MEAN VALUE REPRESENTATIONS AND CURVATURES OF COMPACT CONVEX HYPERSURFACES

S. L. LEE

Abstract. It is shown that the kernels for mean value representations of points in $\mathbb{R}^n$ in terms of the integrals over piecewise smooth hypersurfaces are divergence free vector fields defined by homogeneous functions of degree $-(n+1)$, whose restrictions to the unit sphere are positive and orthogonal to the first harmonics. By Minkowski problem, such a function is the reciprocal of the composition of the Gaussian curvature of a compact strictly convex hypersurface with its inverse Gauss map. For a compact strictly convex hypersurface, M, we define a dual surface M^* via its support function. It is shown that the homogeneous extension of degree $-(n+1)$ of the reciprocal of the composition of the curvature of M with its inverse Gauss map, as a function on the unit normal sphere, is equal to a homogeneous function defined by the curvature of M^*.

1. Introduction

Representation of points and functionals on a convex set by its extreme points is an important problem in mathematics and its applications. Barycentric coordinates, the Krein-Millman Theorem and Choquet's Theorem are examples of such a representation. Particularly useful in approximation, geometric modelling and numerical computation is the representation of points in a convex polyhedron or in the kernel of a star-shaped polyhedron in terms of its vertices. More precisely if the vertices are v_i, $i = 1, 2, \ldots, k$, and v is any point in the convex polyhedron or in the kernel of the star-shaped polyhedron then the problem is to compute coefficients $\lambda_i \geq 0$ (strictly positive if v is not in the boundary) with desired properties and satisfy

$$\sum_{i=1}^{k} \lambda_i = 1 \text{ and } v = \sum_{i=1}^{k} \lambda_i v_i.$$

The λ_i *'s* are referred to as *coordinates* of v with respect to v_i, $i = 1, \ldots, n$. For an n-simplex, which is the convex hull of $n+1$ affinely independent set of points that form

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its vertices, the coordinates are unique and they are the *barycentric coordinates* for the simplex. For an n-dimensional polyhedron with number of vertices $k > n+1$, the coordinates $\lambda_i\,'s$ are not unique. There are many extensions of barycentric coordinates to convex polygon and polyhedron, for instance, E. Wachpress ([18], 1975), R. Sibson ([16], 1981) for the construction of finite element approximation and scattered data interpolation and C. Loop and T. DeRose ([13], 1989), S. Lodha ([12], 1993), J.Warren ([19], 1996 and [20], 2002), J. Warren, Scheafer, Hirani and Desbrun ([21], 2007) for geometric modelling and computer graphics.

Recently, in conjunction with the construction of one-one transformations and parametrizations of meshes in $\mathbb{R}^3$, Floater [2] has found an explicit formula for the representation of points that lie in the kernel of star-shaped polygons in $\mathbb{R}^2$ in terms of the vertices of the polygon. Because of its simplicity and usefulness in computer graphics and its potential applications in functional approximation and interpolation, the idea has been quickly extended to star-shaped polyhedrons in $\mathbb{R}^3$ (see [4], [5], [8], [9], [10]). Floater's original idea was motivated by the mean value property of harmonic functions, i.e. if ϕ is harmonic in a region $\Omega \subset \mathbb{R}^2$, then for any circle $C(v,r) \subset \Omega$ with centre at v and radius r,

$$\phi(v) = \frac{1}{2\pi r}\int_C \phi\, dC. \tag{1.1}$$

He observed that a class of real-valued piecewise linear functions defined on a triangular mesh in $\mathbb{R}^2$, which he calls *convex combination functions*, share discretely some properties of harmonic functions [2]. Forcing the mean value property (1.1) on the convex combination functions produces the new coordinates, which he calls the *mean value coordinates.* The problem of computing these coordinates then reduces to integrating elementary trigonometric functions over the unit circle or the unit sphere (see [2], [4], [10]).

However, the mean value coordinates are more naturally associated with conservative vector fields on $\mathbb{R}^2$ and divergence free vector fields on $\mathbb{R}^3$. This relationship was established in [11], thereby providing a mathematical foundation and puts mean value coordinates in a mathematical framework for further development. In particular, a class of conservative vector fields F are constructed that provide the representations

$$v = \frac{\int_C x\ F(x-v)\cdot N dC}{\int_C F(x-v)\cdot N dC} \tag{1.2}$$

for any piecewise smooth closed curve $C \subset \mathbb{R}^2$ and for any point $v \in \mathbb{R}^2$, and a class of divergence free vector fields F are constructed such that

$$v = \frac{\int_S x\ F(x-v)\cdot N dS}{\int_S F(x-v)\cdot N dS} \tag{1.3}$$

for any piecewise smooth closed surface $S \subset \mathbb{R}^3$ and $v \in \mathbb{R}^3$. Here and hereafter, dS denotes the standard surface area element of a surface S and N is its unit outward normal. All surfaces are assumed to be closed and piecewise smooth, unless otherwise stated. We also use "*dot* " for inner product. We shall refer to representations of the form (1.2), (1.3) as *mean value representations* and say that the vector field F provides mean value representations. The special case, $F(x) = x/||x||^{n+1}$, $n = 2, 3$, produces Floater's mean value coordinates. Even in this special case the coordinates derived are for very general configurations of points on the curves and surfaces [11]. We emphasize that the surfaces for the mean value representations need not be convex nor star-shaped nor even simply connected and the representations hold for any piecewise smooth hypersurface and for any point in $\mathbb{R}^n$. However, the vector fields that produce Floater's mean value coordinates do not appear to inherit any intrinsic property of the curves or surfaces in their mean value representations. To address this problem, Warren et al [21] have recently introduced a vector field defined in terms of the curvature of the boundary of a given compact convex region for the representation of points in the region in terms of integrals over its boundary.

