Numerical Methods for solving singularly perturbed third order ordinary differential equations.

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First, to distinguish between the regular perturbation problems and singular perturbation problems, consider a family of BVPs P_ε, depending on a small parameter ε. Under certain conditions, a 'solution' $y_\varepsilon(x)$ of P_ε can be constructed by the well known 'method of perturbation' ; that is, as a power series in ε with its first term y_0 being the solution of the problem P_0 (obtained by putting $\varepsilon = 0$ in P_ε). When such a power series expansion converges as $\varepsilon \to 0$ uniformly in x, it is a regular perturbation problem. When $y_\varepsilon(x)$ does not have a uniform limit in x as $\varepsilon \to 0$, this regular perturbation method fails and it is called a singular perturbation problem. For some problems the solution may converge uniformly but not its derivatives.

A singular perturbation problem is said to be of convection - diffusion type, if the order of the differential equation is reduced by one when the perturbation parameter ε is set equal to zero. If the order reduces by two, it is known as a reaction-diffusion type problem. Further, if the order of the differential equation in an SPP is greater than two, it is said to be a higher order SPP.

Typically these problems arise in various fields of applied mathematics such as fluid dynamics (boundary layer problems), elasticity (edge effect in shells), quantum mechanics (WKB problems), electrical networks, chemical reactions, control theory, gas porous electrodes theory and many other areas. The Navier - Stokes equation with a large Reynolds number is one of the most striking examples of SPPs, which led to the idea of boundary layer, introduced by Prandtl. Convective heat transport problem with large Peclet number is another important example to be noted.

It is well known that the solutions of SPPs are nonsmooth with singularities related to boundary layers. When the perturbation parameter ε is close to a critical value, even the most contemporary numerical methods fail to be robust and layer-resolving. Careful examination of the numerical results from the various finite difference schemes on uniform grids show that, for fixed (small) value of the parameter ε , the maximum pointwise error usually increases as the mesh is refined, untill the mesh parameter and the perturbation parameter ε have the same order of magnitude. This is due to the presence of the so called ' boundary or interior layers ' exhibited by the solution.

Hence one has to look for robust computational methods which will give numerical approximations which inherit the stability properties of the exact solution by preserving the monotonicity of the original problem.

Miller et al. [MRS96] have suggested Fitted mesh methods and established their parameter-uniform covergence for problems of this type. They have suggested both a finite difference operator and a piecewise uniform fitted mesh, to achieve the parameter-uniform numerical methods.

Roos et al. [RST96] conducted a systematic study on numerical methods for singularly perturbed differential equations. A good number of robust computational methods for solving SPPs for both ODEs and PDEs are included.

Farrell et al. [FHM$^+$00]developed techniques for constructing robust layer resolving methods, and in addition, have developed further experimental techniques for justifying computationally

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that in practice the approximations generated by these methods display the required robust and layer-resolving convergence. They have also applied these techniques in developing new numerical methods for solving Prandtl problem.

O'Malley [O'M74], [O'M91], Zhao Weili [Zha90], [Zha94] Howes [How82], [How83] and Nayfeh [Nay85] have presented asymptotic and numerical methods for solving SPPs of higher order and in particular third order.

Motivated by the works of O'Malley [O'M74], [O'M91], Zhao Weili [Zha90], [Zha94] Howes [How82], [How83], Miller et al. [MRS96], Roos et al. [RST96] and Farrell et al. [FHM+00] computational methods for solving SPBVPs for third order ODEs are developed and presented here.

More pricisely robust and layer resolving numerical methods are suggested for solving the SPBVPs of the form

$$-\varepsilon y'''(x) + a(x)y''(x) + b(x)y'(x) + c(x)y(x) = f(x), \quad x \in \Omega = (0,1) \tag{0.1}$$

$$y(0) = p, \quad y'(0) = q, \quad y'(1) = r \tag{0.2}$$

where $a(x)$, $b(x)$, $c(x)$ and $f(x)$ belong to $C^{(4)}(\overline{\Omega})$ such that $a(x) \leq -\alpha, \ \alpha > 0, \ b(x) \geq 0, \ 0 \geq c(x) \geq -\gamma, \ \gamma > 0, \ \alpha - 3\gamma > \eta$ for some $\eta > 0, \ y \in C^{(3)}(\Omega), \ \Omega = (0,1)$ and $\overline{\Omega} = [0,1]$.

The study of singular perturbation problems will help the researchers to understand the problems that occur in the Geophysical Fluid Dynamics in a better way.

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