinates. By Newton's law, the equations satisfied by $\boldsymbol{q}$ should look something like

$$\ddot{\boldsymbol{q}} = \boldsymbol{F}(\boldsymbol{q}, t), \tag{2.1}$$

where a dȯt denotes derivative with respect to t, two döts denote second time derivative, etc. So in order to study a mechanical system (2.1), we need to compute and study the forces $\boldsymbol{F}$. As the engineers can tell you, this is a messy subject.

2.1. *Variational Principle of Mechanics*

It was realised a long time ago (see a "classical" text such as [10, 15, 39] for the historical account) that, for many (the physicists would say all) physical systems, there is an easier and more elegant way to describe the dynamics. Instead of the (vector) force $\boldsymbol{F}(\boldsymbol{q}, t)$, we consider a scalar function $L(\boldsymbol{q}, \dot{\boldsymbol{q}}, t)$ called the Lagrangian. For a mechanical system, one usually has $L = T - U$, where T is the kinetic energy and U is the potential energy. The equations of motion are found by *Hamilton's principle*: we form an action functional

$$I(\boldsymbol{q}, \dot{\boldsymbol{q}}) := \int_{t_0}^{t_1} L(\boldsymbol{q}(t), \dot{\boldsymbol{q}}(t), t)\, \mathrm{d}t \tag{2.2}$$

and require that the variation of I vanishes for every (permissible) variations $\delta\boldsymbol{q}$. Explicitly, we compute

$$\begin{aligned} \delta I &= \int_{t_0}^{t_1} \Big\{ \frac{\partial L}{\partial \boldsymbol{q}} \delta\boldsymbol{q} + \frac{\partial L}{\partial \dot{\boldsymbol{q}}} \delta\dot{\boldsymbol{q}} \Big\}\, \mathrm{d}t \\ &= \int_{t_0}^{t_1} \Big\{ \frac{\partial L}{\partial \boldsymbol{q}} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{\boldsymbol{q}}} \Big\} \delta\boldsymbol{q}\, \mathrm{d}t \end{aligned} \tag{2.3}$$

where the second line has been obtained by writing $\delta\dot{\boldsymbol{q}} = \mathrm{d}\delta\boldsymbol{q}/\mathrm{d}t$ and integrating by parts. If δI is to vanish for any possible $\delta\boldsymbol{q}$, we must have

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{\boldsymbol{q}}} - \frac{\partial L}{\partial \boldsymbol{q}} = 0, \tag{2.4}$$

which is known as the Euler–Lagrange equation.

For reasons that will become more apparent as we go along, it is advantageous to pass on to the Hamiltonian formulation of the dynamics, which we do as follows. Define the momentum $\boldsymbol{p}$ by

$$\boldsymbol{p} = \frac{\partial L}{\partial \dot{\boldsymbol{q}}}, \tag{2.5}$$

let (some of you may know that this is a Legendre transformation)

$$H = \boldsymbol{p} \cdot \dot{\boldsymbol{q}} - L(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) \tag{2.6}$$

and express H as a function of $\boldsymbol{q}$ and $\boldsymbol{p}$ using (2.5). The object $H(\boldsymbol{q}, \boldsymbol{p}, t)$ is called the *Hamiltonian*; for a mechanical system, it is usually the energy. As may be verified directly, the evolution of $\boldsymbol{p}$ and $\boldsymbol{q}$ are given by *Hamilton's equation*,

$$\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = -\frac{\partial H}{\partial \boldsymbol{q}} \qquad \text{and} \qquad \frac{\mathrm{d}\boldsymbol{q}}{\mathrm{d}t} = \frac{\partial H}{\partial \boldsymbol{p}}. \tag{2.7}$$

It is clear that in Hamiltonian systems, the dynamics is determined completely by the Hamiltonian $H(\boldsymbol{p}, \boldsymbol{q})$.

We note that, unlike in Lagrangian dynamics where we have $\boldsymbol{q}$ and $\dot{\boldsymbol{q}}$, in the Hamiltonian formulation the coordinate $\boldsymbol{q}$ and the momentum $\boldsymbol{p}$ have become mathematical 'equals'. This may be better appreciated by changing variables to $\boldsymbol{Q} = \boldsymbol{p}$ and $\boldsymbol{P} = -\boldsymbol{q}$ and writing Hamilton's equations in these new variables.

2.2. *Symplectic Structure*

We introduce another formalism that will be useful later. For any $F(\boldsymbol{q}, \boldsymbol{p})$ and $G(\boldsymbol{q}, \boldsymbol{p})$, we define their *Poisson bracket* as

$$\{F, G\} = \sum_i \frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i} - \frac{\partial G}{\partial q_i}\frac{\partial F}{\partial p_i}. \tag{2.8}$$

It is clear that that the Poisson bracket is completely specified by its values for $\boldsymbol{p}$ and $\boldsymbol{q}$,

$$\{q_i, q_j\} = \{p_i, p_j\} = 0 \qquad \text{and} \qquad \{q_i, p_j\} = -\{p_j, q_i\} = \delta_{ij} \tag{2.9}$$

where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ otherwise. A pair of variables (q_j, p_j) that satisfies (2.9) are said to be *conjugate*.

We note three special properties. For any F, G and K functions of $(\boldsymbol{p}, \boldsymbol{q})$, the Poisson bracket (2.8)

(i) is antisymmetric, $\{G, F\} = -\{F, G\}$;

(ii) is a derivation, $\{FG, K\} = F\{G, K\} + \{F, K\}G$;

(iii) satisfies Jacobi's identity, $\{F, \{G, K\}\} + \{G, \{K, F\}\} + \{K, \{F, G\}\} = 0$.

Simple though these may seem, they reflect deep properties of the space where $\boldsymbol{p}$ and $\boldsymbol{q}$ live, many of which were not discovered until quite recently (and many others are probably still awaiting discovery).

The evolution equation (2.7) can now be written in terms of the Poisson bracket as $\dot{\boldsymbol{p}} = \{\boldsymbol{p}, H\}$ and $\dot{\boldsymbol{q}} = \{\boldsymbol{q}, H\}$; in fact, using the chain rule, we have for any $F(\boldsymbol{p}, \boldsymbol{q}, t)$,

$$\frac{\mathrm{d}F}{\mathrm{d}t} = \{F, H\} + \partial_t F. \tag{2.10}$$

It follows that if the Hamiltonian H does not depend on t explicitly, $\partial_t H = 0$, then it is constant under the dynamics. In what follows, our Hamiltonians are assumed time-independent unless stated otherwise.

Let $\boldsymbol{z} = (\boldsymbol{p}, \boldsymbol{q})$. Maps $T : \boldsymbol{z} \mapsto \tilde{\boldsymbol{z}}$ such that $\{\tilde{z}_i, \tilde{z}_j\} = \{z_i, z_j\}$ are called *symplectic maps*. It can be shown that the equations of motion in the new variables $\tilde{\boldsymbol{z}}$ are again Hamilton's canonical equations (2.7).

2.3. *Symmetries, Conservation Laws and Adiabatic Invariance*

The close connection between symmetries and conservation laws arguably appears most clearly in the context of Hamiltonian systems. Returning to Hamilton's equations (2.7), we notice that if $\partial H/\partial q_k = 0$, which means that the dynamics is invariant under changes of q_k, then the corresponding momentum p_k is a constant of motion, $\mathrm{d}p_k/\mathrm{d}t = 0$.

One can regard p_k as the 'generator of translation', parameterised by s, in the q_k direction by taking p_k as the 'Hamiltonian' as follows,

$$\frac{\mathrm{d}q_i}{\mathrm{d}s} = \{q_i, p_k\} = \delta_{ik} \qquad \text{and} \qquad \frac{\mathrm{d}p_i}{\mathrm{d}s} = \{p_i, p_k\} = 0. \tag{2.11}$$

Now the change in $F(\boldsymbol{p}, \boldsymbol{q})$ as we translate in the q_k direction is evidently given by

$$\frac{\mathrm{d}F}{\mathrm{d}s} = \{F, p_k\}. \tag{2.12}$$

Taking $F = H$, if H (and thus the dynamics) is invariant under translations in the q_k direction, we must have $\{H, p_k\} = 0$. More interesting is the converse: Suppose that we have an $M(\boldsymbol{p}, \boldsymbol{q})$ such that $\{H, M\} = 0$. Then we can find (perhaps with a lot of work) a direction in which the dynamics can be invariantly translated. We conclude that, in canonical Hamiltonian systems, the existence of a conserved quantity corresponds to a continuous symmetry of the dynamics. This is a special case of *Noether's theorem*, a more general case of which we will see later; see [24] for a full discussion.

Even more interesting is the behaviour of *approximate symmetries* in Hamiltonian systems. Suppose now that

$$H = H_0 + \varepsilon H_1 \tag{2.13}$$

where $\partial H_0/\partial q_k = \{H_0, p_k\} = 0$ and ε is small. Then in many interesting cases (subject to some rather technical assumptions detailed in the references below), one can show that the variable p_k is an *adiabatic invariant*, in the sense that its *total* variation is of order ε over very long timescales; more precisely,

$$|q_k(t) - q_k(0)| \le \varepsilon C \qquad \text{for } t \le \mathrm{e}^{\kappa/\varepsilon}. \tag{2.14}$$

Although such behaviour has been observed in nature for a long time, its mathematical proof, known as Neishtadt or Nekhoroshev theory, is quite involved and was not discovered until the 1960s; not surprisingly, these are closely related to the more famous Kolmogorov–Arnold–Moser (KAM) theory. We refer the readers to [16, 18] for further discussions.

We note that the existence of an invariant p_k allows us to reduce the number of 'active' variables by *two*: first $p_k = p_k(0)$ can be regarded as a parameter in (2.7), which can then be integrated as a system with $2n-2$ variables $(q_1, \cdots, q_{k-1}, q_{k+1}, q_n, p_1, \cdots, p_{k-1}, p_{k+1}, \cdots, p_n)$; the solution of this reduced system can be used to integrate the non-autonomous equation $\mathrm{d}q_k/\mathrm{d}t = \partial H/\partial p_k$.

3. An Example: the Free Rigid Body

It is instructive to first look at the dynamics of/on a rigid body with no external forces acting on it. We use kinematic considerations to derive equations of motion in a rotating frame which will be useful later, the dynamics gives us an example of Lagrangian and Hamiltonian formalism we've just discussed, and the reduction procedure serves as a simple(r) illustration of the analogous procedure for fluids.

The connection between the rigid body and inviscid fluid dynamics actually goes much deeper than we can describe in these notes. The Poisson bracket of the rigid body is that of the Lie algebra so(3) or, equivalently, su(2), while the Poisson bracket of the 2d Euler equation of fluid dynamics can be thought of as the limit $n \to \infty$ of su$(2n+1)$. See [3, 26] for more details.

3.1. *Kinematics: Rotating Frames*

In what follows, we will denote by $\boldsymbol{r}(t)$ the position of a particle relative to a coordinate system which is inertial (that is, a system which is "fixed relative to the stars"). We consider a rotating rigid body whose centre of mass is fixed at $\boldsymbol{r} = 0$. We denote by $\boldsymbol{x}(t)$ the position of our particle relative to a coordinate system which is fixed on this body, chosen such that the centre of mass of the body is also at $\boldsymbol{x} = 0$. It is clear that $\boldsymbol{x}$ and $\boldsymbol{r}$ are related by a rotation matrix $\boldsymbol{O}(t)$ (cf. [4]),

$$\boldsymbol{r}(t) = \boldsymbol{O}(t)\boldsymbol{x}(t). \tag{3.1}$$

Let $\boldsymbol{x}$ be fixed for now. Recall that the velocity and angular velocity are defined as (denoting time derivative by a dot)

$$\boldsymbol{v} := \dot{\boldsymbol{r}} \qquad \text{and} \quad \boldsymbol{\omega} := \frac{\boldsymbol{r} \times \boldsymbol{v}}{|\boldsymbol{r}|^2}. \tag{3.2}$$

Since the body is rigid and the centre of mass is fixed, we have

$$\boldsymbol{r} \times \boldsymbol{v} = 0, \tag{3.3}$$

so after a little computation, we find that

$$\dot{\boldsymbol{r}} = \boldsymbol{v} = \boldsymbol{\omega} \times \boldsymbol{r}. \tag{3.4}$$

Recall that a vector product $\boldsymbol{a} \times \boldsymbol{b}$ can be written as $M_{\boldsymbol{a}}\boldsymbol{b}$ where $M_{\boldsymbol{a}}$ is an antisymmetric matrix (what is it?); conversely, if M is antisymmetric, we can find a vector $\boldsymbol{a}_M$ such that $M\boldsymbol{b} = \boldsymbol{a} \times \boldsymbol{b}$ for every vector $\boldsymbol{b}$. Now equating (3.4) with d/dt of (3.1),

$$\dot{\boldsymbol{O}}\boldsymbol{x} = \boldsymbol{\omega} \times (\boldsymbol{O}\boldsymbol{x}) = M_{\boldsymbol{\omega}}\boldsymbol{O}\boldsymbol{x} \tag{3.5}$$

or, since $\boldsymbol{x}$ is arbitrary,

$$\dot{\boldsymbol{O}} = M_{\boldsymbol{\omega}}\boldsymbol{O}. \tag{3.6}$$

We define the *angular velocity in the body* as the vector $\boldsymbol{\Omega}$ corresponding to the antisymmetric matrix

$$M_{\boldsymbol{\Omega}} := \boldsymbol{O}^{-1}M_{\boldsymbol{\omega}}\boldsymbol{O}. \tag{3.7}$$

Now let $\boldsymbol{x}$ vary in time. Taking time derivative of (3.1), we find

$$\begin{aligned} \dot{\boldsymbol{r}} &= \dot{\boldsymbol{O}}\boldsymbol{x} + \boldsymbol{O}\dot{\boldsymbol{x}} \\ &= \boldsymbol{O}(\dot{\boldsymbol{x}} + M_{\boldsymbol{\Omega}}\boldsymbol{x}) \\ &= \boldsymbol{O}(\dot{\boldsymbol{x}} + \boldsymbol{\Omega} \times \boldsymbol{x}). \end{aligned} \tag{3.8}$$

Taking time derivative again,

$$\begin{aligned} \ddot{\boldsymbol{r}} &= \dot{\boldsymbol{O}}(\dot{\boldsymbol{x}} + M_{\boldsymbol{\Omega}}\boldsymbol{x}) + \boldsymbol{O}(\ddot{\boldsymbol{x}} + \dot{M}_{\boldsymbol{\Omega}} + M_{\boldsymbol{\Omega}}\dot{\boldsymbol{x}}) \\ &= \boldsymbol{O}M_{\boldsymbol{\Omega}}(\dot{\boldsymbol{x}} + M_{\boldsymbol{\Omega}}\boldsymbol{x}) + \boldsymbol{O}(\ddot{\boldsymbol{x}} + \dot{M}_{\boldsymbol{\Omega}}\boldsymbol{x} + M_{\boldsymbol{\Omega}}\dot{\boldsymbol{x}}) \\ &= \boldsymbol{O}(\ddot{\boldsymbol{x}} + M_{\boldsymbol{\Omega}}M_{\boldsymbol{\Omega}}\boldsymbol{x} + 2M_{\boldsymbol{\Omega}}\dot{\boldsymbol{x}} + \dot{M}_{\boldsymbol{\Omega}}\boldsymbol{x}). \end{aligned} \tag{3.9}$$

Newton's second law tells us that, in an inertial frame (i.e. for $\boldsymbol{r}(t)$), the acceleration of a point particle (or a parcel of fluid, etc) is equal to the force $\boldsymbol{F}$ acting on it, so we have

$$\ddot{\boldsymbol{x}} = -2\boldsymbol{\Omega} \times \dot{\boldsymbol{x}} - \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{x}) - \dot{\boldsymbol{\Omega}} \times \boldsymbol{x} + \boldsymbol{f} \tag{3.10}$$

where $\boldsymbol{f} = \boldsymbol{O}^{-1}\boldsymbol{F}$ is the external force seen in our rotating frame.