Our object is to continue the investigation of the divergence free vector fields introduced in [11] for mean value representations and their relations with the curvatures of compact strictly convex hypersurfaces in $\mathbb{R}^n$. The divergence free vector fields are closely related to homogeneous functions. A function $h : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ is a *homogeneous function* of degree m, if

$$h(tx) = t^m h(x), \text{ for all } t > 0 \text{ and } x \in \mathbb{R}^n \setminus \{0\}. \tag{1.4}$$

In Section 2, it is shown that for a continuously differentiable vector field $F = (F_1, \dots, F_n)$ that vanishes at infinity, $x_j F(x)$, $j = 1, \dots, n$, are divergence free if and only if $F(x) = h(x)x$ for some homogeneous function h of degree $-(n+1)$. It is then shown that a homogeneous function h of degree $-(n+1)$ whose restriction to $\mathbb{S}^{n-1}$ is non-vanishing and orthogonal to the first harmonics, i.e.

$$\int_{\mathbb{S}^{n-1}} x_j h(x)\, x \cdot N d\mathbb{S}^{n-1} = 0, \quad j = 1, \dots, n, \tag{1.5}$$

defines a vector field that provides mean value representations. However, functions on the unit sphere that are nonvanishing and orthogonal to the first harmonics are curvatures, as functions on the unit normal sphere, of compact strictly convex hypersurfaces via the Minkowski problem (see [1], [15]). Section 3 develops the relationship between mean value representations and curvatures of compact strictly convex hypersurfaces and shows that homogeneous extensions of degree $-(n+1)$ of their curvatures, as functions on the unit normal sphere, provide a large class of homogeneous functions for mean value representations. In this context the vector fields, $F(x) = x/||x||^{n+1}$, $x \in \mathbb{R}^n \setminus \{0\}$, that produce Floater's mean value coordinates for $n = 2, 3$ are defined by homogeneous extensions of the curvature of the unit $(n-1)$-dimensional sphere. In Section 4 we show that the homogeneous extension of degree $-(n+1)$ of the reciprocal of the composition of the curvature of a compact strictly convex hypersurface M with its inverse Gauss map, as a function on the unit normal sphere, is equal to a homogeneous function defined by the curvature of a dual surface M^*. In particular, the mean value representations by the corresponding vector field over M^* coincides with the continuous barycentric coordinates in [21].

2. Homogeneous vector fields and mean value representations

Let S be a compact piecewise smooth hypersurface in $\mathbb{R}^n$ and assume that it is oriented with its normal in the direction away from the bounded region. The object is to find a vector field $F = [F_1, \ldots F_n]^T$ in $\mathbb{R}^n$ that satisfies

$$\int_S x\ F(x) \cdot N dS = 0 \quad \text{and} \quad \int_S F(x) \cdot N dS \neq 0, \tag{2.1}$$

for any S. Equation (2.1) would then give

$$\int_S (x - v)\ F(x - v) \cdot N dS = 0,$$

for any $v \in \mathbb{R}^n$, so that the following mean value representation holds:

$$v = \frac{\int_S x\ F(x - v) \cdot N dS}{\int_S F(x - v) \cdot N dS}\ .$$

2.1. **Divergence free vector fields and homogeneous functions.** For a vector field $G = (G_1, \ldots, G_n)$ that is continuously differentiable on a region Ω with piecewise smooth boundary S, the generalized Stoke's theorem or Divergence theorem says that

$$\int_S G \cdot N dS = \int_\Omega div(G) dx_1 \cdots dx_n,$$

where $div(G) := \sum_{i=1}^n \partial_i G_i$. It follows that the first integral in (2.1), which is a vector integral, vanishes if $div(x_j F(x)) = 0$ on Ω for $j = 1, \ldots, n$. Therefore, we want to find vector fields F for which $x_j F(x)$, $j = 1, \ldots, n$, are divergence free on "most" of $\mathbb{R}^n$. Such vector fields are related to homogeneous functions. A homogeneous function h of degree m, which is defined by (1.4), i.e. $h(tx) = t^m h(x)$ for $t > 0$ and $x \in \mathbb{R}^n \setminus \{0\}$, is uniquely determined by its values on the unit $(n-1)$-dimensional sphere $\mathbb{S}^{n-1}$. If h is continuously differentiable, taking the derivatives of the expressions in (1.4) with respect to t and then setting $t = 1$ gives the Euler's formula:

$$x \cdot \bigtriangledown h(x) = m h(x), \quad x \in \mathbb{R}^n \setminus \{\mathbf{0}\}, \tag{2.2}$$

where $\bigtriangledown := (\partial_1, \dots, \partial_n)$ is the gradient operator. Further, $\partial_j h$, $j = 1, \dots, n$, are homogeneous functions of degree $m - 1$.

Theorem 2.1. *For a continuously differentiable vector field $F = (F_1, \dots, F_n)$ that vanishes at infinity, $x_j F(x)$, $j = 1, \dots, n$, are divergence free on $\mathbb{R}^n \setminus \{0\}$ if and only if $F(x) = h(x)x$ for some homogeneous function h of degree $-(n+1)$.*

Proof. For a homogeneous function $g : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ of degree m,

$$div(g(x)x) = x \cdot \bigtriangledown g(x) + g(x) div(x) = (m+n)g(x), \tag{2.3}$$

by (2.2), so that $g(x)x$ is divergence free if and only if g is homogeneous of degree $-n$.

Suppose $F(x) = h(x)x$, $x \in \mathbb{R}^n \setminus \{0\}$, for some homogeneous function h of degree $-(n+1)$. Since $x_j h(x)$, $j = 1, \dots, n$, is homogeneous of degree $-n$, $x_j F(x) = x_j h(x) x$, $j = 1, \dots, n$, are divergence free on $\mathbb{R}^n \setminus \{0\}$.

