# Approximation using scattered shifts of a multivariate function *

Ronald DeVore and Amos Ron

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We consider approximation by scattered shifts of a basis function $\phi$. We start with a countable set $\Xi$ of points in $\mathbb{R}^d$ and define $S_\Xi(\phi)$ to be the set of all functions which are finite linear combinations of the shifts $\phi(\cdot - \xi)$, $\xi \in \Xi$. We are interested in how well a given function $f \in L_p(\mathbb{R}^d)$ can be approximated by the elements of $S_\Xi(\phi)$. Such approximation problems have been well studied, but the known error bounds are given in terms of a global mesh density parameter. In contrast, error bounds that depend on the local density of the scattered centers (i.e., provide improved error bounds on subdomains that contain dense clusters of centers) are less studied and less understood, even though it is often the natural setting in applications.[1]

We shall present results on two types of problems for scattered center approximation. In the first, we assume that the set $\Xi$ is fixed and we derive results that show improved approximation in regions where the density is high. There are two challenges in this part of the theory. The first stems form the fact that it is, perhaps, impossible to derive error bounds based on the local density of the centers: if the local density of the centers changes too fast, one needs to replace the local density parameter by a suitable majorant. The second challenge is that one must decompose the approximand into its "good" part and its "bad" part, and this decomposition must reflect the local density of the centers. We use the wavelet decomposition of the given approximand in order to preform the aforementioned decomposition.

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[1]The most notable exception, is, of course, spline approximation in one variable: a key property of univariate spline approximation is the fact that the error bounds in linear approximation by splines reflects the local mesh ratio.

The second setting that we consider allows the centers to be chosen dependent on the function $f$. The basic goal is to establish error bounds that depend on the cardinality $N$ of the chosen center set $\Xi$. This is a form of nonlinear approximation known as $N$-term approximation which has been well studied in other settings, primarily for wavelet bases. Our result here is similar to the results on nonlinear wavelet approximation. We show that a function can be approximated in $L_p(\mathbb{R}^d)$ with error $O(N^{-s/d})$ once it lies in the Triebel-Lizorkin space $F^s_{\tau,q}(\mathbb{R}^d))$ where $s$, $p$, and $\tau$ are related (as in the Sobolev embedding theorem) by $\frac{1}{\tau} - \frac{1}{p} = \frac{s}{d}$ and $q = (1 + \frac{s}{d})^{-1}$. From this result and standard embeddings for Triebel-Lizorkin spaces, we derive corresponding theorems for $N$-term approximation in terms of the Besov classes. While our actual results in this direction are close in nature to the wavelet results, the non-linear approximation algorithm that leads to the above error bounds differs from its wavelet counterpart: the thresholding algorithm that is employed in the wavelet case is sub-optimal in the present case; as such, we introduce and analyse a more sophisticated algorithm.

Ronald DeVore, Department of Mathematics, University of South Carolina, Columbia, SC 29208, USA, `devore@math.sc.edu`
Amos Ron, Computer Science Department, University of Wisconsin-Madison, Madison, WI 53706, USA, `amos@cs.wisc.edu`