Thus we see that there are several "spurious" forces (or "pseudoforces") in a rotating frame. The first one, and the most important for us, is the *Coriolis force* $-2\boldsymbol{\Omega} \times \dot{\boldsymbol{x}}$. This force depends on the velocity and causes the deflection of objects which would otherwise move along straight lines (or geodesics on the rotating body). Next, we also have the *centrifugal force* $-\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{x})$. In geophysical applications, this force is usually incorporated into the gravity $g\hat{\boldsymbol{z}}$; in fact, the shape of a rotating self-gravitating fluid (such as planets) is such that this "effective gravity" is normal to its (mean) surface. Finally, we have the inertial force of rotation $\dot{\boldsymbol{\Omega}} \times \boldsymbol{x}$, which is obviously not important for geophysical applications.

3.2. *Dynamics: the Free Rigid Body*

In the absence of external forces (in an inertial frame), our rigid body conserves its angular momentum

$$\boldsymbol{M} := \int \rho(\boldsymbol{r}) \frac{\boldsymbol{r} \times \dot{\boldsymbol{r}}}{|\boldsymbol{r}|^2} \, \mathrm{d}\boldsymbol{r}, \tag{3.11}$$

where ρ is the mass density and the integral is taken over the body. (Note an inconsistency in our notation: for angular momentum, $\boldsymbol{M}$ is in space and $\boldsymbol{m} = \boldsymbol{O}^{-1}\boldsymbol{M}$ is in the body, but for angular velocity, $\boldsymbol{\Omega}$ is in the body but $\boldsymbol{\omega}$ is in space; this is to conform with the "usual notation" in GFD.) It is proved in mechanics that the angular momentum is related to the angular velocity by

$$\boldsymbol{m} = \boldsymbol{I}\boldsymbol{\Omega}, \tag{3.12}$$

where the moment of inertia tensor $\boldsymbol{I}$ is positive-definite and symmetric. Since $\boldsymbol{I}$ is symmetric, there exists a coordinate system such that it is diagonal; we shall henceforth choose our body coordinates such that this is the case and write $\boldsymbol{I} = \mathrm{diag}(I_1, I_2, I_3)$. In these *principal axis* coordinates, we simply have $m_i = I_i\,\Omega_i$ for $i = 1, 2, 3$.

The orientation of a rigid body in space can be given in terms of the *Euler angles* $\boldsymbol{\xi} = (\xi_1, \xi_2, \xi_3)$. As realised by Euler, the equations of motion satisfied by these angles are most easily obtained in the body coordinate system defined above. It can be shown (consult a mechanics book) that the time derivative $\dot{\boldsymbol{\xi}}$ are related to the angular velocity in the body $\boldsymbol{\Omega}$ by

$$\begin{aligned} \dot{\xi}_1 \sin\xi_1 &= \sin\xi_1 \cos\xi_3\,\Omega_1 - \sin\xi_1 \sin\xi_3\,\Omega_2 \\ \dot{\xi}_2 \sin\xi_1 &= \sin\xi_3\,\Omega_1 + \cos\xi_3\,\Omega_2 \\ \dot{\xi}_3 \sin\xi_1 &= -\cos\xi_1 \sin\xi_3\,\Omega_1 - \cos\xi_1 \cos\xi_3\,\Omega_2 + \sin\xi_1\,\Omega_3. \end{aligned} \tag{3.13}$$

Recall that for a mechanical system, the Lagrangian $L = T - V$, where T is the kinetic energy and V the potential energy. Since no external forces act on our rigid body, $V = 0$. The kinetic energy takes a particularly simple form in the principal-axis body coordinates, so we can immediately write down the Lagrangian as

$$L(\boldsymbol{\xi}, \dot{\boldsymbol{\xi}}) = \tfrac{1}{2}(I_1\Omega_1^2 + I_2\Omega_2^2 + I_3\Omega_3^2). \tag{3.14}$$

To compute the Euler–Lagrange equations, $\boldsymbol{\Omega}$ needs to be expressed in terms of $\boldsymbol{\xi}$ and $\dot{\boldsymbol{\xi}}$ using the inverse of (3.13), viz.,

$$\begin{aligned} \Omega_1 &= \cos\xi_3\,\dot{\xi}_1 + \sin\xi_1 \sin\xi_3\,\dot{\xi}_2 \\ \Omega_2 &= -\sin\xi_3\,\dot{\xi}_1 + \sin\xi_1 \cos\xi_3\,\dot{\xi}_2 \\ \Omega_3 &= \cos\xi_1\,\dot{\xi}_2 + \dot{\xi}_3. \end{aligned} \tag{3.15}$$

For convenience below, we write these equations as $\boldsymbol{\Omega} = \boldsymbol{N}^{-1}\dot{\boldsymbol{\xi}}$ and (3.13) as $\dot{\boldsymbol{\xi}} = \boldsymbol{N}\boldsymbol{\Omega}$, where the matrix $\boldsymbol{N} = (N_{ij})$ is a function of the angles $\boldsymbol{\xi}$.

Now let us pass on to Hamiltonian dynamics. The momenta $\boldsymbol{p}$ are

$$p_i = \frac{\partial L}{\partial \dot{\xi}_i} = \textstyle\sum_j \dfrac{\partial L}{\partial \Omega_j}\dfrac{\partial \Omega_j}{\partial \dot{\xi}_i} = \sum_j I_j\Omega_j \dfrac{\partial \Omega_j}{\partial \dot{\xi}_i} = \sum_j I_j \Omega_j N_{ji}. \tag{3.16}$$

The Hamiltonian is obtained as usual by Legendre transformation,

$$\begin{aligned} H(\boldsymbol{\xi}, \boldsymbol{p}) &= \textstyle\sum_i p_i \dot{\xi}_i - L \\ &= \textstyle\sum_{ij} I_j \Omega_j N_{ji} (N^{-1})_{ij} \Omega_j - \frac{1}{2}\sum_i I_j \Omega_j^2 \\ &= \textstyle\frac{1}{2}\sum_i I_j \Omega_j^2, \end{aligned} \tag{3.17}$$

where $\boldsymbol{\Omega}$ is to be expressed in terms of $(\boldsymbol{\xi}, \boldsymbol{p})$ using the inverse of (3.16). Hamilton's canonical equations in the variables $(\boldsymbol{p}, \boldsymbol{\xi})$ follows.

3.3. *Reduction: the Euler Equations*

The particularly simple form of the Hamiltonian (3.17) suggests that we work in the (principal-axis) body coordinates (cf. [21]). So let us take as our variables $(\boldsymbol{m}, \boldsymbol{\theta})$, where $\boldsymbol{m}$ is the angular momentum in body coordinates and $\boldsymbol{\theta}$, which as we will see shortly is actually redundant, is the variable conjugate to $\boldsymbol{m}$. The Hamiltonian is now

$$H(\boldsymbol{m},\boldsymbol{\theta}) = \frac{1}{2}\Big(\frac{m_1^2}{I_1} + \frac{m_2^2}{I_2} + \frac{m_3^2}{I_3}\Big), \tag{3.18}$$

which is independent of $\boldsymbol{\theta}$. The equations of motion are

$$\frac{\mathrm{d}m_i}{\mathrm{d}t} = \{m_i, H\} = \textstyle\sum_j \{m_i, m_j\} \displaystyle\frac{\partial H}{\partial m_j}, \tag{3.19}$$

plus something for $\dot{\boldsymbol{\theta}}$ that we won't need. So it remains to compute the Poisson brackets $\{m_i, m_j\}$.

It follows from (3.16) and the definition of the matrix $\boldsymbol{N}$ above that

$$m_k = \textstyle\sum_l p_l N_{lk}\,. \tag{3.20}$$

Using this, we compute

$$\begin{aligned} \{m_i, m_j\} &= \sum_k \frac{\partial m_i}{\partial \xi_k}\frac{\partial m_j}{\partial p_k} - \frac{\partial m_i}{\partial p_k}\frac{\partial m_j}{\partial \xi_k} \\ &= \sum_{kl} p_l \Big(\frac{\partial N_{li}}{\partial \xi_k} N_{kj} - \frac{\partial N_{lj}}{\partial \xi_k} N_{ki}\Big) \\ &= \sum_{kln} m_n N_{nl}^{-1} \Big(\frac{\partial N_{li}}{\partial \xi_k} N_{kj} - \frac{\partial N_{lj}}{\partial \xi_k} N_{ki}\Big). \end{aligned} \tag{3.21}$$

Using the identity

$$\sum_l N_{nl}^{-1}\frac{\partial N_{lj}}{\partial \xi_k} + \frac{\partial N_{nl}^{-1}}{\partial \xi_k} N_{lj} = 0 \tag{3.22}$$

obtained by differentiating the identity $\boldsymbol{N}\boldsymbol{N}^{-1} = \mathbf{1}$, we find

$$\{m_i, m_j\} = \sum_{kln} m_n \big(N_{lj}N_{ki} - N_{li}N_{kj}\big)\frac{\partial N_{nl}^{-1}}{\partial \xi_k}. \tag{3.23}$$

Evaluating this explicitly, we find

$$\{m_i, m_j\} = -\epsilon_{ijk}\, m_k \tag{3.24}$$

where $\epsilon_{ijk} = +1$ if ijk is an even permutation of 123, $\epsilon_{ijk} = -1$ if ijk is an odd permutation of 123, and $\epsilon_{ijk} = 0$ otherwise.

Equation (3.24) defines a *non-canonical* Poisson bracket in the *reduced variables* $\boldsymbol{m}$. For any two functions F and G of $\boldsymbol{m}$ only, their Poisson bracket is defined to be

$$\{F, G\} := \sum_{ij} \frac{\partial F}{\partial m_i}\frac{\partial G}{\partial m_j}\{m_i, m_j\}. \tag{3.25}$$

This bracket is often written in the form

$$\{F, G\} := \sum_{ij} J_{ij}(\boldsymbol{m}) \frac{\partial F}{\partial m_i} \frac{\partial G}{\partial m_j}, \tag{3.26}$$

where $J(\boldsymbol{m})$ is often called the *cosymplectic matrix*. Let K be another arbitrary function of $\boldsymbol{m}$. It is easily verified that the Poisson bracket (3.25) is (i) antisymmetric, (ii) is a derivation,

$$\{FK, G\} = \{F, G\}K + F\{K, G\}, \tag{3.27}$$

and (iii) satisfies Jacobi's identity,

$$\{F, \{G, K\}\} + \{G, \{K, F\}\} + \{K, \{F, G\}\} = 0. \tag{3.28}$$

Unlike the canonical bracket (2.8), the last property is no longer trivial.

Also unlike in the canonical case, the cosymplectic matrix $J(\boldsymbol{m})$ may now be degenerate. In our present example, the matrix

$$J(\boldsymbol{m}) = \begin{pmatrix} 0 & -m_3 & m_2 \\ m_3 & 0 & -m_1 \\ -m_2 & m_1 & 0 \end{pmatrix} \tag{3.29}$$

is singular (or degenerate): we have

$$J_{ij}(\boldsymbol{m}) \frac{\partial C}{\partial m_j} = 0 \tag{3.30}$$

for

$$C(\boldsymbol{m}) = |\boldsymbol{m}|^2. \tag{3.31}$$

In general, functions C that satisfy

$$\{C, F\} = 0 \tag{3.32}$$

for any F are called *Casimir invariants*. Putting the Hamiltonian H for F, it follows that Casimirs are indeed invariant,

$$\frac{\mathrm{d}C}{\mathrm{d}t} = \{C, H\} = 0. \tag{3.33}$$

Casimir invariants are a property of the singular Poisson bracket, not of the Hamiltonian. We note a somewhat subtle point here: in general, a null vector of a singular matrix cannot be written as the gradient of a function. The fact that we can write (3.30) is a consequence of Jacobi's property (iii) of the Poisson bracket; see [39].

If the cosymplectic matrix $J(\boldsymbol{m})$ is linear in $\boldsymbol{m}$, that is, if

$$\{m_i, m_j\} = c_{ij}^k m_k \tag{3.34}$$

for some constants c_{ij}^k, the bracket is often called a Lie–Poisson bracket. This is because c_{ij}^k are actually the structure constants of a Lie algebra. As we shall see, this is an important special case.

Considering the Hamiltonian in (3.18) as a function of $\boldsymbol{m}$ only, $H = H(\boldsymbol{m})$, the equations of motion follow from (3.19),

$$\frac{\mathrm{d}m_1}{\mathrm{d}t} = \frac{I_2 - I_3}{I_2 I_3}\, m_2 m_3\,, \tag{3.35}$$

with the equations for m_2 and m_3 obtained by cyclic permutations. So now we have obtained an equation for $\boldsymbol{m}$, but what about the angles $\boldsymbol{\xi}$? Well, once we have solved (3.35) for $\boldsymbol{m}(t)$, we can sub the now-known $\boldsymbol{\Omega}(t)$ into (3.13) which can then be integrated for $\boldsymbol{\xi}(t)$. The point is that (3.35) has fewer degrees of freedom than the original system involving $(\boldsymbol{\xi}, \dot{\boldsymbol{\xi}})$, and that the variables eliminated in the process ($\boldsymbol{\xi}$ in our case) can be recovered from the solution of the reduced system (3.35) by integrating a non-autonomous system such as (3.13). This is the idea of *reduction*; we shall encounter this again in the fluid context below. Of course if we have a rigid body under external forces (that presumably depend on $\boldsymbol{\xi}$), a reduction cannot be performed and we are stuck with the original six-variable situation.

4. Hamiltonian Models of Fluid Dynamics

In our discussion of fluids, we adopt an approach opposite to that in section 3, starting in this section with the *reduced* description of the dynamics and only discussing the "full" description in section 5 below. There are two reasons for this: one is that the reduced (Eulerian) description of fluid dynamics is often felt more intuitive than the unreduced (Lagrangian) description, another is that the machinery we use in the Eulerian description will be useful to effect the reduction from the Lagrangian description.

We note that fluid equations are Hamiltonian at several levels. In these lectures, we will be mostly concerned with the structure of the *dynamical* fluid equations. If we are given a non-divergent velocity field in two dimensions, presumably obtained as a solution of some fluid equations, the dynamics of particles moving in that velocity field will again have a Hamiltonian structure; this is the subject of Prof. Legras' lectures.