Conversely, suppose $div(x_j F(x)) = 0$, $j = 1, \dots, n$. Then

$$F_j(x) + x_j \sum_{i=1}^{n} \frac{\partial F_i(x)}{\partial x_i} = div(x_j \mathbf{F}(x)) = 0, \quad j = 1, \dots, n \ ,$$

which implies that

$$\frac{F_j(x)}{x_j} = -\sum_{i=1}^{n} \frac{\partial F_i(x)}{\partial x_i}, \quad j = 1, \dots, n \tag{2.4}$$

and hence for each $j = 1, \dots, n$,

$$F_i(x) = \frac{x_i}{x_j} F_j(x), \quad i = 1, \dots, n. \tag{2.5}$$

Differentiating (2.5) with respect to x_i and substituting the resulting derivatives back into (2.4) shows that all F_j, $j = 1, \dots, n$, are solutions of the differential equation

$$x \cdot \bigtriangledown U(x) = -nU(x). \tag{2.6}$$

The general solution of the corresponding homogeneous equation is an arbitrary homogeneous function of degree 0 and the general solution of (2.6) that vanishes at infinity is an arbitrary homogeneous function of degree $-n$. It then follows from the relations in (2.4), that $F_j(x) = x_j h(x)$, $j = 1, \dots, n$, and hence $F(x) = h(x)x$, $x \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ for some homogeneous function h of degree $-(n+1)$. ♠

2.2. **Homogeneous functions of degree** 1. Homogeneous functions of degree 1 have interesting properties and appear naturally in conjunction with the study of compact strictly convex hypersurfaces. We shall first establish some formulas that will provide a crucial relationship between the curvature of a compact strictly convex hypersurface and that of its dual introduced in Section 3.

Proposition 2.2. *Let h be a homogeneous function of degree 1, which has continuous derivatives up to second order, $A_{ij} \equiv A_{ij}^h$ be the cofactor of $\partial_i\partial_j h$ in the Hessian matrix $H \equiv H^h := (\partial_i\partial_j h)$ of h and $Adj(H) = (A_{ij})^T$ be the adjoint of H. Then for all $i = 1, \ldots, n$,*

$$\frac{A_{ii}(x)}{x_i^2} = \Phi(x), \quad x_i \neq 0, \tag{2.7}$$

for some homogeneous function Φ of degree $-(n+1)$.

If g is a differentiable homogeneous function of degree m and $g(x) \neq 0$ for $x \neq 0$, then

$$(\bigtriangledown g)^T Adj(H)(\bigtriangledown g) = m^2 g^2 \, \Phi \, . \tag{2.8}$$

Proof. Since h is a homogeneous function of degree 1, $\partial_i h$, $i = 1, \ldots, n$, are homogeneous of degree 0. By (2.2)

$$\sum_{i=1}^n x_i \frac{\partial^2 h(x)}{\partial x_i \partial x_j} = 0, \quad j = 1, \ldots, n \, . \tag{2.9}$$

Since the linear system (2.9) has nontrivial solutions, $\det(\partial_i\partial_j h) = 0$ and for $j = 1, \ldots, n$,

$$\frac{A_{ij}}{x_i} = \frac{A_{jj}}{x_j}, \quad i = 1, \ldots, n \, . \tag{2.10}$$

Therefore, for any $j = 1, \ldots, n$,

$$\frac{A_{ij}}{x_i x_j} = \frac{A_{jj}}{x_j^2}, \quad i = 1, \ldots, n$$

and because of symmetry, this gives, in particular,

$$\frac{A_{ii}}{x_i^2} = \frac{A_{jj}}{x_j^2}, \quad i = 1, \ldots, n \, .$$

Since A_{ii} is a determinant of order $n-1$, whose entries are homogeneous functions of degree -1, it is a homogeneous function of degree $-(n-1)$. Hence A_{ii}/x_i^2 is homogeneous of degree $-(n+1)$, and we have (2.7).

To prove (2.8), we note that

$$\sum_{i=1}^n x_i \frac{\partial g(x)}{\partial x_i} = mg(x), \quad x \in \mathbb{R}^n \setminus \{0\} \, . \tag{2.11}$$

Consider the linear system formed by (2.11) and any of the $n-1$ equations in (2.9). If $g(x) \neq 0$ for $x \neq 0$, such a system is nonsingular for all $x \neq 0$. Then for any $k = 1, \ldots, n$,

$$x_i = \frac{mg(x)\, A_{ik}(x)}{\det(H_k(x))}, \quad i = 1, \ldots, n, \tag{2.12}$$

where H_k is the matrix of coefficients of the linear system and is derived from the Hessian of h by replacing the kth row by $(\partial_1 g, \ldots, \partial_n g)$.

Equation (2.12) gives

$$\begin{aligned}\det(H_k(x)) &= mg(x)\,\frac{A_{ik}(x)}{x_i}\\ &= mg(x)\,\frac{A_{kk}(x)}{x_k}, \quad k=1,\ldots,n,\end{aligned}$$

by (2.10). Expanding the determinant in the last equation and multiplying by $\frac{\partial g(x)}{\partial x_k}$ gives

$$\begin{aligned}\frac{\partial g(x)}{\partial x_k}\sum_{j=1}^{n} A_{kj}(x)\frac{\partial g(x)}{\partial x_j} &= mg(x)\,\frac{A_{kk}(x)}{x_k^2}\,x_k\frac{\partial g(x)}{\partial x_k}\\ &= mg(x)\;\Phi(x)\;x_k\frac{\partial g(x)}{\partial x_k}, \quad k=1,\ldots,n\;.\end{aligned}$$

Summing these equations and using Euler's formula (2.2) leads to

$$\begin{aligned}\sum_{k=1}^{n}\sum_{j=1}^{n}\frac{\partial g(x)}{\partial x_k}A_{kj}(x)\frac{\partial g(x)}{\partial x_j} &= mg(x)\;\Phi(x)\;\sum_{k=1}^{n} x_k\frac{\partial g(x)}{\partial x_k}\\ &= m^2 g(x)^2\;\Phi(x)\;,\end{aligned}$$

which establishes (2.8). ♠

2.3. **Mean value representations in $\mathbb{R}^n$.** Let h be a homogeneous function of degree $-(n+1)$ and define

$$F(x) := h(x)x\;. \tag{2.13}$$

By Proposition 2.1, $x_jF(x) = x_jh(x)x$, $j = 1,\ldots,n$, are divergence free on any region not containing the origin. Therefore their integrals over the boundary of any n-dimensional region not containing the origin are zero. Hence $\int_S xF(x)\cdot N dS = 0$, for any S that does not enclose the origin. We also require their integrals over any $(n-1)$-dimensional surface that encloses the origin to be zero. Since $x_jF(x) = x_jh(x)x$ are divergence free, it suffices to integrate over the unit $(n-1)$-dimensional sphere, $\mathbb{S}^{n-1}$ and to require (1.5) to hold, i.e.

$$\int_{\mathbb{S}^{n-1}} x_jh(x)\;x\cdot N d\mathbb{S}^{n-1} = 0, \quad j=1,\ldots,n.$$

The following generalized spherical polar coordinates provide a useful description of (1.5) as well as a parametrization for the unit sphere and compact strictly convex hyperfaces:

$$x_j := \cos\theta_{j-1}\prod_{i=j}^{n-1}\sin\theta_i,\;\; \rho>0,\;\; 0\le\theta_1<2\pi,\;\; 0\le\theta_i\le\pi,\;\; i=2,\ldots,n-1, \tag{2.14}$$

$$\text{for } j=1,\ldots,n-1,$$

where $\cos\theta_0 := 1$, as a convention. The functions $x_j(\theta)$, $j = 1,\ldots,n$, will be referred to as the *first order harmonics* of the $(n-1)$-dimensional spherical harmonics. The tangent vectors, $\frac{\partial x}{\partial \theta_j}$, $j = 1, \ldots, n-1$, provide an orthogonal frame and their Grammian, $\left(\frac{\partial x}{\partial \theta_i} \cdot \frac{\partial x}{\partial \theta_j}\right)_{i,j=1}^{n-1}$ is a diagonal matrix with diagonal entries, $\frac{\partial x}{\partial \theta_j} \cdot \frac{\partial x}{\partial \theta_j} = \prod_{i=j+1}^{n-1} \sin^2 \theta_i$, $j = 1, \ldots, n-1$, so that

$$\sqrt{\det\left(\frac{\partial x}{\partial \theta_i} \cdot \frac{\partial x}{\partial \theta_j}\right)} = \prod_{k=1}^{n-2} \sin^k \theta_{k+1}.$$

The condition (1.5) is equivalent to

$$\int_{D_\theta} x_j(\theta) h(x(\theta)) \prod_{k=1}^{n-2} \sin^k \theta_{k+1} d\theta_1 \cdots \theta_{n-1} = 0, \quad j = 1, \ldots, n. \tag{2.15}$$

We shall refer to condition (2.15) by saying that $h|_{\mathbb{S}^{n-1}}$ is *orthogonal to the first harmonics.*

Theorem 2.3. *If $h : \mathbb{R}^n \to \mathbb{R}$ is a homogeneous function of degree $-(n+1)$ whose restriction to $\mathbb{S}^{n-1}$ is non-vanishing and orthogonal to the first harmonics, then the vector field $h(x)x$ provides mean value represemtations, i.e. for any closed $(n-1)$-dimensional piecewise smooth surface $S \subset \mathbb{R}^n$ and any $v \in \mathbb{R}^n$,*

$$v = \frac{\int_S xh(x-v)\ (x-v) \cdot NdS}{\int_S h(x-v)\ (x-v) \cdot NdS}\ . \tag{2.16}$$

Proof. We have shown that for any $(n-1)$-dimensional surface $S \subset \mathbb{R}^n$ and $0 \notin S$,

$$\int_S xh(x)\ x \cdot NdS = \mathbf{0}. \tag{2.17}$$

To show that

$$\int_S h(x)\ x \cdot NdS \neq \mathbf{0}\ , \tag{2.18}$$

we apply Divergence theorem and consider two cases: (i) 0 not in the region Ω enclosed by S, (ii) $0 \in \Omega$. In the first case,

$$\begin{aligned} \int_S h(x)\ x \cdot \mathbf{n} dS &= \int_\Omega div(h(x)x) dx_1 \cdots dx_n \\ &= -\int_\Omega h(x) dx_1 \cdots dx_n \neq 0, \end{aligned}$$

by (2.3) since h is homogeneous of degree $-(n+1)$. For the second case, taking a ball B_R with centre at the origin and arbitrary large radius R that contains Ω, and applying Divergence theorem on $B_R \setminus \Omega$ give

$$\int_{\partial B_R} h(x)\ x \cdot Nd(\partial B_R) - \int_S h(x)\ x \cdot NdS = -\int_{B_R \setminus \Omega} h(x) dx_1 \cdots dx_n. \tag{2.19}$$

Since $h(x)$ is homogeneous of degree $-(n+1)$, the first integral on the left of (2.19) tends to 0 as $R \to \infty$. Taking the limit in (2.19) gives

$$\int_S h(x)\ x \cdot NdS = \int_{\mathbb{R}^n \setminus \Omega} h(x) dx_1 \cdots dx_n \neq 0.$$

Since (2.17) and (2.18) hold for integrals over any $(n-1)$-dimensional surface, it follows that for any S and $v \in \mathbb{R}^n \setminus S$,

$$\int_S (x-v)h(x-v)\ (x-v)\cdot N dS = \mathbf{0}$$

and

$$\int_S h(x-v)\ (x-v)\cdot N dS \neq \mathbf{0},$$

which gives the representation (2.16).

Equation (2.16) also holds if $v \in S$. This follows by taking a ball B_δ with centre at v and arbitrary small radius δ. Then (2.16) holds for the boundary of $\Omega_\delta := \Omega \setminus B_\delta$ for all $\delta > 0$. where Ω is the region with boundary S. Hence, it holds in the limit as $\delta \to 0$. ♠

Remark 1. *Floater's mean value coordinates* ([2], [4]) *are derived from mean value representations by the homogeneous vector fields,* $F(x) = h(x)x$*, where* $h(x) = 1/||x||^{n+1}$*,* $n = 2, 3$*. In* [11]*, the mean value coordinates are constructed from their mean value representations for this particular case for* $n = 2, 3$*, in terms of very general configurations of points that lie on the piecewise smooth boundary of any compact* n*-dimensional region. The construction in* [11] *can be extended to any dimension.*

Theorem 2.3 shows that the condition for the vector fields $x_j F(x)$, $j = 1, \ldots, n$, be divergence free produces strong results that the mean value representation (2.16) holds for any $(n-1)$-dimensional surface $S \subset \mathbb{R}^n$ and any $v \in \mathbb{R}^n$, which provides a useful tool for the analysis and construction of barycentric type coordinates for a very general configurations of points. On the other hand the divergence free condition is a strong requirement. In spite of this, the homogeneous function h in Theorem 2.3 provides important flexibility for the construction of divergence free vector fields with geometric properties. Theorem 2.3 says that a homogeneous function h of degree $-(n+1)$ defines a vector field that provides mean value representations if its restriction to the unit sphere $\mathbb{S}^{n-1}$ is non-vanishing and orthogonal to the first harmonics. On the other hand, a positive function on the unit sphere is the curvature of a compact strictly convex hypersurface, unique up to a translation, if and only if its reciprocal is orthogonal to the first harmonics. Therefore, the homogeneous extension of degree $-(n+1)$ of the curvature of a given compact strictly convex hypersurface defines a divergence free vector field that provides mean value reprsentations. The following section gives a detailed account of this relationship.