4.1. *Hamilton's Principle for PDEs*

Recall that in the case of ODEs in section 2, our (dependent) variable is $\boldsymbol{q}$ with one independent variable t. For models described by partial differential equations, we consider a dependent variable u and several independent variables, say, $\boldsymbol{x} \in \mathcal{D} \subset \mathbb{R}^n$ and t. Our Lagrangian now takes the form

$$L(u, u_t; t) = \int_{\mathcal{D}} l(u, u_{\boldsymbol{x}}, u_t; \boldsymbol{x}, t) \, \mathrm{d}\boldsymbol{x}^n, \tag{4.1}$$

where the scalar quantity $l(u, \cdots)$ is often called the Lagrangian density, and $u_{\boldsymbol{x}} = \partial_{\boldsymbol{x}} u$ and $u_t = \partial_t u$. Objects such as L, taking functions (u and u_t here[a]) as argument and giving a real value, are called *functionals*. The *functional derivative* of a functional with respect to one of its argument is defined as

$$\frac{\delta L}{\delta u}(u, \cdots) := \lim_{\varepsilon \to 0} \frac{L(u + \varepsilon v, \cdots) - L(u, \cdots)}{\varepsilon} \tag{4.2}$$

for all possible v (in a suitable space, etc); it is usually useful to compare this to the definition of the derivative of a function of many variables. The functional derivative $\delta L/\delta u$ is usually "the same type of object" as u, as will be seen below.

In analogy with the finite-dimensional case, Hamilton's principle here states that the variation of the action,

$$\delta I = \delta \int_0^T L(u, u_t; t) \, \mathrm{d}t = \delta \int_0^T \int_{\mathcal{D}} l(u, u_{\boldsymbol{x}}, u_t; \boldsymbol{x}, t) \, \mathrm{d}\boldsymbol{x}^n \, \mathrm{d}t, \tag{4.3}$$

vanishes when u is a solution of the dynamics. To compute the last quantity, we take the leading-order part of

$$\begin{aligned} &\int_0^T \int_{\mathcal{D}} \big\{ l(u + \delta u, u_{\boldsymbol{x}} + \delta u_{\boldsymbol{x}}, u_t + \delta u_t; \boldsymbol{x}, t) - l(u, u_{\boldsymbol{x}}, u_t; \boldsymbol{x}, t) \big\} \, \mathrm{d}\boldsymbol{x}^n \, \mathrm{d}t \\ &\quad = \int_0^T \int_{\mathcal{D}} \Big\{ \frac{\partial l}{\partial u} \delta u + \frac{\partial l}{\partial u_{\boldsymbol{x}}} \delta \partial_{\boldsymbol{x}} u + \frac{\partial l}{\partial u_t} \delta \partial_t u \Big\} \, \mathrm{d}\boldsymbol{x}^n \, \mathrm{d}t \\ &\quad = \int_0^T \int_{\mathcal{D}} \Big\{ \frac{\partial l}{\partial u} - \frac{\partial}{\partial \boldsymbol{x}} \cdot \Big(\frac{\partial l}{\partial u_{\boldsymbol{x}}} \Big) - \frac{\partial}{\partial t} \Big(\frac{\partial l}{\partial u_t} \Big) \Big\} \, \delta u \, \mathrm{d}\boldsymbol{x}^n \, \mathrm{d}t, \end{aligned} \tag{4.4}$$

where we have integrated by parts in $\boldsymbol{x}$ and t to arrive at the last line (assuming as we've done so far that boundary terms vanish as needed). The functional derivative of L with respect to u (holding t and u_t fixed) is

$$\frac{\delta L}{\delta u} = \frac{\partial l}{\partial u} - \frac{\partial}{\partial \boldsymbol{x}} \cdot \Big(\frac{\partial l}{\partial u_{\boldsymbol{x}}} \Big). \tag{4.5}$$

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[a]There are of course functionals which depend on more complicated objects derived from u, but we won't need them in this course.

As with the usual partial derivatives, it is important to remember which quantities are held fixed when computing derivatives as the notation can be ambiguous.

In the Hamiltonian formulation, the evolution of any functional F is given by the usual-looking equation

$$\frac{\mathrm{d}F}{\mathrm{d}t} = \{F, H\}, \tag{4.6}$$

where now H is a functional and the Poisson bracket involves an integral over $\mathcal{D}$ of functional derivatives of F and H. As before, we require that the bracket $\{\cdot,\cdot\}$ (i) be antisymmetric, (ii) be a derivation, and (iii) satisfy Jacobi's identity. These are probably best understood using the examples below.

4.2. *Vorticity-Based Models*

Here we make the somewhat ad-hoc separation of models into those that can be described by a scalar dependent variable, usually related to the vorticity, and those having a larger set of dependent variables. In this subsection we give examples of the first category.

4.2.1. *Two-dimensional Euler Equations*

The dynamics of an incompressible inviscid fluid in two space dimensions is governed by the Euler equations,

$$\begin{aligned} &\partial_t \boldsymbol{v} + \boldsymbol{v}\cdot\boldsymbol{\nabla}\boldsymbol{v} = -\boldsymbol{\nabla}p \\ &\boldsymbol{\nabla}\cdot\boldsymbol{v} = 0, \end{aligned} \tag{4.7}$$

where $\boldsymbol{v} = (u, v)$ is the velocity vector, with $\boldsymbol{v} = \boldsymbol{v}(\boldsymbol{x}, t)$ and $\boldsymbol{x} = (x, y) \in \mathcal{D}$, and where the pressure p can be regarded as the Lagrange multiplier needed to enforce the incompressibility constraint $\boldsymbol{\nabla}\cdot\boldsymbol{v} = 0$. On the boundary $\partial\mathcal{D}$, we must have $\boldsymbol{v}\cdot\hat{\boldsymbol{n}} = 0$ where $\hat{\boldsymbol{n}}$ is the outward normal to $\partial\mathcal{D}$.

A natural way to write (4.7) is using the vorticity–streamfunction formulation: Assuming that our domain $\mathcal{D}$ is not too exotic topologically, we can write our incompressible velocity $\boldsymbol{v}$ as

$$\boldsymbol{v} = \boldsymbol{\nabla}^{\perp}\psi := (-\partial_y\psi, \partial_x\psi) \tag{4.8}$$

for a scalar $\psi(\boldsymbol{x}, t)$ called the *streamfunction*. We define the *vorticity* by

$$\omega := \boldsymbol{\nabla}^{\perp}\cdot\boldsymbol{v} = \partial_x v - \partial_y u. \tag{4.9}$$

Taking $\boldsymbol{\nabla}^\perp \cdot$ (4.8), we find that the vorticity is the Laplacian of the stream-function,

$$\omega = \Delta\psi. \tag{4.10}$$

Taking $\boldsymbol{\nabla}^\perp \cdot$ (4.7a), we find

$$\partial_t \omega + \boldsymbol{v} \cdot \boldsymbol{\nabla}\omega = \partial_t \omega + \boldsymbol{\nabla}^\perp \psi \cdot \boldsymbol{\nabla}\omega = 0. \tag{4.11}$$

The boundary conditions are now

$$\psi = c_i \qquad \text{on } \partial\mathcal{D}_i \tag{4.12}$$

where $\partial\mathcal{D}_i$ denotes each connected part of the boundary. If our domain is simply connected, we can simply set $\psi = 0$ on the boundary. In most of these notes, we will assume that the boundary conditions vanish when doing integration by parts. Often these require certain (implicit) assumptions on the domains and the boundary conditions imposed; these should be verified for each problem at hand.

The first Hamiltonian description for a continuous fluid model in Eulerian variables (in more than one space dimension) was obtained by Morrison and Greene [22] for the 2d Euler equations (and the virtually identical 1d Vlasov–Poisson equations of magnetohydrodynamics). The Hamiltonian is just the kinetic energy,

$$H(\omega) = \frac{1}{2} \int_{\mathcal{D}} |\boldsymbol{\nabla}\psi|^2 \, \mathrm{d}\boldsymbol{x}^2, \tag{4.13}$$

to be considered as a functional of ω. The Poisson bracket of two functionals F and G is given by

$$\{F, G\} = -\int_{\mathcal{D}} \frac{\delta F}{\delta\omega} \partial\Big(\omega, \frac{\delta G}{\delta\omega}\Big) \, \mathrm{d}\boldsymbol{x}^2, \tag{4.14}$$

where $\partial(f, g) := \boldsymbol{\nabla}^\perp f \cdot \boldsymbol{\nabla} g = f_x g_y - f_y g_x$ for any two functions f and g defined in $\mathcal{D}$.

To obtain the evolution equation from Hamilton's equation (4.6), we need to compute the variation of the Hamiltonian,

$$\begin{aligned} \delta \int_{\mathcal{D}} \tfrac{1}{2} |\boldsymbol{\nabla}\psi|^2 \, \mathrm{d}\boldsymbol{x}^2 &= \delta \int_{\mathcal{D}} \tfrac{1}{2} \boldsymbol{\nabla}\psi \cdot \boldsymbol{\nabla}\delta\psi \, \mathrm{d}\boldsymbol{x}^2 \\ &= -\delta \int_{\mathcal{D}} \psi \Delta\psi \, \mathrm{d}\boldsymbol{x}^2 = -\int_{\mathcal{D}} \psi \, \delta\omega \, \mathrm{d}\boldsymbol{x}^2. \end{aligned} \tag{4.15}$$

This gives us the functional derivative of H with respect to ω,

$$\frac{\delta H}{\delta\omega} = -\psi. \tag{4.16}$$

We then fix $\boldsymbol{x} \in \mathcal{D}$ and take as our functional F

$$\omega(\boldsymbol{x}) = \int_{\mathcal{D}} \omega(\boldsymbol{x}')\, \delta(\boldsymbol{x} - \boldsymbol{x}')\, \mathrm{d}x'\, \mathrm{d}y' \tag{4.17}$$

where $\delta(\boldsymbol{x} - \boldsymbol{x}')$ is the Dirac distribution. The equation of motion (4.11) thus follows,

$$\begin{aligned} \partial_t \omega(\boldsymbol{x}) &= \{\omega(\boldsymbol{x}), H\} \\ &= \int_{\mathcal{D}} \delta(\boldsymbol{x} - \boldsymbol{x}')\, \partial\big(\omega(\boldsymbol{x}'), \psi(\boldsymbol{x}')\big)\, \mathrm{d}x'\, \mathrm{d}y' = -\partial(\psi, \omega)(\boldsymbol{x}). \end{aligned} \tag{4.18}$$

The antisymmetry and Leibniz property of the bracket defined in (4.14) are straightforward to verify; Jacobi's identity can be verified by integration by parts. This Poisson bracket has an infinite number of Casimir invariants, they being all functionals of the form

$$C_f(\omega) = \int_{\mathcal{D}} f(\omega)\, \mathrm{d}\boldsymbol{x}^2 \tag{4.19}$$

for any scalar function f. Using the facts that $\delta C_f/\delta\omega = f'(\omega)$ and $\partial(\omega, g(\omega)) = 0$ for any function g, one can see that indeed

$$\{C_f, F\} = \int_{\mathcal{D}} \frac{\delta F}{\delta \omega} \partial\Big(\omega, \frac{\delta C_f}{\delta \omega}\Big)\, \mathrm{d}\boldsymbol{x}^2 = \int_{\mathcal{D}} \frac{\delta F}{\delta \omega} \partial\Big(\omega, f'(\omega)\Big)\, \mathrm{d}\boldsymbol{x}^2 = 0 \tag{4.20}$$

for any functional F.

The effects of planetary rotation can be included in this model by simply replacing the vorticity ω by $\omega + \Omega(\boldsymbol{x})$, where $\Omega(\boldsymbol{x})$ is the normal component of the planet's angular velocity at $\boldsymbol{x}$. For the geophysicists: this includes both f-plane and β-plane approximations as well as the full spherical case.

The *2d quasi-geostrophic equation* is given by

$$\begin{aligned} \partial_t q + \partial(\psi, q) &= 0 \\ q &= (\Delta - F)\, \psi, \end{aligned} \tag{4.21}$$

where the rotational Froude number F is a constant and $\boldsymbol{v} = \boldsymbol{\nabla}^\perp \psi$. The Poisson bracket is again (4.14) with ω replaced by q, while the Hamiltonian is

$$H(q) = \frac{1}{2} \int \big\{ |\boldsymbol{\nabla}\psi|^2 + F\psi^2 \big\}\, \mathrm{d}x\, \mathrm{d}y. \tag{4.22}$$

The details are left for exercise.

4.2.2. *Layered Quasi-Geostrophic Models*

A "poor person's" model of the stratified QGE below can be obtained by coupling several layers of the 2d QGE above,

$$\partial_t q_i + \partial(\psi_i, q_i) = 0 \tag{4.23}$$

where $q_i(\boldsymbol{x}, t)$ is the potential vorticity in layer i. The coupling enters through the definition of the streamfunction ψ_i. For simplicity, we consider the two-layer model, where

$$q_i = \Delta\psi_i + (-1)^i F_i(\psi_1 - \psi_2) + f + \beta y, \tag{4.24}$$

for $i = 1, 2$, where we have included the rotation $f + \beta y$.

The Hamiltonian is given by

$$H(q_1, q_2) = \frac{1}{2}\int \big\{ d_1|\psi_1|^2 + d_2|\psi_2|^2 + d_1F_1(\psi_1 - \psi_2)^2 \big\}\, \mathrm{d}x\, \mathrm{d}y, \tag{4.25}$$

where d_i is the depth of layer i and where $d_1F_1 = d_2F_2$. Computing the variations of H, we find

$$\begin{aligned} \delta H &= \int \big\{ d_1 \boldsymbol{\nabla}\psi_1 \cdot \boldsymbol{\nabla}\delta\psi_1 + d_2 \boldsymbol{\nabla}\psi_2 \cdot \boldsymbol{\nabla}\delta\psi_2 \\ &\qquad\qquad + d_1F_1(\psi_1 - \psi_2)(\delta\psi_1 - \delta\psi_2) \big\}\, \mathrm{d}x\, \mathrm{d}y \\ &= \int \big\{ -d_1\psi_1\Delta\delta\psi_1 + d_1F_1\psi_1(\delta\psi_1 - \delta\psi_2) \\ &\qquad\qquad - d_2\psi_2\Delta\delta\psi_2 + d_2F_2\psi_1(\delta\psi_1 - \delta\psi_2) \big\}\, \mathrm{d}x\, \mathrm{d}y, \end{aligned} \tag{4.26}$$

where again the fact that $d_1F_1 = d_2F_2$ has been used. From this, we find

$$\frac{\delta H}{\delta q_i} = -d_i\psi_i\,. \tag{4.27}$$

For $F(q_1, q_2)$ and $G(q_1, q_2)$, the Poisson bracket is

$$[F, G] = -\int \Big\{ \frac{1}{d_1}\frac{\delta F}{\delta q_1}\partial\Big(q_1, \frac{\delta G}{\delta q_1}\Big) + \frac{1}{d_1}\frac{\delta F}{\delta q_1}\partial\Big(q_1, \frac{\delta G}{\delta q_1}\Big) \Big\}\, \mathrm{d}x\, \mathrm{d}y. \tag{4.28}$$

This bracket has as Casimirs all functionals of the form

$$C_{fg}(q_1, q_2) = \int \big\{ f(q_1) + g(q_2) \big\}\, \mathrm{d}x\, \mathrm{d}y \tag{4.29}$$

for scalar functions f and g. The verifications of this and of the equations of motion (4.23) are left as exercises.

4.2.3. *Stratified Quasi-Geostrophic Equations*

Let $\rho_0(z)$ be a prescribed reference density, $N(z) := [(g/\rho_0)\rho_0'(z)]^{1/2}$ the buoyancy (Brunt–Väisälä) frequency, and $S(z) := N(z)^2/f^2$ the stratification function. We take as domain a horizontal layer, with $\boldsymbol{x} = (x, y, z) \in \mathbb{R}^2 \times [0, 1]$. Let $q = q(\boldsymbol{x}, t)$. In the interior, $z \in (0, 1)$, the continuously stratified quasigeostrophic model is

$$\begin{aligned} &\partial_t q + \partial(\psi, q) = 0 \\ &q = \Delta_2 \psi + \frac{1}{\rho_0}\frac{\partial}{\partial z}\Big(\frac{\rho_0}{S}\partial_z\psi\Big) + f + \beta y, \end{aligned} \tag{4.30}$$

where $\Delta_2 = \partial_{xx} + \partial_{yy}$ is the 2d (horizontal) Laplacian and f and β are constants. We note that the advection is purely horizontal, arising from $\partial(\psi, q)$, with the vertical coupling arising from the relation of q and ψ. On the boundary, we have an advection equation for the temperature ψ_z,

$$\partial_t \psi_z + \partial(\psi, \psi_z) = 0 \qquad \text{on } z = 0, 1. \tag{4.31}$$

It is useful to take as our dependent variables q for $z \in (0, 1)$ and $\lambda_i := \rho_0\psi_z/S\big|_{z=i}$ for $i = 0, 1$. The Hamiltonian for this model is

$$H(q, \lambda_1, \lambda_2) = \int \frac{\rho_0}{2}\Big\{|\boldsymbol{\nabla}_2\psi|^2 + \frac{1}{S}\psi_z^2\Big\}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z, \tag{4.32}$$

where $\boldsymbol{\nabla}_2 = (\partial_x, \partial_y)$ is the horizontal gradient. The variation of H is

$$\begin{aligned} \delta H &= \int \rho_0\Big\{\boldsymbol{\nabla}_2\psi\cdot\boldsymbol{\nabla}_2\delta\psi + \frac{1}{S}\psi_z\delta\psi_z\Big\}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\ &= \int\Big\{-\rho_0\psi\Delta_2\delta\psi + \frac{\partial}{\partial z}\Big(\frac{\rho_0}{S}\psi\delta\psi_z\Big) - \psi\frac{\partial}{\partial z}\Big(\frac{\rho_0}{S}\delta\psi_z\Big)\Big\}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\ &= \Big[\int\frac{\rho_0}{S}\psi\delta\psi_z\,\mathrm{d}x\,\mathrm{d}y\Big]_{z=0}^{z=1} - \int\rho_0\psi\,\delta q\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z, \end{aligned} \tag{4.33}$$

from which we get

$$\frac{\delta H}{\delta q} = -\rho_0\psi, \qquad \frac{\delta H}{\delta\lambda_0} = -\psi\big|_{z=0} \qquad \text{and} \qquad \frac{\delta H}{\delta\lambda_1} = \psi\big|_{z=1}. \tag{4.34}$$

For F and G functionals of $(q, \lambda_1, \lambda_2)$, their Poisson bracket is

$$\begin{aligned} &= -\int\frac{1}{\rho_0}\frac{\delta F}{\delta q}\partial\Big(q, \frac{\delta G}{\delta q}\Big)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\ &\qquad + \int\Big\{\frac{\delta F}{\delta\lambda_1}\partial\Big(\lambda_1, \frac{\delta G}{\delta\lambda_1}\Big) - \frac{\delta F}{\delta\lambda_0}\partial\Big(\lambda_0, \frac{\delta G}{\delta\lambda_0}\Big)\Big\}\,\mathrm{d}x\,\mathrm{d}y. \end{aligned} \tag{4.35}$$

It has as Casimirs

$$C(q,\lambda_0,\lambda_1) = \int f(q)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z + \int \big\{g(\lambda_0) + h(\lambda_1)\big\}\,\mathrm{d}x\,\mathrm{d}y \qquad (4.36)$$

for arbitrary functions f, g and h.

Historically, this stratified quasi-geostrophic model was used as an approximation for more general fluid models in the limit of strong rotation (i.e. large f). It was proved in [7] that the solution of the rotating stratified Boussinesq equations does converge in this limit to the solution of the stratified QG.

4.2.4. *Semigeostrophic Models*

The semigeostrophic model is widely used in mesoscale dynamics, because it represents realistic frontal structures. This model can also be cast in the prototypical form

$$\partial_t q + \partial(\psi, q) = 0 \qquad \text{where } \psi = -\delta H/\delta q \qquad (4.37)$$

for some Hamiltonian $H(q)$, when expressed in isentropic-geostrophic coordinates. In these coordinates, rigid boundaries appear to move in time. This leads to dynamical degrees of freedom (as with QG on horizontal boundaries). In the special case of lateral boundaries, these degrees of freedom correspond to coastal Kelvin waves, which are analogous in some respects to the Eady edge waves represented in both the QG and SG models. These degrees of freedom must be taken into account in the variational calculations, and enter into many of the resulting expressions.

The semigeostrophic model has a particularly rich geometric structure related to the contact transformation used in its derivation, but discussion of it is well beyond the scope of these notes. We refer interested readers to [11] and the references therein.

4.3. *Other Fluid Models*

Here we give two examples of "more realistic" geophysical models; these require more than one scalar field for their description. The first example will appear again when we consider reduction from the Lagrangian description in section 5.3.

4.3.1. *Rotating Shallow-Water Equations*

Let $\boldsymbol{x} = (x, y)$ and $\boldsymbol{v} = (u, v)$. The rotating shallow-water equations are

$$\begin{aligned} &\partial_t \boldsymbol{v} + \boldsymbol{v} \cdot \boldsymbol{\nabla} \boldsymbol{v} + f \boldsymbol{v}^\perp = -g \boldsymbol{\nabla} h \\ &\partial_t h + \boldsymbol{\nabla} \cdot (\boldsymbol{v} h) = 0, \end{aligned} \tag{4.38}$$

where f and g are constants (what are they?) and $\boldsymbol{v}^\perp = (-v, u)$ as with $\boldsymbol{\nabla}^\perp$. These equations describe the dynamics of a shallow layer of fluid with no vertical structure (i.e. the fluid moves in columns whose height is h), but they may also be interpreted to describe the dynamics of a compressible two-dimensional fluid whose density is h, in which case they are sometimes called the compressible 2d Euler equations.

The energy of this model is

$$H(\boldsymbol{v}, h) = \frac{1}{2} \int \{ h |\boldsymbol{v}|^2 + g h^2 \} \, \mathrm{d}x \, \mathrm{d}y, \tag{4.39}$$

which will serve as our Hamiltonian. By considering variations of the Hamiltonian as before, we find

$$\frac{\delta H}{\delta \boldsymbol{v}} = h \boldsymbol{v} \qquad \text{and} \qquad \frac{\delta H}{\delta h} = \frac{1}{2} |\boldsymbol{v}|^2 + g h. \tag{4.40}$$

The potential vorticity

$$q = \frac{\boldsymbol{\nabla}^\perp \cdot \boldsymbol{v} + f}{h}, \tag{4.41}$$

which is materially conserved, $\partial_t q + \boldsymbol{v} \cdot \boldsymbol{\nabla} q = 0$, has an important place in the dynamics. It can be verified that the equations of motion are recovered using the Poisson bracket

$$\{F, G\} = \int \Big\{ q \, \frac{\delta F}{\delta \boldsymbol{v}^\perp} \cdot \frac{\delta G}{\delta \boldsymbol{v}} - \frac{\delta F}{\delta \boldsymbol{v}} \cdot \boldsymbol{\nabla} \frac{\delta G}{\delta h} + \frac{\delta G}{\delta \boldsymbol{v}} \cdot \boldsymbol{\nabla} \frac{\delta F}{\delta h} \Big\} \, \mathrm{d}x \, \mathrm{d}y, \tag{4.42}$$

whose Casimir invariants will be found below.

This is a good place to show how a general change of variable works for noncanonical Hamiltonian systems. Suppose that instead of $(\boldsymbol{v}, h)$ we want to use (q, d, h) where $d := \boldsymbol{\nabla} \cdot \boldsymbol{v}$ is the divergence, as dependent variables. Using the relation

$$\boldsymbol{v} = \boldsymbol{\nabla} \Delta^{-1} d + \boldsymbol{\nabla}^\perp [\Delta^{-1}(q h) - f], \tag{4.43}$$

one can readily express the Hamiltonian in terms of (q, d, h). To compute the Poisson bracket in the new (q, d, h) variable, we first note the relations

$$\delta q = \frac{1}{h} \boldsymbol{\nabla}^\perp \cdot \delta \boldsymbol{v} - \frac{q}{h} \delta h. \qquad \text{and} \qquad \delta d = \boldsymbol{\nabla} \cdot \delta \boldsymbol{v}. \tag{4.44}$$

Then we consider a functional $F(q,d,h)$ and its variation,

$$\begin{aligned}\delta F &= \int \Big\{ \frac{\delta F}{\delta q}\delta q + \frac{\delta F}{\delta d}\delta d + \frac{\delta F}{\delta h}\Big|_{(q,d)}\delta h \Big\}\,\mathrm{d}x\,\mathrm{d}y \\ &= \int \Big\{ \frac{\delta F}{\delta q}\Big(\frac{1}{h}\boldsymbol{\nabla}^{\perp}.\delta \boldsymbol{v} - \frac{q}{h}\delta h\Big) + \frac{\delta F}{\delta d}\boldsymbol{\nabla}.\delta\boldsymbol{v} + \frac{\delta F}{\delta h}\Big|_{(q,d)}\delta h \Big\}\,\mathrm{d}x\,\mathrm{d}y \qquad (4.45) \\ &= \int \Big\{ -\delta\boldsymbol{v}\cdot\Big[\boldsymbol{\nabla}^{\perp}\Big(\frac{1}{h}\frac{\delta F}{\delta q}\Big) + \boldsymbol{\nabla}\frac{\delta F}{\delta d}\Big] + \delta h\Big[\frac{\delta F}{\delta h}\Big|_{(q,d)} - \frac{q}{h}\frac{\delta F}{\delta q}\Big]\Big\}\,\mathrm{d}x\,\mathrm{d}y,\end{aligned}$$

where the last line was obtained using integration by parts. The functional derivatives in $(\boldsymbol{v},h)$ and (q,d,h) are then related by

$$\frac{\delta F}{\delta \boldsymbol{v}} = -\boldsymbol{\nabla}^{\perp}\Big(\frac{1}{h}\frac{\delta F}{\delta q}\Big) - \boldsymbol{\nabla}\frac{\delta F}{\delta d} \quad \text{and} \quad \frac{\delta F}{\delta h}\Big|_{\boldsymbol{v}} = \frac{\delta F}{\delta h}\Big|_{(q,d)} - \frac{q}{h}\frac{\delta F}{\delta q}. \qquad (4.46)$$

Substituting these into (4.42), we find the Poisson bracket in the (q,d,h) variables,

$$\begin{aligned}\{F,G\} = \int \Big\{ q\Big[\partial\Big(\frac{1}{h}\frac{\delta F}{\delta q},\frac{1}{h}\frac{\delta G}{\delta q}\Big) + \partial\Big(\frac{\delta F}{\delta d},\frac{\delta G}{\delta d}\Big) \\ + \boldsymbol{\nabla}\frac{\delta F}{\delta d}\cdot\boldsymbol{\nabla}\Big(\frac{1}{h}\frac{\delta G}{\delta d}\Big) - \boldsymbol{\nabla}\frac{\delta G}{\delta d}\cdot\boldsymbol{\nabla}\Big(\frac{1}{h}\frac{\delta F}{\delta d}\Big)\Big] \qquad (4.47) \\ + \Big(\frac{\delta F}{\delta h} - \frac{q}{h}\frac{\delta F}{\delta q}\Big)\Delta\frac{\delta G}{\delta d} - \Big(\frac{\delta G}{\delta h} - \frac{q}{h}\frac{\delta G}{\delta q}\Big)\Delta\frac{\delta F}{\delta d}\Big\}\,\mathrm{d}x\,\mathrm{d}y.\end{aligned}$$

The Casimirs C can now be computed as follows. Putting $F = C$ and $G = h$ in (4.42), we find

$$0 = \{C,h\} = \boldsymbol{\nabla}\cdot\frac{\delta C}{\delta \boldsymbol{v}}, \qquad (4.48)$$

so by (4.46b), C cannot depend on the divergence d. Putting $G = \boldsymbol{v}$ in (4.42) gives

$$\begin{aligned}0 = \{C,\boldsymbol{v}\} &= q\frac{\delta C}{\delta \boldsymbol{v}^{\perp}} + \boldsymbol{\nabla}\frac{\delta C}{\delta h} \\ &= \boldsymbol{\nabla}\Big(\frac{1}{h}\frac{\delta C}{\delta q}\Big) - \boldsymbol{\nabla}^{\perp}\frac{\delta C}{\delta d} + \boldsymbol{\nabla}\Big(\frac{\delta C}{\delta h} - \frac{q}{h}\frac{\delta C}{\delta q}\Big) \qquad (4.49) \\ &= -\frac{1}{h}\frac{\delta C}{\delta q}\boldsymbol{\nabla} q + \boldsymbol{\nabla}\frac{\delta C}{\delta h},\end{aligned}$$

where we have used (4.46) for the second line and the fact that $\delta C/\delta d = 0$ for the last line. The last line can be satisfied if

$$\frac{\delta C}{\delta h} = f(q) \qquad \text{and} \qquad \frac{\delta C}{\delta q} = hf'(q), \qquad (4.50)$$

which gives us the Casimirs

$$C(q,h) = \int h f(q)\, \mathrm{d}x\, \mathrm{d}y \tag{4.51}$$

for any function f.

4.3.2. *Stratified 3d Euler Equations*

We now consider a stratified compressible perfect fluid in three space dimensions under the influence of rotation and gravity, with the Coriolis parameter f taken to be constant for simplicity. Let $\boldsymbol{x} = (x, y, z)$ and let $\boldsymbol{u} = (u, v, w)$ be a function of $(\boldsymbol{x}, t)$. Our system reads

$$\begin{aligned} &\partial_t \boldsymbol{u} + \boldsymbol{u}.\boldsymbol{\nabla}\boldsymbol{u} + f\hat{\boldsymbol{z}} \times \boldsymbol{u} = \frac{1}{\rho}\boldsymbol{\nabla} p - g\hat{\boldsymbol{z}} \\ &\partial_t \rho + \boldsymbol{\nabla}.(\boldsymbol{u}\rho) = 0 \\ &\partial_t s + \boldsymbol{u}.\boldsymbol{\nabla} s = 0. \end{aligned} \tag{4.52}$$

Here $\rho(\boldsymbol{x}, t)$ is the (mass) density and $s(\boldsymbol{x}, t)$ the entropy (density); the pressure p is a given function of ρ and s. It will prove convenient to rewrite the advective term as

$$\boldsymbol{u}.\boldsymbol{\nabla}\boldsymbol{u} = \tfrac{1}{2}\boldsymbol{\nabla}|\boldsymbol{u}|^2 + \boldsymbol{\omega} \times \boldsymbol{u} \tag{4.53}$$

where $\boldsymbol{\omega} = \boldsymbol{\nabla} \times \boldsymbol{u}$ is the vorticity. For simplicity, we will ignore boundary terms by assuming that all relevant quantities tend to zero (or constant, as appropriate) as $|\boldsymbol{x}| \to \infty$.