3. Mean value representations associated with curvatures of compact strictly convex hypersurfaces

This section studies the relation between mean value representations and curvatures of compact strictly convex hypersurfaces. We first develop some preliminaries.

3.1. **Preliminaries on curvatures of compact strictly convex hypersurfaces.**

For a compact strictly convex hypersurface M in $\mathbb{R}^n$, the *Gauss map* $G : M \to \mathbb{S}^{n-1}$ takes a point $y \in M$ to a point $x =: G(y) \in \mathbb{S}^{n-1}$, whose position vector is the unit normal to M at y. Because M is strictly convex, its Gauss map is a diffeomorphism of M onto $\mathbb{S}^{n-1}$. The inverse $G^{-1} : \mathbb{S}^{n-1} \to M$ provides a parametrization of M on the unit sphere. The function $h : \mathbb{S}^{n-1} \to \mathbb{R}$ defined by

$$h(x) = \sum_{i=1}^{n} x_i y_i, \quad x \in \mathbb{S}^{n-1}\ , \tag{3.1}$$

where $y = G^{-1}(x)$, is called the *support function* of M. The value $h(x)$ is the distance from the origin to the tangent plane at the point $y \in M$ where the normal is x. The support function h can also be considered as a function on M, in which case it will be denoted by h_M to avoid confusion. We shall first consider h as a function on $\mathbb{S}^{n-1}$, so that is can be extended to a homogeneous function of degree 1 on $\mathbb{R}^n \setminus \{0\}$ by defining $h(x) := ||x||h(x/||x||)$. In this extension the coordinates $y_i = y_i(x)$, $i = 1, \dots, n$, of points $y = G^{-1}(x)$ of M are naturally extended to homogeneous functions on $\mathbb{R}^n \setminus \{0\}$ of degree 0. The support function h is uniquely determined by M. Since

$$h(x) = \sup_{y \in M} \left(\sum_{i=1}^{n} x_i y_i \right),$$

it follows that h is convex. Indeed h is strictly convex on any hyperplane $x_i = -1$, $i = 1, \dots, n$ (see [1]). All quantities associated with M can be described in terms of its support function. The following lemma summarizes some known results, which are useful formulas for theoretical development as well as for computations (see [1], [14]).

Lemma 3.1. *Let h be the support function of a compact strictly convex hypersurface M, $x = G(y)$, $y \in M$, be the Gauss map of M onto $\mathbb{S}^{n-1}$, where $y = G^{-1}(x)$ is extended to a homogeneous function of degree* 0 *on $\mathbb{R}^n$ and A_{ij}, $i, j = 1 \dots, n$, be the cofactor of $\partial_i \partial_j h$ in the Hessian matrix $H = (\partial_i \partial_j h)_{i,j=1}^{n}$, of h.*

1. *The coordinate functions of points $y \in M$ are given by*

$$y_i = \frac{\partial h}{\partial x_i}, \quad i = 1, \dots, n. \tag{3.2}$$

2. *The curvature K of M and the cofactors of the Hessian of h as functions on the unit sphere $\mathbb{S}^{n-1}$ satisfy the relation:*
$$\sum_{i=1}^{n} A_{ii}(x) = \frac{1}{K \circ G^{-1}(x)}, \quad x \in \mathbb{S}^{n-1} \tag{3.3}$$
and as homogeneous functions they satisfy
$$\sum_{i=1}^{n} A_{ii}(x) = \frac{1}{||x||^{n-1} K \circ G^{-1}(x)}, \quad x \in \mathbb{R}^n \setminus \{0\} . \tag{3.4}$$
3. *For $i = 1, \dots, n$,*
$$\frac{A_{ii}(x)}{x_i^2} = \frac{1}{||x||^{n+1}\, K \circ G^{-1}(x)}, \quad x \in \mathbb{R}^n,\ x_i \neq 0 . \tag{3.5}$$
4. *If $\mathbb{S}^{n-1}$ is parametrized by the polar coordinates $x = x(\theta)$ defined by (2.14), then*
$$\det \left(\frac{\partial^2 h(x(\theta))}{\partial\theta_i \partial\theta_j} + h(x(\theta))\delta_{ij} \right)_{i,j=1}^{n-1} = \frac{1}{K \circ G^{-1}(x(\theta))}, \tag{3.6}$$
$$\textit{for } \theta_1 \in [0, 2\pi),\ \theta_j \in [0, \pi],\ j = 2, \dots, n-1.$$

We remark that equation (3.2) follows from the definition $h(x) = \sum_{i=1}^{n} x_i y_i$ in (3.1), Euler's formula $h(x) = \sum_{i=1}^{n} x_i \frac{\partial h(x)}{\partial x_i}$ and the fact that x is orthogonal to the vectors $\frac{\partial y(x)}{\partial x_j}$, $j = 1, \dots, n$, which lie on the tangent plane to M at $y(x)$. Equation (3.5) follows from Lemma 2.2 and (3.4). The relation (3.6) is obtained by computing the Hessian of h in polar coordinates and using Euler's formula for the homogeneous function h and (3.3) in polar form.

It is important to note that the curvature function $K \circ G^{-1}(x)$, $x \in \mathbb{R}^n \setminus \{\mathbf{0}\}$, in (3.4) and (3.5), which is a homogeneous extension of the curvature of M as a function on the unit sphere, is a homogeneous function of degree 0.