As before, the Hamiltonian is just the energy of the fluid,

$$H = \int \big\{ \tfrac{1}{2}\rho|\boldsymbol{u}|^2 + U(\rho, s) + \rho g z \big\}\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z, \tag{4.54}$$

where the internal energy U is a given function of ρ and s, subject to the condition that

$$\frac{\partial U}{\partial \rho} = \frac{p}{\rho^2}. \tag{4.55}$$

From thermodynamics, the temperature is $T = \partial U/\partial s$. Taking $(\boldsymbol{u}, \rho, s)$ as variables, the functional derivatives of the Hamiltonian are

$$\frac{\delta H}{\delta \boldsymbol{u}} = \rho\boldsymbol{u}, \qquad \frac{\delta H}{\delta \rho} = \tfrac{1}{2}|\boldsymbol{u}|^2 + U_\rho + gz \qquad \text{and} \qquad \frac{\delta H}{\delta s} = U_s. \tag{4.56}$$

The equations of motion can be recovered from (4.55) and the Poisson bracket,

$$\{F,G\} = \int \Big\{ \frac{\boldsymbol{\omega}}{\rho} \cdot \Big(\frac{\delta F}{\delta \boldsymbol{u}} \times \frac{\delta G}{\delta \boldsymbol{u}} \Big) + \frac{\delta G}{\delta \rho} \boldsymbol{\nabla} \cdot \frac{\delta F}{\delta \boldsymbol{u}} - \frac{\delta F}{\delta \rho} \boldsymbol{\nabla} \cdot \frac{\delta G}{\delta \boldsymbol{u}} + \frac{1}{\rho} \frac{\delta G}{\delta s} \frac{\delta F}{\delta \boldsymbol{u}} \cdot \boldsymbol{\nabla} s - \frac{1}{\rho} \frac{\delta F}{\delta s} \frac{\delta G}{\delta \boldsymbol{u}} \cdot \boldsymbol{\nabla} s \Big\} \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z. \tag{4.57}$$

The Casimirs of the bracket (4.57) are functionals of the form

$$C = \int \rho g(q,s) \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z \tag{4.58}$$

where the function g is arbitrary and

$$q = \frac{(\boldsymbol{\omega} + f\hat{\boldsymbol{z}}) \cdot \boldsymbol{\nabla} s}{\rho} \tag{4.59}$$

is the (Ertel) potential vorticity, which is materially conserved,

$$\partial_t q + \boldsymbol{u} \cdot \boldsymbol{\nabla} q = 0. \tag{4.60}$$

5. More on Hamiltonian Fluid Dynamics

Having seen many models of fluid flows, all of which can be cast in Hamiltonian form (in the absence of forcing and dissipation, as assumed throughout), we will now discuss several important properties of their Hamiltonian structure. Among other things, this will illustrate the geometric origin of potential vorticity.

5.1. *Continuous Symmetries*

Compared to finite-dimensional dynamical systems, now we have the possibility of varying the dependent variables as well as the independent ones. As was realised by E. Noether, even more complicated, or interesting, depending on one's point of view, possibilities exist; see [24] for (much) more details. As before, for each symmetry "translation", there corresponds a "generator". If the dynamics governed by a Hamiltonian H is invariant under the translation generated by a functional M, the corresponding generator is conserved under the dynamics, $\{M,H\} = 0$. We illustrate this by several examples.

Let us consider the 2d Euler equations (4.13)–(4.14) and take

$$M(\omega) = \int_{\mathcal{D}} \omega y \, \mathrm{d}\boldsymbol{x}^2. \tag{5.1}$$

Coinsidering a functional $F(\omega)$ with $\delta F/\delta\omega = \phi$, the change in F under the translation generated by M is [cf. (2.12)],

$$\begin{aligned}\frac{\mathrm{d}F}{\mathrm{d}s} = \{F, M\} &= \int_{\mathcal{D}} \frac{\delta M}{\delta\omega}\partial\Big(\omega, \frac{\delta F}{\delta\omega}\Big)\,\mathrm{d}\boldsymbol{x}^2 \\ &= \int_{\mathcal{D}} y\,\partial(\omega,\phi)\,\mathrm{d}\boldsymbol{x}^2 = \int_{\mathcal{D}} \omega\,\partial(\phi, y)\,\mathrm{d}\boldsymbol{x}^2 = \int_{\mathcal{D}} \omega\,\partial_x\phi\,\mathrm{d}\boldsymbol{x}^2.\end{aligned} \tag{5.2}$$

This implies that $\mathrm{d}\phi/\mathrm{d}s = \partial_x\phi$, so the translation generated by M has the effect

$$\phi \mapsto \phi + s\,\partial_x\phi, \tag{5.3}$$

which is precisely what happens under a translation in the x-direction. Writing M in a slightly different way, and integrating by parts as usual,

$$M = \int_{\mathcal{D}} (v_x - u_y)\,y\,\mathrm{d}\boldsymbol{x}^2 = \int_{\mathcal{D}} \Big\{u\frac{\partial y}{\partial y} - v\frac{\partial y}{\partial x}\Big\}\,\mathrm{d}\boldsymbol{x}^2 = \int_{\mathcal{D}} u\,\mathrm{d}\boldsymbol{x}^2, \tag{5.4}$$

we recover the x-momentum of the fluid.

This is a good place to consider the boundary condition we've neglected so far. Taking $F = M$ in (4.6) and keeping the boundary term in the integral, we find

$$\begin{aligned}\frac{\mathrm{d}M}{\mathrm{d}t} = \{M, H\} &= \int_{\mathcal{D}} y\partial(\omega,\psi)\,\mathrm{d}\boldsymbol{x}^2 \\ &= -\int_{\partial\mathcal{D}} y\Delta\psi\boldsymbol{\nabla}\psi\cdot\mathrm{d}\boldsymbol{l} + \int_{\mathcal{D}} \partial_x\psi\Delta\psi\,\mathrm{d}\boldsymbol{x}^2,\end{aligned} \tag{5.5}$$

where $\mathrm{d}\boldsymbol{l}$ is the length element along $\partial\mathcal{D}$. The first term vanishes thanks to the boundary condition (4.12), so we have

$$\{M, H\} = \int_{\partial\mathcal{D}} \partial_x\psi\,\boldsymbol{\nabla}^\perp\psi\cdot\mathrm{d}\boldsymbol{l} - \int_{\mathcal{D}} \tfrac{1}{2}\partial_x|\boldsymbol{\nabla}\psi|^2\,\mathrm{d}\boldsymbol{x}^2. \tag{5.6}$$

This will vanish only if our domain $\mathcal{D}$ has a special property, namely symmetry in the x-direction. As an example, one could take a channel $y \in [0, 1]$ with either periodic boundary conditions in x or suitable vanishing of the velocity as $|x| \to \infty$; the details are left to the reader.

If we now take

$$M(\omega) = \int_{\mathcal{D}} -\tfrac{1}{2}|\boldsymbol{x}|^2\omega\,\mathrm{d}\boldsymbol{x}^2, \tag{5.7}$$

we find

$$\begin{aligned}-\tfrac{1}{2}\int_{\mathcal{D}} |\boldsymbol{x}|^2\partial(\omega,\phi)\,\mathrm{d}\boldsymbol{x}^2 &= \tfrac{1}{2}\int_{\mathcal{D}} \omega\,\partial(x^2 + y^2, \phi)\,\mathrm{d}\boldsymbol{x}^2 \\ &= \int_{\mathcal{D}} \omega\,(x\,\partial_y\phi - y\,\partial_x\phi)\,\mathrm{d}\boldsymbol{x}^2.\end{aligned} \tag{5.8}$$

The change in $\phi = \delta F/\delta\omega$ is thus

$$\frac{\mathrm{d}\phi}{\mathrm{d}s} = x\,\partial_y\phi - y\,\partial_x\phi, \tag{5.9}$$

which is the result of a rotation about $\boldsymbol{x} = \boldsymbol{0}$. An alternative expression for M can be computed thus,

$$\begin{aligned} M(\omega) &= -\tfrac{1}{2}\int_{\mathcal{D}} (x^2+y^2)\,(v_x - u_y)\,\mathrm{d}\boldsymbol{x}^2 \\ &= \int_{\mathcal{D}} \{xv - yu\}\,\mathrm{d}\boldsymbol{x}^2 = \int_{\mathcal{D}} \boldsymbol{x}\cdot\boldsymbol{v}^{\perp}\,\mathrm{d}\boldsymbol{x}^2, \end{aligned} \tag{5.10}$$

which is the angular momentum of the fluid, as expected. As with the x-momentum above, for the integration by parts to work, $\mathcal{D}$ must have rotational symmetry.

Finally, let us take

$$M(\omega) = \int_{\mathcal{D}} f(\omega)\,\mathrm{d}\boldsymbol{x}^2 \tag{5.11}$$

for some function f. We find

$$\int_{\mathcal{D}} f'(\omega)\,\partial(\omega,\phi)\,\mathrm{d}\boldsymbol{x}^2 = \int_{\mathcal{D}} \phi\,\partial(f'(\omega),\omega)\,\mathrm{d}\boldsymbol{x}^2 = 0. \tag{5.12}$$

One would think that a non-trivial functional such as M would generate a non-trivial symmetry. It does, but the symmetry is "hidden", as we will see below.

5.2. *Lagrangian Description of Fluid Dynamics*

In this section we present the Lagrangian picture of fluid dynamics, which can be thought of as the limit of finite-dimensional particle dynamics of mechanics as the particle label (or index) assumes continuous values.

Instead of our usual model, the 2d Euler equations, here we will consider a model governing an inviscid compressible isentropic fluid in two space dimensions (often called compressible Euler equations),

$$\begin{aligned} &\partial_t\boldsymbol{v} + \boldsymbol{v}\cdot\boldsymbol{\nabla}\boldsymbol{v} = -\frac{1}{\rho}\boldsymbol{\nabla}p \\ &\partial_t\rho + \boldsymbol{\nabla}\cdot(\boldsymbol{v}\rho) = 0, \end{aligned} \tag{5.13}$$

where the pressure $p(\rho)$ is a given function of the density ρ. This will serve to illustrate the central place of the density and the *potential vorticity*

$$q = \frac{\boldsymbol{\nabla}^{\perp}\cdot\boldsymbol{v}}{\rho}, \tag{5.14}$$

which is a material invariant of the dynamics, $\partial_t q + \boldsymbol{v} \cdot \boldsymbol{\nabla} q = 0$. Readers interested in other models are referred to [31] for more details.

Let us start by introducing the *particle label* $\boldsymbol{a} = (a, b)$, which is a continuous variable stuck to fluid particles and chosen in such a way that

$$\mathrm{d(mass)} = \mathrm{d}\boldsymbol{a}^2. \tag{5.15}$$

We denote by $\boldsymbol{x}(\boldsymbol{a}, \tau)$ the physical position of the particle whose label is $\boldsymbol{a}$ at time τ. Similarly, by $\boldsymbol{a}(\boldsymbol{x}, t)$ we mean the label of the particle whose physical position is $\boldsymbol{x}$ at time t. We take the times τ and t to be identical, but by ∂_τ we mean derivative with respect to time with $\boldsymbol{a}$ held fixed while by ∂_t we mean derivative with respect to time with $\boldsymbol{x}$ held fixed. Let $\boldsymbol{v} := \boldsymbol{x}_\tau$ be the velocity of a fluid particle, then these derivatives are related by

$$\partial_\tau = \partial_t + \boldsymbol{v} \cdot \partial_{\boldsymbol{x}}\,, \tag{5.16}$$

which you may recognise as the definition of material derivative following a fluid parcel.

Since we will need to change from $(\boldsymbol{a}, \tau)$ to $(\boldsymbol{x}, t)$, the matrices

$$\Bigl(\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{a}}\Bigr) = \begin{pmatrix} x_a & x_b \\ y_a & y_b \end{pmatrix} \qquad \text{and} \qquad \Bigl(\frac{\partial \boldsymbol{a}}{\partial \boldsymbol{x}}\Bigr) = \begin{pmatrix} a_x & a_y \\ b_x & b_y \end{pmatrix}, \tag{5.17}$$

where $a_x := \partial a/\partial x$, etc., will be very useful. These are evidently related by

$$\Bigl(\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{a}}\Bigr)\Bigl(\frac{\partial \boldsymbol{a}}{\partial \boldsymbol{x}}\Bigr) = \mathbf{1}. \tag{5.18}$$

The density ρ of the fluid is then given by

$$\rho = \Bigl|\frac{\partial \boldsymbol{a}}{\partial \boldsymbol{x}}\Bigr| := a_x b_y - a_y b_x = \Bigl|\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{a}}\Bigr|^{-1} := \frac{1}{x_a y_b - x_b y_a}. \tag{5.19}$$

Using these, we can derive the transformation rules for partial derivatives,

$$\begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix} = \rho \begin{pmatrix} y_b \partial_a - y_a \partial_b \\ x_a \partial_b - x_b \partial_a \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \partial_a \\ \partial_b \end{pmatrix} = \frac{1}{\rho} \begin{pmatrix} b_y \partial_x - b_x \partial_y \\ a_x \partial_y - a_y \partial_x \end{pmatrix}. \tag{5.20}$$

While not central to our discussion, it is instructive to consider the following. Taking $\partial_\tau \rho$, we have using (5.19),

$$\partial_\tau \rho = \partial_\tau \Bigl(\Bigl|\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{a}}\Bigr|^{-1}\Bigr) = -\rho^2 \partial_\tau \Bigl(\Bigl|\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{a}}\Bigr|\Bigr). \tag{5.21}$$

Now

$$\partial_\tau (x_a y_b - x_b y_a) = \frac{1}{\rho}(\partial_x x_\tau + \partial_y y_\tau), \tag{5.22}$$

where (5.20a) has been used, so we have

$$\partial_\tau \rho = -\rho\,(\partial_x x_\tau + \partial_y y_\tau). \tag{5.23}$$

Using the relation (5.16) to write this in terms of $(\boldsymbol{x}, t)$, we find

$$\partial_t \rho + \partial_{\boldsymbol{x}} \cdot (\rho \boldsymbol{v}) = 0, \tag{5.24}$$

which is just the continuity equation. From this derivation (as well as the more "usual" one in fluid dynamics), it is clear that the origin of this equation is purely kinematical.

Moving on to the dynamics, let us take $(\boldsymbol{a}, \tau)$ as our independent variables and $\boldsymbol{x}(\boldsymbol{a}, \tau)$ as our dependent (i.e. dynamical) variables. The Lagrangian is given by

$$L(\boldsymbol{x}, \boldsymbol{x}_\tau) = \int \left\{ \tfrac{1}{2}|\boldsymbol{x}_\tau|^2 - U\big(|\partial \boldsymbol{x}/\partial \boldsymbol{a}|^{-1}\big) \right\} \mathrm{d}\boldsymbol{a}^2. \tag{5.25}$$

The significance of the first term is clear: it is the kinetic energy of the fluid particle. In the second term, U is the internal energy (density), which we take for simplicity to be a function of the (mass) density $\rho = |\partial \boldsymbol{x}/\partial \boldsymbol{a}|^{-1}$ only. The situation is therefore just as in mechanics of particles.

As before, we compute the variation of the action

$$\begin{aligned} \delta \int L \,\mathrm{d}\tau &= \int\!\!\int \Big\{ \boldsymbol{x}_\tau \cdot \delta \boldsymbol{x}_\tau + U'(\rho)\rho^2 \delta \Big|\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{a}}\Big| \Big\} \,\mathrm{d}\boldsymbol{a}^2 \,\mathrm{d}\tau \\ &= \int\!\!\int \Big\{ -\boldsymbol{x}_{\tau\tau} \cdot \delta \boldsymbol{x} + p(\rho)\,\delta \Big|\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{a}}\Big| \Big\} \,\mathrm{d}\boldsymbol{a}^2 \,\mathrm{d}\tau, \end{aligned} \tag{5.26}$$

where the first term has been obtained by integration by parts as usual and where we have defined the pressure[b] $p(\rho) := \rho^2 U'(\rho)$. The variation of the Jacobian is

$$\delta \Big|\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{a}}\Big| = y_b \delta x_a + x_a \delta y_b - y_a \delta x_b - x_b \delta y_a\,, \tag{5.27}$$

[b] This is evidently the same pressure we've seen in (5.13a).

so computing the second term in (5.26), we find

$$\begin{aligned}\int p(\rho)\,\delta\Big|\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{a}}\Big|\;\mathrm{d}\boldsymbol{a}^2 &= \int p(\rho)\,\big\{y_b\delta x_a + x_a\delta y_b - y_a\delta x_b - x_b\delta y_a\big\}\;\mathrm{d}\boldsymbol{a}^2\\ &= -\int\big\{\big[\partial_a(py_b) - \partial_b(py_a)\big]\delta x\\ &\qquad + \big[\partial_b(px_a) - \partial_a(px_b)\big]\delta y\big\}\;\mathrm{d}\boldsymbol{a}^2 \qquad (5.28)\\ &= -\int\big\{(y_bp_a - y_ap_b)\delta x + (x_ap_b - x_bp_a)\delta y\big\}\;\mathrm{d}\boldsymbol{a}^2\\ &= -\int\Big\{\frac{\partial_x p}{\rho}\,\delta x + \frac{\partial_y p}{\rho}\,\delta y\Big\}\;\mathrm{d}\boldsymbol{a}^2,\end{aligned}$$

where we have integrated by parts and used (5.20). Thus,

$$\delta\int L\;\mathrm{d}\tau = -\iint\Big\{\boldsymbol{x}_{\tau\tau} + \frac{1}{\rho}\partial_{\boldsymbol{x}}p(\rho)\Big\}\;\mathrm{d}\boldsymbol{a}^2\;\mathrm{d}\tau. \qquad (5.29)$$

Setting this to zero, we find the equations of motion in the Lagrangian picture,

$$\boldsymbol{x}_{\tau\tau} = -\Big|\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{a}}\Big|\,\partial_{\boldsymbol{x}}p\Big(\Big|\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{a}}\Big|^{-1}\Big), \qquad (5.30)$$

where the $\partial_{\boldsymbol{x}}$ is to be expressed in terms of $\partial_{\boldsymbol{a}}$ using (5.20a). Recalling the definition $\boldsymbol{v} = \boldsymbol{x}_\tau$ and (5.16), this is equivalent to

$$\partial_t\boldsymbol{v} + \boldsymbol{v}\cdot\partial_{\boldsymbol{x}}\boldsymbol{v} = \partial_\tau\boldsymbol{v} = -\frac{1}{\rho}\partial_{\boldsymbol{x}}p(\rho), \qquad (5.31)$$

which is just the momentum equation in the Eulerian picture.

In this Lagrangian picture, the Hamiltonian formulation is canonical. Noting that the canonical momentum is just

$$\frac{\partial L}{\partial \boldsymbol{x}_\tau} = \boldsymbol{v}, \qquad (5.32)$$

the Hamiltonian is given by

$$\begin{aligned}H(\boldsymbol{x},\boldsymbol{v}) &= \int\big\{\boldsymbol{v}\cdot\boldsymbol{x}_\tau - L\big\}\;\mathrm{d}\boldsymbol{a}^2\\ &= \int\big\{\tfrac{1}{2}|\boldsymbol{v}|^2 + U\big(|\partial\boldsymbol{x}/\partial\boldsymbol{a}|^{-1}\big)\big\}\;\mathrm{d}\boldsymbol{a}^2.\end{aligned} \qquad (5.33)$$

The Poisson bracket can be thought of as the sum of the canonical bracket (2.8) over particle labels (i.e. as an integral over $\boldsymbol{a}$),

$$\{F,G\} = \int\Big\{\frac{\delta F}{\delta \boldsymbol{x}}\cdot\frac{\delta G}{\delta \boldsymbol{v}} - \frac{\delta F}{\delta \boldsymbol{v}}\cdot\frac{\delta G}{\delta \boldsymbol{x}}\Big\}\;\mathrm{d}\boldsymbol{a}^2. \qquad (5.34)$$

Verification of the equations of motion is left as an exercise.

5.3. *Particle-Relabelling Symmetry and Reduction*

Let us keep our attention on the 2d compressible Euler equations (5.13). The variation of the density is

$$\delta\Big|\frac{\partial \boldsymbol{a}}{\partial \boldsymbol{x}}\Big| = \delta\,(a_x b_y - a_y b_x) = a_x \delta b_y + b_y \delta a_x - a_y \delta b_x - b_x \delta a_y. \tag{5.35}$$

Using (5.20b), this can be written as

$$\delta\rho = \rho\,(\partial_a \delta a + \partial_b \delta b). \tag{5.36}$$

So variations which preserve density, $\partial_a \delta a + \partial_b \delta b = 0$, must be of the form

$$(\delta a, \delta b) = (-\partial_b \phi, \partial_a \phi) \qquad \text{or} \qquad \delta \boldsymbol{a} = \boldsymbol{\nabla}_{\boldsymbol{a}}^{\perp} \phi \tag{5.37}$$

for some $\phi(\boldsymbol{a})$. But what does this variation do? It changes the labels on fluid particles, in a continuous manner, of course, without changing the density.

Now let us consider a Lagrangian of the form

$$L(\boldsymbol{x}, \boldsymbol{x}_\tau) = \int \big\{ \tfrac{1}{2}|\boldsymbol{x}_\tau|^2 - U\big(|\partial \boldsymbol{x}/\partial \boldsymbol{a}|^{-1}\big) \big\}\, \mathrm{d}\boldsymbol{a}^2, \tag{5.38}$$

where the particle label $\boldsymbol{a}$ enters only through the density $\rho = |\partial \boldsymbol{x}/\partial \boldsymbol{a}|^{-1}$. This Lagrangian is therefore invariant under variations of the form (5.37), or it has a *particle-relabelling symmetry*. Under variations of the form (5.37), we find

$$\begin{aligned}
\delta \int L\, \mathrm{d}\tau &= \iint \boldsymbol{x}_\tau \cdot \delta \boldsymbol{x}_\tau\, \mathrm{d}\boldsymbol{a}^2\, \mathrm{d}\tau = \iint \boldsymbol{x}_\tau \cdot \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{a}} \delta \boldsymbol{a}_\tau\, \mathrm{d}\boldsymbol{a}^2\, \mathrm{d}\tau \\
&= -\iint \boldsymbol{x}_\tau \cdot \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{a}} \boldsymbol{\nabla}_{\boldsymbol{a}}^{\perp} \delta\phi_\tau\, \mathrm{d}\boldsymbol{a}^2\, \mathrm{d}\tau \\
&= \iint \boldsymbol{\nabla}_{\boldsymbol{a}}^{\perp} \cdot \Big(\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{a}} \boldsymbol{x}_\tau\Big) \delta\phi_\tau\, \mathrm{d}\boldsymbol{a}^2\, \mathrm{d}\tau \\
&= -\iint \partial_\tau \boldsymbol{\nabla}_{\boldsymbol{a}}^{\perp} \cdot \Big(\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{a}} \boldsymbol{x}_\tau\Big) \delta\phi\, \mathrm{d}\boldsymbol{a}^2\, \mathrm{d}\tau \\
&= \iint \partial_\tau \Big(\frac{\partial_y x_\tau - \partial_x y_\tau}{\rho}\Big) \delta\phi\, \mathrm{d}\boldsymbol{a}^2\, \mathrm{d}\tau,
\end{aligned} \tag{5.39}$$

where we have used (5.20) to arrive at the last line. Since $\delta\phi$ is arbitrary, we must have

$$\partial_\tau \Big(\frac{\partial_y x_\tau - \partial_x y_\tau}{\rho}\Big) = 0, \tag{5.40}$$

which states that the potential vorticity $q = (\boldsymbol{\nabla}^{\perp} \cdot \boldsymbol{v})/\rho$ is conserved for $\boldsymbol{a}$ fixed (i.e. following fluid particles). Here the potential vorticity q can be regarded as the generator of the particle-relabelling symmetry.

We now show how the "full" Lagrangian description of fluid dynamics can be expressed in a "reduced" Eulerian one, in the context of the compressible 2d Euler equations; for a more general treatment, see [25]. Returning to the Hamiltonian formulation in Lagrangian coordinates (5.33)–(5.34), we restrict our attention to functionals of the form

$$F = \int \frac{1}{\rho}\,\varphi(\boldsymbol{v},\rho)\,\mathrm{d}\boldsymbol{a}^2 \qquad \text{and} \qquad G = \int \frac{1}{\rho}\,\gamma(\boldsymbol{v},\rho)\,\mathrm{d}\boldsymbol{a}^2 \tag{5.41}$$

for any functions φ and γ, the motivation being that any functional in the Eulerian picture must be expressible in terms of the velocity and the density only[c]. The functional derivatives can be computed as usual,

$$\frac{\delta F}{\delta \boldsymbol{x}} = \frac{1}{\rho}\boldsymbol{\nabla}\Big(\rho\frac{\partial\varphi}{\partial\rho} - \varphi\Big) \qquad \text{and} \qquad \frac{\delta F}{\delta \boldsymbol{v}} = \frac{1}{\rho}\frac{\partial\varphi}{\partial\boldsymbol{v}}, \tag{5.42}$$

and similarly for those of G. Let us now consider

$$\{F,G\} = \int \Big\{\frac{1}{\rho^2}\frac{\partial\gamma}{\partial\boldsymbol{v}}\cdot\boldsymbol{\nabla}\Big(\rho\frac{\partial\varphi}{\partial\rho} - \varphi\Big) - \frac{1}{\rho^2}\frac{\partial\varphi}{\partial\boldsymbol{v}}\cdot\boldsymbol{\nabla}\Big(\rho\frac{\partial\gamma}{\partial\rho} - \gamma\Big)\Big\}\,\mathrm{d}\boldsymbol{a}^2. \tag{5.43}$$

In physical space (i.e. using $(\boldsymbol{x},t)$ instead of $(\boldsymbol{a},\tau)$ as independent coordinates), (5.41) can be written as

$$F = \int_{\mathcal{D}} \varphi(\boldsymbol{v},\rho)\,\mathrm{d}\boldsymbol{x}^2 \qquad \text{and} \qquad G = \int_{\mathcal{D}} \gamma(\boldsymbol{v},\rho)\,\mathrm{d}\boldsymbol{x}^2, \tag{5.44}$$

where the factor of ρ is the Jacobian of the coordinate change, $\mathrm{d}\boldsymbol{a}^2 = \rho\,\mathrm{d}\boldsymbol{x}^2$. If we now consider F as a functional of $(\boldsymbol{v},\rho)$, we have

$$\frac{\delta F}{\delta \boldsymbol{v}} = \frac{\partial\varphi}{\partial\boldsymbol{v}} \qquad \text{and} \qquad \frac{\delta F}{\delta\rho} = \frac{\partial\varphi}{\partial\rho}; \tag{5.45}$$

the case for G is analogous. We can now rewrite (5.43) as

$$\{F,G\} = \int_{\mathcal{D}} \frac{1}{\rho}\Big\{\frac{\delta G}{\delta\boldsymbol{v}}\cdot\boldsymbol{\nabla}\Big(\rho\frac{\delta F}{\delta\rho} - \varphi\Big) - \frac{\delta F}{\delta\boldsymbol{v}}\cdot\boldsymbol{\nabla}\Big(\rho\frac{\delta G}{\delta\rho} - \gamma\Big)\Big\}\,\mathrm{d}\boldsymbol{x}^2, \tag{5.46}$$

where the factor $1/\rho$ on $\delta G/\delta\boldsymbol{v}$ has cancelled the Jacobian ρ. After some more work, we recover the Poisson bracket

$$\{F,G\} = \int_{\mathcal{D}} \Big\{q\frac{\delta F}{\delta\boldsymbol{v}^\perp}\cdot\frac{\delta G}{\delta\boldsymbol{v}} + \frac{\delta G}{\delta\boldsymbol{v}}\cdot\boldsymbol{\nabla}\frac{\delta F}{\delta\rho} - \frac{\delta F}{\delta\boldsymbol{v}}\cdot\boldsymbol{\nabla}\frac{\delta G}{\delta\rho}\Big\}\,\mathrm{d}\boldsymbol{x}^2, \tag{5.47}$$

where $q = (\boldsymbol{\nabla}^\perp\cdot\boldsymbol{v})/\rho$ is the usual potential vorticity. We note that this is just the Poisson bracket (4.42) for the shallow-water equations (which can be regarded as a special case of the 2d compressible Euler equations).

[c] As well as their physical space derivatives $\partial_{\boldsymbol{x}}\rho$, $\partial_{\boldsymbol{x}}\boldsymbol{v}$, etc.; this is left for exercise.

The fact that we could derive this bracket from general (mainly kinematic) considerations without assuming any equations of motion strongly points to the underlying *geometric* aspects of fluid dynamics; roughly speaking, one could think of the Poisson bracket $\{\cdot,\cdot\}$ as determined by the geometry and the Hamiltonian H as determined by the dynamics, but the actual picture is more complicated than that.[d]

It can be verified directly that the Casimirs of the bracket (5.47) are functionals of the form

$$C_f(\boldsymbol{v},\rho) = \int_{\mathcal{D}} \rho f(q)\;\mathrm{d}\boldsymbol{x}^2 \tag{5.48}$$

for some arbitrary function f. Alternately, using the expression $q = x_a u_b - x_b u_a - y_b v_a + y_a v_b$, we can check that any functional of the form

$$C_f(\boldsymbol{x},\boldsymbol{v}) = \int f(q)\;\mathrm{d}\boldsymbol{a}^2 \tag{5.49}$$

commutes with any functional of the form (5.41),

$$\{C_f, F\} = 0. \tag{5.50}$$

6. Nonlinear Hydrodynamic Stability

In this section we discuss a method to prove nonlinear stability of fluid flows following a method first discovered by Arnold [2] for the 2d Euler equations and developed further for various other models. In the interest of clarity, our discussion will be based on the rigid body and the 2d Euler equations as much as possible. For extensions to other fluid (and plasma) models, see, e.g., [12, 13, 14, 20, 23, 32, 36] and the references therein.