Finally we state a formula for the Gaussian curvature of a compact strictly convex hypersurface M defined implicitly by $g(x) = 1$ where g is a homogeneous function on $\mathbb{R}^n$ of degree 1 with continuous partial derivatives up to second order. Let H^g be the Hessian of g, and $\mathrm{Adj}(H^g)$ be the adjoint of H^g. Define $K^g : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ by
$$K^g := \frac{g^{n-1}\, (\triangledown g)^T \mathrm{Adj}(H^g)(\triangledown g)}{|| \triangledown g||^{n+1}} . \tag{3.7}$$
The function K^g is a homogeneous function of degree 0, since g is homogeneous of degree 1, $\triangledown g$ and $\partial_i \partial_j g$ are homogeneous of degree 0 and -1 respectively and the entries of $\mathrm{Adj}(H^g)$ are homogeneous functions of degree $-n+1$. For $y \in M$, the value $K^g(y)$ is the curvature of M at y. The factor g^{n-1} has been introduced in (3.7) to ensure that K^g is a zero degree homogeneous extension of the curvature K as a function defined on M for consistency with formulas based on support functions. The formula (3.7) can be derived

from Proposition 2.2 and Lemma 3.1 or directly from the Weingarten map (see [17]). A more general result without the restriction that g be homogeneous can be found in [7].

3.2. **Homogeneous extension for functions defined on compact convex hypersurfaces.** A function defined on the unit sphere can be easily extended to a homogeneous function on $\mathbb{R}^n \setminus \{0\}$. Since a compact strictly convex hypersurface M is homeomorphic to the unit sphere, any function defined on M can also be extended to $\mathbb{R}^n \setminus \{0\}$. Here, we shall construct a method for such an extension. To do this we introduce a homogeneous function that provides an implicit equation for the hypersurface as well as homogeneous extensions of functions defined on the hypersurface. We shall assume that the origin is in the convex set enclosed by M. For each $y \in \mathbb{R}^n \setminus \{\mathbf{0}\}$, let $m(y)$ be the point of intersection of the ray $\{\lambda y : \lambda > 0\}$ with M and define

$$\phi(y) := \frac{||y||}{||m(y)||}, \quad y \in \mathbb{R}^n \setminus \{0\} \ . \tag{3.8}$$

Lemma 3.2. *The function ϕ is a positive homogeneous function of degree* 1 *and for $y \neq 0$, $m(y) = y/\phi(y)$. Further, M has an implicit representation given by $\phi(y) = 1$.*

Proof. Since $m(ty) = m(y)$, $t > 0$,

$$\phi(ty) = \frac{||ty||}{||m(ty)||} = \frac{t\,||y||}{||m(y)||} = t\phi(y).$$

Since $m(y)$ lies on the ray $\{\lambda y : \lambda > 0\}$, $m(y) = \lambda y$ for some $\lambda > 0$. By (3.8), $\lambda = 1/\phi(y)$. Since $m(y) = y/\phi(y)$, it follows that $y \in M$ if and only if $\phi(y) = 1$. ♠

Example: The ray $\{\lambda y : \lambda > 0\}$ intersects the ellipse $\sum_{i=1}^n (y_i/a_i)^2 = 1$ when

$$1 = \sum_{i=1}^n (\lambda y_i/a_i)^2 = \lambda^2 \sum_{i=1}^n (y_i/a_i)^2 \ ,$$

so that $\phi(y) = 1/\lambda = \sqrt{\sum_{i=1}^n (y_i/a_i)^2}$. In particular, for the unit sphere, $\phi(y) = ||y||$.

Since ϕ is homogeneous of degree 1, any function f defined on M can be extended to a homogeneous function of degree m, for any m, by defining

$$f(y) := \phi(y)^m f\left(\frac{y}{\phi(y)}\right), \quad y \in \mathbb{R}^n \setminus \{0\}.$$

Since M is defined implicitly by $\phi(y) = 1$, and ϕ is homogeneous of degree 1 the function $K : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ defined as in (3.7) for ϕ, i.e.

$$K := \frac{\phi^{n-1}\,(\bigtriangledown\phi)^T \mathrm{Adj}(H^\phi)(\bigtriangledown\phi)}{||\bigtriangledown\phi||^{n+1}} \ , \tag{3.9}$$

is the curvature function of M.

3.3. **Minkowski problem and mean value representations.** The following theorem is commonly known as the *Minkowski problem* (see [1], [14] and [15]):

Theorem 3.3 (Minkowski problem)**.** *Given $f > 0$ on $\mathbb{S}^{n-1}$, there exists a unique (up to translation) compact strictly convex hypersurface M in $\mathbb{R}^n$ such that $f \circ G$ is the Gaussian curvature on M if and only if $1/f$ is orthogonal to the first harmonics on $\mathbb{S}^{n-1}$.*

Theorem 3.3 establishes a one-one correspondence between positive functions on $\mathbb{S}^{n-1}$ whose reciprocals are orthogonal to the first harmonics and the curvatures of compact strictly convex hypersurfaces in $\mathbb{R}^n$. In other words, K is the curvature of a compact strictly convex hypersurface M if and only if $K \circ G^{-1}$ is a positive function on $\mathbb{S}^{n-1}$ and $1/K \circ G^{-1}$ is orthogonal to the first harmonics. The following is a corollary of Theorem 2.3 and Theorem 3.3.

Corollary 3.4. *Let K and G be the curvature and Gauss map, respectively, of a compact strictly convex hypersurface M in $\mathbb{R}^n$. Then the homogeneous extension $1/\{||x||^{n+1}K \circ G^{-1}(x)\}$ of $1/K \circ G^{-1}$ of degree $-(n+1)$ defines the vector field, $x/\{||x||^{n+1}K \circ G^{-1}(x)\}$, that provides mean value representations.*

In the next section we show that $1/\{||x||^{n+1}K \circ G^{-1}(x)\}$ can be expressed in terms of the curvature of the surface defined implicitly by $h(x) = 1$, where h is the support function of the hypersurface M. We shall denote this surface by M^* and show that M^* and M have a dual relationship, which is crucial in our construction.

4. Dual surfaces and mean value representations

4.1. **Support duals of compact strictly convex hypersurfaces.** Let M be a compact strictly convex hypersurface in $\mathbb{R}^n$ described by the implicit equation $\phi(y) = 1$, $y \in \mathbb{R}^n \setminus \{0\}$, where ϕ is the homogeneous function of degree 1 defined by (3.8). Its Gauss map is given by $G = \bigtriangledown\phi/|| \bigtriangledown \phi||$. Let h be the support function of M, which is the first degree homogeneous extension of the support function defined on the unit sphere. Recall that $h(x) = x \cdot y$, where $y = G^{-1}(x)$, and $h_M(y) = x \cdot y$, where $x = G(y)$. Therefore,

$$h_M(y) = y \cdot x = \frac{y \cdot \bigtriangledown\phi(y)}{|| \bigtriangledown \phi(y)||} = \frac{\phi(y)}{|| \bigtriangledown \phi(y)||} \; . \tag{4.1}$$

On the other hand equation (3.2), i.e. $y = \bigtriangledown h(x)$, shows that the position vector y of a point of M is normal to the level set surfaces $h(x) = C$. In particular $y/||y||$ is a unit normal to the hypersurface defined by $h(x) = 1$, which we denote by M^*. Since h is convex and strictly convex on the planes $x_i = -1$, $i = 1, \ldots, n$, and $h(x) \to \infty$ with $||x||$, M^* is a compact strictly convex hypersurface in $\mathbb{R}^n$. Its Gauss map is given by $G^* = \bigtriangledown h/|| \bigtriangledown h||$.

Let ϕ^* be the homogeneous function of degree 1 defined by M^* in the same way as ϕ is defined by M. Since both ϕ^* and h are homogeneous of degree 1 and $\phi^*(x) = 1$ if and only if $h(x) = 1$, they must be equal. So the function h plays dual roles: as the support function for M as well as defining an implicit equation for M^*. We shall call M^* the *support dual* of M. Let h^* be the support function of M^* given by

$$h^*(y) = \sum_{i=1}^{n} y_i x_i, \quad y = G^*(x) \in \mathbb{S}^{n-1}$$

and extended homogeneously to $\mathbb{R}^n \setminus \{0\}$ by $h^*(y) = ||y|| h^*(y/||y||)$. Since $x = \nabla h^*(y)$, it follows that x is normal to the level surfaces $h^*(y) = C$. In particular, any vector $x \in M^*$, i.e. that satisfies $h(x) = 1$ is normal to the surface defined by $h^*(y) = 1$. But x is the normal to M which is defined implicitly by $\phi(y) = 1$. Hence $h^* = \phi$. We summarize the results of the above discussion as

Lemma 4.1. *Let M be a compact strictly convex hypersurface in $\mathbb{R}^n$ defined by $\phi(y) = 1$, where ϕ is the homogeneous function defined by (3.8), $h(x) = x \cdot y$ be the degree one homogeneous support function of M, where $x = G(y)$ is the unit normal to M at $y \in M$, M^* be the surface defined by $h(x) = 1$ and h^* be the corresponding support function for M^*.*

1. *If ϕ^* is defined by M^* in the same way as ϕ is defined by M as in (3.8), then*

$$h^* = \phi \text{ and } h = \phi^* . \tag{4.2}$$

2. *Vectors $x \in M^*$ are normal to M and $y \in M$ are normal to M^* .*
3. *The support functions h_M and h^*_M satisfy*

$$h_M = \frac{h^*}{|| \nabla h^*||} \text{ and } h^*_{M^*} = \frac{h}{|| \nabla h||} . \tag{4.3}$$

Remark 2. *The surfaces M and M^* can be expressed by*

$$M = \{y \in \mathbb{R}^n : h^*(y) = 1\} \text{ and } M^* = \{x \in \mathbb{R}^n : h(x) = 1\},$$

where h is the support function of M and h^* is the support function of M^*.

4.2. **Homogeneous extensions of curvatures of hypersurfaces and curvatures of their support duals.** Let K and K^* be the curvature functions of M and M^* respectively. Equation (2.8) in Proposition 2.2 gives a useful connection between K and K^*, for if the h in (2.8) is the support function of M (respectively M^*) then the function Φ in (2.8) is the curvature function K of M (respectively K^* of M^*). The following lemma follows from Proposition 2.2 and Lemma 3.1 .

Lemma 4.2. *Let G, h and K be the Gauss map, support function and curvature, respectively, of M. Then for any homogeneous function g of degree m,*

$$(\bigtriangledown g(x))^T Adj(H(x))(\bigtriangledown g(x)) = \frac{m^2 g(x)^2}{||x||^{n+1}\ K \circ G^{-1}(x)}, \quad x \in \mathbb{R}^n \setminus \{0\}\ . \tag{4.4}$$

Proof. If h is the support function of M, (2.7) and (3.5) give

$$\Phi(x) = \frac{A_{ii}(x)}{x_i^2} = \frac{1}{||x||^{n+1}\ K \circ G^{-1}(x)}, \quad x \in \mathbb{R}^n,\ x_i \neq 0\ .$$

Then (4.4) follows from (2.8). ♠

Theorem 4.3. *For a compact strictly convex hypersurface M, let G, h K be as in Lemma 4.2. Let M^* be the support dual of M and G^*, h^* and K^* be the corresponding Gauss map, support function and curvature, respectively. Then*

$$\frac{K(y)}{h_M(y)^{n+1}} = \frac{1}{||y||^{n+1}\ K^* \circ (G^*)^{-1}(y)}, \quad y \in \mathbb{R}^n \setminus \{0\} \tag{4.5}$$

and

$$\frac{K^*(x)}{h^*_{M^*}(x)^{n+1}} = \frac{1}{||x||^{n+1}\ K \circ G^{-1}(x)}, \quad x \in \mathbb{R}^n \setminus \{0\}\ . \tag{4.6}$$