Besides confirming results obtained by linearised analysis, it turns out that this method, which is deeply connected with the Hamiltonian structure of the equations, also give us results that cannot be obtained using the more traditional linear analysis, such as the generalised Rayleigh–Fjørtoft theorem, saturation bounds, etc. We conclude the section by discussing several stability-related issues.

[d]The Korteweg–de Vries equation has a *bihamiltonian* structure, meaning it can be described by two different brackets (with different Hamiltonians); this leads to very interesting consequences.

6.1. *Review of Concepts on Stability*

Let us review a few basic concepts for a dynamical system

$$\frac{\mathrm{d}\boldsymbol{z}}{\mathrm{d}t} = \boldsymbol{F}(\boldsymbol{z}). \tag{6.1}$$

A *fixed point* $\boldsymbol{z}_0$ of this sytem is defined by the condition $\boldsymbol{F}(\boldsymbol{z}_0) = 0$. To study the *stability* of a fixed point $\boldsymbol{z}_0$, we need to consider $\boldsymbol{\nabla F}$; indeed, for $\boldsymbol{z}$ sufficiently close to $\boldsymbol{z}_0$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}|\boldsymbol{z} - \boldsymbol{z}_0|^2 \simeq \langle(\boldsymbol{z} - \boldsymbol{z}_0), \boldsymbol{\nabla F}(\boldsymbol{z}_0).(\boldsymbol{z} - \boldsymbol{z}_0)\rangle. \tag{6.2}$$

This means that the behaviour of the solution near $\boldsymbol{z}_0$ is determined by the eigenvalues of $\boldsymbol{\nabla F}(\boldsymbol{z}_0)$. If the real part of the eigenvalues are all negative, the fixed point $\boldsymbol{z}_0$ is said to be *asymptotically stable* since all solutions near $\boldsymbol{z}_0$ approach it as $t \to \infty$.

We note that if $\boldsymbol{z}$ is finite dimensional, consideration of the *linearised* system (6.2) is often (e.g., when the real parts of the eigenvalues are non-zero) sufficient to know what happens when $\boldsymbol{z}$ is at a *finite* distance from $\boldsymbol{z}_0$: If $\boldsymbol{F}$ is sufficiently regular (e.g., twice differentiable), one can find a $B > 0$ such that, for any solution $\boldsymbol{z}(t)$ with $|\boldsymbol{z}(0)-\boldsymbol{z}_0| \le B$, one has $|\boldsymbol{z}(t)-\boldsymbol{z}_0| \to 0$ as $t \to \infty$. As we will see below, this is not true for infinite-dimensional systems such as most fluid models.

Now let $\boldsymbol{z} = (\boldsymbol{p}, \boldsymbol{q})$ and consider the canonical Hamiltonian system (2.7). The condition for $\boldsymbol{z}_0 = (\boldsymbol{p}_0, \boldsymbol{q}_0)$ to be a fixed point is that $\partial H/\partial \boldsymbol{p} = 0$ and $\partial H/\partial \boldsymbol{q} = 0$ there. Another way to obtain this (which will be useful below) is to consider the *variation*

$$\delta H = \frac{\partial H}{\partial \boldsymbol{p}}\delta \boldsymbol{p} + \frac{\partial H}{\partial \boldsymbol{q}}\delta \boldsymbol{q} \tag{6.3}$$

and set it to zero. At a fixed point $\boldsymbol{z}_0$, we have

$$\boldsymbol{\nabla F} = \begin{pmatrix} 0 & \partial^2 H/\partial \boldsymbol{p}^2 \\ -\partial^2 H/\partial \boldsymbol{q}^2 & 0 \end{pmatrix}. \tag{6.4}$$

By diagonalising the symmetric matrices $\partial^2 H/\partial \boldsymbol{p}^2$ and $\partial^2 H/\partial \boldsymbol{q}^2$, we find that the eigenvalues λ_i of $\boldsymbol{\nabla F}$ satisfy

$$\lambda_i^2 = \mu_i \nu_i \tag{6.5}$$

where μ_i and ν_i are the (real) eigenvalues of $\partial^2 H/\partial \boldsymbol{p}^2$ and $\partial^2 H/\partial \boldsymbol{q}^2$, respectively. It follows that the eigenvalues of a Hamiltonian system occur in $\pm$ pairs, and thus a Hamiltonian system can never be asymptotically stable.

It is still possible for a fixed point of a Hamiltonian system to be stable, but for this we need another definition: A fixed point $\boldsymbol{z}_0$ is stable in the sense of Lyapunov if for any $\varepsilon \in (0, \varepsilon_0)$ there exists a $\delta > 0$ such that

$$\|\boldsymbol{z}(0) - \boldsymbol{z}_0\| \leq \delta \quad \Rightarrow \quad \|\boldsymbol{z}(t) - \boldsymbol{z}_0\| \leq \varepsilon \ \forall t \geq 0. \tag{6.6}$$

Now suppose that the Hamiltonian H is positive definite in a neighbourhood of $\boldsymbol{z}_0$, then the level sets of H form codimension-one surfaces enclosing $\boldsymbol{z}_0$. Since any solution stays on a level set of H, it stays near $\boldsymbol{z}_0$ according the above definition if the level sets are sufficiently smooth, and is therefore stable in the sense of Lyapunov.

Note that we have not specified the norm used in (6.6), although one usually assumes the usual Euclidean (i.e. l_2) norm. This is because in finite dimensions, all norms[e] are equivalent, in the sense that, for any two norms $\|\cdot\|_1$ and $\|\cdot\|_2$, there exists constants c_1, $c_2 \in (0, \infty)$ such that

$$c_1 \, \|\boldsymbol{u}\|_1 \leq \|\boldsymbol{u}\|_2 \leq c_2 \, \|\boldsymbol{u}\|_1 \tag{6.7}$$

for any $\boldsymbol{u} \in \mathbb{R}^n$. This is however not true when our "vector" is infinite dimensional, such as the case when studying fluid dynamics. In fact, this is a large part of the difficulty—one could say without too much exaggeration that the study of partial differential equations can be boiled down the the search of suitable norms (or suitable "function spaces").

6.2. *The Rigid Body Revisited*

Let us return to the reduced formulation of the rigid body (§3.3) for the moment. We would like to find the fixed points of this system and to study their stability. Following the standard way to find fixed points of a canonical Hamiltonian system, we consider the variation of the Hamiltonian (3.18)

$$\delta H = \sum_i \frac{m_i}{I_i} \delta m_i \,. \tag{6.8}$$

But setting this to zero gives us $\boldsymbol{m} = 0$. So what went wrong?

The problem came from the singular nature of the Poisson bracket (3.26). To obtain all the fixed points of the dynamics, we need to consider the *constrained* variations

$$\delta A := \delta H - \alpha \, \delta C \tag{6.9}$$

[e]We recall the properties that must be satisfied by any norm $\|\cdot\|$. For any vectors $\boldsymbol{u}$, $\boldsymbol{v} \in \mathbb{R}^n$ (or $\mathbb{C}^n$): (i) $\|\boldsymbol{u}\| \geq 0$ with $\|\boldsymbol{u}\| = 0 \Rightarrow \boldsymbol{u} = 0$, (ii) $\|\alpha \boldsymbol{u}\| = |\alpha| \, \|\boldsymbol{u}\|$ for any $\alpha \in \mathbb{R}$ (or $\mathbb{C}$), (iii) $\|\boldsymbol{u} + \boldsymbol{v}\| \leq \|\boldsymbol{u}\| + \|\boldsymbol{v}\|$.

where α is a constant and C is the Casimir in (3.31). Setting

$$\delta A = \sum_i \Big(\frac{m_i}{I_i} - 2\alpha m_i\Big)\delta m_i = 0, \tag{6.10}$$

we find that $m_i = 0$ unless $\alpha = 1/(2I_i)$. Assuming that $I_1 \neq I_2 \neq I_3 \neq I_1$, which is the generic situation, the only fixed points are $(m_1, 0, 0)$, $(0, m_2, 0)$ and $(0, 0, m_3)$. These correspond to rotations around the three principal axes, just as we expected on physical grounds, or directly from the equations of motion (3.35).

We note that different choices of the constant α, which can be regarded as different choices for the Casimir αC, give us different fixed points. We will encounter an analogous situation for fluids below. We also note that these fixed points are *not* states of no motion; rather, they correspond to *steady motions*. This is because we have removed the coordinate variables (i.e. the Euler angles) in the reduction process.

Now let us look at a fixed point, say, $(m_1, 0, 0)$, which implies that we've fixed $\alpha = 1/(2I_1)$. The stability of this fixed point is determined by the second variation [this is the analogue of $\boldsymbol{\nabla F}$ in (6.2)]

$$\frac{\delta^2 A}{\delta \boldsymbol{m}^2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1/I_2 - 1/I_1 & 0 \\ 0 & 0 & 1/I_3 - 1/I_1 \end{pmatrix}. \tag{6.11}$$

This matrix is semi-definite (which implies stability of the system) when I_1 is either larger or smaller than both I_2 and I_3. This is just what is expected on physical grounds: free rotations of a rigid body about the axes of largest and smallest moments of inertia are stable, but free rotation about the axis with the middle moment of inertia is unstable.

6.3. *Stability of the 2d Euler Equation*

Now we turn to the 2d Euler equation of §4.2.1 and study its fixed points. As with the rigid body example, if we naïvely set $\delta H/\delta\omega = -\psi = 0$, we only find the zero flow $\psi = 0$ as a fixed point. To obtain all possible fixed points, we need to consider the *augmented functional* $A = H + C_f$ and consider its variation [using (4.19) and (4.16)],

$$\delta A(\omega) = \int \big\{-\psi + f'(\omega)\big\}\delta\omega \,\mathrm{d}\boldsymbol{x}^2. \tag{6.12}$$

If this is to vanish for all possible $\delta\omega$, we find

$$\psi = f'(\omega), \tag{6.13}$$

which is the (familiar?) condition for the solution of the 2d Euler equation to be steady: the streamfunction ψ and vorticity ω are functionally related. This can be seen directly from the equation of motion (4.11): setting $\partial_t\omega = 0$ implies $\boldsymbol{\nabla}^{\perp}\psi.\boldsymbol{\nabla}\omega = 0$, which in turn implies that ψ and ω are collinear, $\boldsymbol{\nabla}\omega = \alpha\boldsymbol{\nabla}\psi$ for a scalar $\alpha(\boldsymbol{x})$, and thus (locally) functionally related.

In analogy to the situation with the rigid body, the fixed points of the 2d Euler equation are not the zero flow, but steady flows. We note that while solving for the fixed points of a finite-dimensional system is usually quite straightforward, to find the fixed points of infinite-dimensional systems such as the 2d Euler equation we need to solve PDEs like (6.13), which we can rewrite as

$$\Delta\psi = g(\psi). \tag{6.14}$$

This is a semi-linear elliptic equation. Solving this type of equation directly is a difficult (and largely open) problem [17]; methods to obtain its solutions in the context of the 2d Euler equation have been proposed in [33][42].

Now suppose that $\psi_0 = f'(\omega_0)$ is a steady flow. To study its stability, we consider the second variation

$$\begin{aligned}\delta^2 A &= \int \left\{-\delta\psi\,\delta\omega + f''(\omega_0)\,\delta\omega^2\right\}\,\mathrm{d}\boldsymbol{x}^2\\ &= \int \left\{|\boldsymbol{\nabla}\delta\psi|^2 + f''(\omega_0)\,(\delta\Delta\psi)^2\right\}\,\mathrm{d}\boldsymbol{x}^2,\end{aligned} \tag{6.15}$$

where we have integrated by parts to arrive at the second line. There are two cases to consider. First, suppose that $f''(\omega) > 0$. Then $\delta^2 A$ is positive definite, which, as we shall see shortly, implies the stability of the basic flow ψ_0. The second case obtains when the domain is bounded (actually, all we need is that the domain be bounded between two parallel lines). Recall that in a bounded domain $\mathcal{D}$, we have *Poincaré's inequality*: for any smooth function u which vanishes on the boundary, we have

$$\int_{\mathcal{D}} |\boldsymbol{\nabla}u|^2\,\mathrm{d}\boldsymbol{x}^2 \le c_0(\mathcal{D})\int_{\mathcal{D}} |\Delta u|^2\,\mathrm{d}\boldsymbol{x}^2. \tag{6.16}$$

(This inequality can be proved, e.g., by expanding u is the eigenfunctions of the Laplacian Δ in $\mathcal{D}$.) Now if $f''(\omega_0) \le c < -c_0(\mathcal{D}) < 0$, the "quadratic form" (6.15) is negative definite, which again will allow us to prove the stability[f] of the basic flow ψ_0.

[f]When the boundary $\partial\mathcal{D}$ is multiply-connected, one has to take into account the circulation on each connected piece of $\partial\mathcal{D}$. The development is similar and is left as an exercise.

Unlike with finite-dimensional systems, analysis of the spectrum of the variation $\delta^2 A$ is not sufficient to determine the true (i.e. nonlinear) stability of the system. This is because we need to fix a norm to measure the perturbation quantities and, as mentioned earlier, in infinite dimensions norms are not equivalent, so the definition of stability depends very much on the norm used.[g] Following the method of Arnold [2], we construct such a norm using the conserved functionals H and C.

Given a steady flow ψ_0 with $\psi_0 = f'(\omega_0)$ as before, let

$$A(\psi;\psi_0) := \int_{\mathcal{D}} \big\{ \tfrac{1}{2}|\boldsymbol{\nabla}\psi|^2 - \tfrac{1}{2}|\boldsymbol{\nabla}\psi_0|^2 + f(\omega) - f(\omega_0) \big\} \,\mathrm{d}\boldsymbol{x}^2. \tag{6.17}$$

Since $A(\psi;\psi_0)$ is a sum of conserved quantities (and a constant), we have $\mathrm{d}A/\mathrm{d}t = 0$. Now let us rewrite A in terms of $\hat\psi := \psi - \psi_0$ and ψ_0,

$$\begin{aligned} A &= \int_{\mathcal{D}} \big\{ \tfrac{1}{2}|\boldsymbol{\nabla}(\psi_0 + \hat\psi)|^2 - \tfrac{1}{2}|\boldsymbol{\nabla}\psi_0|^2 + f(\omega_0 + \hat\omega) - f(\omega_0) \big\} \,\mathrm{d}\boldsymbol{x}^2 \\ &= \int_{\mathcal{D}} \big\{ \boldsymbol{\nabla}\psi_0 . \boldsymbol{\nabla}\hat\psi + \tfrac{1}{2}|\boldsymbol{\nabla}\hat\psi|^2 + f(\omega_0 + \hat\omega) - f(\omega_0) \big\} \,\mathrm{d}\boldsymbol{x}^2. \end{aligned} \tag{6.18}$$

Next, we integrate the first term by parts and use Taylor's theorem to write $f(\omega_0 + \hat\omega) = f(\omega_0) + \hat\omega f'(\omega_0) + \hat\omega^2 f''(\omega_0 + \theta\hat\omega)/2$ with $\theta \in (0,1)$. This gives

$$\begin{aligned} A &= \int_{\mathcal{D}} \big\{ -\psi_0 \Delta\hat\psi + \tfrac{1}{2}|\boldsymbol{\nabla}\hat\psi|^2 + \hat\omega f'(\omega_0) + \frac{\hat\omega^2}{2} f''(\omega_0 + \theta\hat\omega) \big\} \,\mathrm{d}\boldsymbol{x}^2 \\ &= \int_{\mathcal{D}} \tfrac{1}{2} \big\{ |\boldsymbol{\nabla}\hat\psi|^2 + \hat\omega^2 f''(\omega_0 + \theta\hat\omega) \big\} \,\mathrm{d}\boldsymbol{x}^2, \end{aligned} \tag{6.19}$$

where the first and third terms on the first line cancel since $\psi_0 = f'(\omega_0)$.

Regarded as a functional of $\hat\psi$, the last expression [cf. (6.15)] has two important properties. First, it is *conserved*, so

$$\begin{aligned} &\int_{\mathcal{D}} \tfrac{1}{2} \big\{ |\boldsymbol{\nabla}\hat\psi(t)|^2 + \hat\omega(t)^2 f''(\omega_0 + \theta\hat\omega(t)) \big\} \,\mathrm{d}\boldsymbol{x}^2 \\ &\qquad = \int_{\mathcal{D}} \tfrac{1}{2} \big\{ |\boldsymbol{\nabla}\hat\psi(0)|^2 + \hat\omega(0)^2 f''(\omega_0 + \theta\hat\omega(0)) \big\} \,\mathrm{d}\boldsymbol{x}^2. \end{aligned} \tag{6.20}$$

Second, when $f''(\omega_0) \ge c_1 > 0$ for every relevant values of ω_0, the expression is positive definite and vanishes for $\hat\psi = 0$ (the details are left for exercise). It thus defines a *norm* with which we can bound the disturbance $\hat\psi(t)$ in terms of $\hat\psi(0)$. This case is often known as *Arnold's first theorem* and it corresponds to the basic flow ψ_0 being a local *energy minimum* among all

[g] It has been shown in [9] that, given almost *any* basic flow, one could find a norm such that it is unstable.

flows obtained by *isovortical rearrangements* of $\omega_0 = \Delta\psi_0$. The isovortical constraint arises because the vorticity is materially conserved under the dynamics; alternately, this can be seen as a consequence that $\delta^2 A$ in (6.15) is a variation *constrained to isovortical sets*.

The case when $f''(\omega_0) \leq c_2 < -c_0(\mathcal{D}) < 0$ for every relevant value of ω_0 is similar. Here one can prove, with the aid of Poincaré's inequality (6.16), that the last expression in (6.19) is negative definite and vanishes for $\hat{\psi} = 0$. This case of *Arnold's second theorem* corresponds to ψ_0 being a local *energy maximum* among all flows obtained by isovortical rearrangements of $\omega_0 = \Delta\psi_0$.

When the dynamics has a momentum invariant M, one could include it in constructing the functional $A = H + C + \alpha M$ for some constant α. This potentially makes it possible to prove more stable flows. For example, taking $M(\psi) = \partial_{xy}\psi$ in a shear (i.e. parallel) flow in a channel, one recovers the linearised stability criteria of Rayleigh (1880) and Fjørtoft (1950) for shear flows; see, e.g., [8] for more details on stability theory.

We stress the *local* nature of the extrema, particularly in the case of energy minima: while a global energy maximum always exists for a given vorticity distribution (in bounded domains), a global minimum may not be accessible by smooth rearrangement of the vorticity. To see this, one can consider the case with

$$\int_{\mathcal{D}} \omega_0 \, \mathrm{d}\boldsymbol{x}^2 = 0. \tag{6.21}$$

By dividing the vorticity into very thin strips, we can make the flow as close as possible to the zero flow, but this energy infimum can never be reached by smooth rearrangements of vorticity.

Furthermore, we note that there is an important difference between energy maxima and minima when one adds a little dissipation. In this case the steady flow correponding to an energy minimum tends to remain stable, while that corresponding to an energy maximum tends to be *destabilised* by the introduction of dissipation.

6.4. ***Stability Issues***

The functional $A(\psi; \psi_0)$ defined in (6.17) may be useful even when the basic flow ψ_0 is not (provably) stable, since it is *quadratic* in the disturbance $\hat{\psi}$. Since it can be thought of as the "energy" of a disturbance $\hat{\psi}$ over a basic state ψ_0, it is often called *pseudoenergy*. Similarly, in place of the

Hamiltonian H in (6.17), one could use a momentum M to construct a quantity which is quadratic in the disturbance from some mean flow. The resulting functional is then called *pseudomomentum*.

When our problem contains a parameter S (which can be a physical constant, domain size, etc.), one can often find a steady flow ψ_S that depends continuously on S. As S passes a certain critical value, the stability property of ψ_S may change, e.g., from stable to unstable. In this case, an extension of Arnold's method can sometimes be used to show that disturbances in the unstable flow can only grow by a limited amount depending on how far one is from S; see [32, 35, 37] for more.

It turns out that the existence of a symmetry, and the corresponding momentum invariant can put a limitation on the applicability of Arnold's stability method. Suppose that our domain is bounded and our dynamics is invariant under translation, say, in the x-direction. Then *Andrews' theorem* [1] tells us that only basic flows which do not depend on x can be proved stable by Arnold's method. This is because basic flows which are not invariant under x-translation cannot be a strict energy extremum: a flow obtained by translating in x will have exactly the same energy and Casimir as the initial flow, yet the two flows are different.

Here the issue is not Arnold's method, but our definition of stability itself. Instead of using *pointwise* measure of the disturbance, $\|\psi(t) - \psi_0\|$, we might decide to disregard translations of the flow in the direction of symmetry, measuring a quantity like $\min_s \|g_s\psi(t) - \psi_0\|$, where g_s is the translation operator parameterised by s. This problem appears to be much more difficult than standard stability problems, and only a few results have been obtained; see [6] for the case of solitons, [30] for the (linear) stability of modons, and [40] for the stability of the low modes on the torus and the sphere.

It can also be shown [41] that, when the net vorticity in a domain vanishes (as must be the case when the domain has the topology of a sphere), any successful application of Arnold's method must involve the use of a momentum invariant. This implies that, in a domain with no symmetry, any Arnold-stable flow must have nonvanishing net vorticity, and that no Arnold-stable flow exists on a "bumpy sphere". This does not, however, imply that no nonlinearly stable flow exists: a flow whose streamfunction is the gravest eigenfunction of the Laplacian Δ is evidently stable, being an energy minimum.

6.5. *Other Geophysical Models*

The extension (or application) of Arnold's stability method to other models of geophysical fluid dynamics has had a rather mixed success. For vorticity-based models, stability results obtain mostly as in the case of 2d Euler equations; see the references at the beginning of this section.

In compressible models (including the shallow-water equations (4.38)), however, the kinetic energy arises from a term like $\rho|\boldsymbol{u}|^2$, which is cubic in the variables. This seems to make it impossible to construct a norm to bound disturbance quantities out of H and C. However, a formal stability result called *Ripa's theorem* would imply stability if certain conditions (which have not been proved) hold; see [27].

For the 3d Euler equations, it has been proved [28, 29] that the disturbance energy is never positive definite (when the basic flow is non-trivial), so here, too, it seems that no Arnold stability result is possible.

7. Adiabatic Invariance in Fluid Dynamics

TBA

8. Numerical Methods for Hamiltonian Systems

TBA

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References

1. D. G. Andrews, *On the existence of nonzonal flows satisfying sufficient conditions for stability*, Geophys. Astrophys. Fluid Dynamics, 28 (1984), pp. 243–256.
2. V. I. Arnol'd, *Conditions for nonlinear stability of stationary plane curvilinear flows of an ideal fluid (in russian)*, Dokl. Akad. Nauk SSSR, 162 (1965), pp. 975–978. English translation in: Soviet Math. 6, 773–777.

3. ———, *Sur la géometrie différentielle des groupes de Lie de dimension infinie et ses applications à l'hydrodynamique des fluides parfaites*, Ann. Inst. Fourier (Grenoble), 16 (1966), pp. 319–361.
4. V. I. Arnold, *Mathematical methods of classical mechanics, 2ed*, Springer-Verlag, 1989.
5. V. I. Arnol'd and B. A. Khesin, *Topological methods in hydrodynamics*, Springer-Verlag, 1998.
6. T. B. Benjamin, *The stability of solitary waves*, Proc. R. Soc. Lond., A328 (1972), pp. 153–183.
7. A. J. Bourgeois and J. T. Beale, *Validity of the quasigeostrophic model for large-scale flow in the atmosphere and ocean*, SIAM J. Math. Anal., 25 (1994), pp. 1023–1068.
8. P. G. Drazin and W. H. Reid, *Hydrodynamic stability*, Cambridge Univ. Press, 1981.
9. S. Friedlander and A. Shnirelman, *Instability of steady flows of an ideal incompressible fluid*, in Mathematical fluid mechanics, J. Neustupa and P. Penel, eds., 2001, pp. 143–172.
10. H. Goldstein, *Classical mechanics, 2ed*, Addison Wesley, 1980.
11. A. Hamiltonian structure with contact geometry for the semi-geostrophic equations, J. Fluid Mech., 272 (1994), pp. 211–233.
12. D. D. Holm, J. E. Marsden, T. S. Ratiu, and A. Weinstein, *Nonlinear stability of fluid and plasma equilibria*, Phys. Rep., 123 (1985), pp. 1–116.
13. P. J. Kushner and T. G. Shepherd, *Wave-activity conservation laws and stability theorems for semi-geostrophic dynamics. Part 1. Pseudomomentum-based theory*, J. Fluid Mech., 290 (1995), pp. 67–104.
14. ———, *Wave-activity conservation laws and stability theorems for semi-geostrophic dynamics. Part 2. Pseudoenergy-based theory*, J. Fluid Mech., 290 (1995), pp. 105–129.
15. C. Lanczos, *The variational principle of mechanics, 4ed*, Dover, 1970.
16. A. J. Lichtenberg and M. A. Lieberman, *Regular and chaotic dynamics, 2ed*, Springer-Verlag, 1992.
17. P.-L. Lions, *On the existence of positive solutions of semilinear elliptic equations*, SIAM Review, 24 (1982), pp. 441–467.
18. P. Lochak and C. Meunier, *Multiphase averaging for classical systems*, Springer-Verlag, 1988.
19. J. E. Marsden and T. S. Ratiu, *Introduction to mechanics and symmetry*, Springer-Verlag, 1994.
20. M. E. McIntyre and T. G. Shepherd, *An exact local conservation theorem for finite-amplitude disturbances to non-parallel shear flows, with remarks on Hamiltonian structure and on Arnol'd stability theorem*, J. Fluid Mech., 181 (1987), pp. 527–565.
21. P. J. Morrison, *Hamiltonian description of the ideal fluid*, Rev. Mod. Phys., 70 (1998), pp. 467–521.
22. P. J. Morrison and J. M. Greene, *Noncanonical Hamiltonian density formulation of hydrodynamics and ideal magnetohydrodynamics*, Phys. Rev. Lett., 45 (1980), pp. 790–794.

23. M. Mu, *Nonlinear stability of two-dimensional quasigeostrophic motions*, Geophys. Astrophys. Fluid Dynamics, 65 (1992), pp. 57–76.
24. P. J. Olver, *Applications of Lie groups to differential equations, 2ed*, Springer-Verlag, 1993.
25. N. Padhye and P. J. Morrison, *Fluid element relabeling symmetry*, Phys. Lett., A 219 (1996), pp. 287–292.
26. C. N. Pope and K. S. Stelle, $\mathrm{su}(\infty)$, $\mathrm{su}_+(\infty)$ *and area-preserving algebras*, Phys. Lett., B226 (1989), pp. 257–263.
27. P. Ripa, *General stability conditions for zonal flows in a one-layer model on the β-plane or the sphere*, J. Fluid Mech., 126 (1983), pp. 463–489.
28. P. Rouchon, *On the Arnol'd stability criterion for steady-state flows of an ideal fluid*, Eur. J. Mech. B/Fluids, 10 (1991), pp. 651–661.
29. L. Sadun and M. Vishik, *The spectrum of the second variation of the energy of an ideal incompressible fluid*, Phys. Lett., A182 (1993), pp. 394–398.
30. H. Sakuma and M. Ghil, *Stability of stationary barotropic modons by Lyapunov's direct method*, J. Fluid Mech., 211 (1990), pp. 393–416.
31. R. Salmon, *Hamiltonian fluid mechanics*, Ann. Rev. Fluid Mech., 20 (1988), pp. 225–256.
32. T. G. Shepherd, *Rigorous bounds on the nonlinear saturation of instabilities to parallel shear flows*, J. Fluid Mech., 196 (1988), pp. 291–322.
33. ———, *A general method for finding extremal states of Hamiltonian dynamical systems, with applications to perfect fluids*, J. Fluid Mech., 213 (1990), pp. 573–587.
34. ———, *Symmetries, conservation laws, and Hamiltonian structure in geophysical fluid dynamics*, Adv. Geophys., 32 (1990), pp. 287–338.
35. ———, *Nonlinear stability and the saturation of instabilities to axisymmetric vortices*, Eur. J. Mech. B/Fluids, 10 (1991), pp. 93–98.
36. T. G. Shepherd, *Arnol'd stability applied to fluid flow: successes and failures*, Springer-Verlag, 1992, pp. 187–206.
37. T. G. Shepherd, *Nonlinear saturation of baroclinic instability. Part III: Bounds on the energy*, J. Atmos. Sci., 50 (1993), pp. 2967–2709.
38. ———, *Hamiltonian geophysical fluid dynamics*, 2003, pp. 63–90. Lecture notes in "Applications of advanced mathematical and computational methods to atmospheric and oceanic problems (MCAO2003); National Center for Atmospheric Research; annotated by D. Wirosoetisno.
39. E. C. G. Sudarshan and N. Mukunda, *Classical dynamics: a modern perspective*, Wiley, 1974.
40. D. Wirosoetisno and T. G. Shepherd, *Nonlinear stability of Euler flows in two-dimensional periodic domains*, Geophys. Astrophys. Fluid Dynamics, 90 (1999), pp. 229–246.
41. ———, *On the existence of two-dimensional Euler flows satisfying energy–Casimir stability criteria*, Phys. Fluids, 12 (2000), pp. 727–730.
42. D. Wirosoetisno and J. Vanneste, *Persistence of steady flows of a two-dimensional perfect fluid in deformed domains*, Nonlinearity, 18 (2005), pp. 2657–2680.