Proof. By (4.4)

$$(\bigtriangledown h^*(y))^T Adj(H^*(y))(\bigtriangledown h^*(y)) = \frac{h^*(y)^2}{||y||^{n+1}\ K^* \circ (G^*)^{-1}(y)}, \quad y \in \mathbb{R}^n \setminus \{0\}\ , \tag{4.7}$$

where H^* is the Hessian of h^*. Since $h^*(y) = 1$ is the implicit equation of M, (3.7) gives

$$K(y) = \frac{(h^*(y))^{n-1}\ (\bigtriangledown h^*(y))^T Adj(H^*(y)) \bigtriangledown h^*(y)}{|| \bigtriangledown h^*(y)||^{n+1}}\ . \tag{4.8}$$

Then by (4.3), (4.8) and (4.7),

$$\begin{aligned} \frac{K(y)}{h_M(y)^{n+1}} &= \frac{K(y)|| \bigtriangledown h^*(y)||^{n+1}}{h^*(y)^{n+1}} \\ &= \frac{(\bigtriangledown h^*(y))^T Adj(H^*(y))(\bigtriangledown h^*(y))}{h^*(y)^2} \\ &= \frac{1}{||y||^{n+1}\ K^* \circ (G^*)^{-1}(y)}\ , \end{aligned}$$

which establishes (4.5). The relation (4.6) follows by duality. ♠

Corollary 4.4. *If K and h_M are the curvature and support function, respectively, of a compact strictly convex hypersurface M in $\mathbb{R}^n$, then $K(x)x/(h_M(x))^{n+1}$ provides mean value representations, i.e. for any closed piecewise smooth hypersurface $S \subset \mathbb{R}^n$ and $v \in \mathbb{R}^n$,*

$$v = \int_S x\, \frac{K(x-v)}{h_M(x-v)^{n+1}}(x-v)\cdot N dS \Big/ \int_S \frac{K(x-v)}{h_M(x-v)^{n+1}}(x-v)\cdot N dS\ .$$

In particular,

$$v = \int_M x \, \frac{K(x-v)}{h_M(x-v)^n} dM \Big/ \int_M \frac{K(x-v)}{h_M(x-v)^n} dM \ .$$

4.3. **Examples.** We illustrate the above results with two simple examples.

1. If M is the unit sphere in $\mathbb{R}^n$, its curvature $K = 1$ on M, G is the identity map and the corresponding homogeneous vector field $x/||x||^{n+1}$ provides the mean value representations

$$v = \int_S x \, \frac{1}{||x-v||^{n+1}} \, (x-v) \cdot N dS \Big/ \int_S \frac{1}{||x-v||^{n+1}} \, (x-v) \cdot N dS$$

that produces Floater's mean value coordinates in two and three dimensions [11].

2. Consider the ellipsoid M, $\sum_{i=1}^n y_i^2/a_i^2 = 1$. Its Gauss map, $G : M \to \mathbb{S}^{n-1}$, is the transformation $x = G(y)$, where

$$x_i = \frac{y_i/a_i^2}{\left(\sum_{i=1}^n y_i^2/a_i^4\right)^{1/2}}, \quad i = 1, \ldots, n \ . \tag{4.9}$$

The inverse transform $y = G^{-1}(x)$ is

$$y_i = \frac{a_i^2 x_i}{\left(\sum_{i=1}^n a_i^2 x_i^2\right)^{1/2}}, \quad i = 1, \ldots, n \ . \tag{4.10}$$

Equations (4.9) and (4.10) define G and G^{-1} on $\mathbb{R}^n \setminus \{0\}$ as homogeneous functions of degree 0. The support function of M,

$$h(x) = \sum_{i=1}^n x_i y_i = \left(\sum_{i=1}^n a_i^2 x_i^2\right)^{1/2}, \quad x \in \mathbb{R}^n \setminus \{0\} \ ,$$

by (4.10). Similarly, by (4.9),

$$h_M(y) = \frac{\sum_{i=1}^n y_i^2/a_i^2}{\left(\sum_{i=1}^n y_i^2/a_i^4\right)^{1/2}}, \quad y \in \mathbb{R}^n \setminus \{0\} \ .$$

The curvature function of M is

$$K(y) = \frac{1}{\prod_{i=1}^n a_i^2 \ \left(\sum_{i=1}^n y_i^2/a_i^4\right)^{(n+1)/2}}, \quad y \in M \ ,$$

As a homogeneous function of degree 0,

$$K(y) = \frac{1}{\prod_{i=1}^n a_i^2} \left\{ \frac{\sum_{i=1}^n y_i^2/a_i^2}{\sum_{i=1}^n y_i^2/a_i^4} \right\}^{(n+1)/2}, \quad y \in \mathbb{R}^n \setminus \{0\} \ . \tag{4.11}$$

By (4.10) and (4.11)

$$K \circ G^{-1}(x) = \frac{\left(\sum_{i=1}^n a_i^2 x_i^2\right)^{(n+1)/2}}{\prod_{i=1}^n a_i^2 \ ||x||^{n+1}}, \quad x \in \mathbb{R}^n \setminus \{0\} \ . \tag{4.12}$$

The homogeneous extension of $1/K \circ G^{-1}$ of degree $-(n+1)$,

$$\frac{1}{||x||^{n+1} \ K \circ G^{-1}(x)} = \frac{\prod_{i=1}^n a_i^2}{\left(\sum_{i=1}^n a_i^2 x_i^2\right)^{(n+1)/2}} \ , \tag{4.13}$$

defines the vector field $F(x) := (\prod_{i=1}^n a_i^2)\ x / (\sum_{i=1}^n a_i^2 x_i^2)^{(n+1)/2}$ that provides mean value representations.

The support dual M^* of M is the ellipsoid $\sum_{i=1}^n a_i^2 x_i^2 = 1$. Its Gauss map $y = G^*(x)$, $y \in \mathbb{S}^{n-1}$, is the transformation

$$y_i = \frac{a_i^2 x_i}{(\sum_{i=1}^n a_i^4 x_i^2)^{1/2}}, \quad i = 1, \ldots, n\ .$$

The inverse $x = (G^*)^{-1}(y)$ is given by

$$x_i = \frac{y_i}{a_i^2\ (\sum_{i=1}^n y_i^2/a_i^2)^{1/2}}, \quad i = 1, \ldots, n\ .$$

A similar consideration as above for M gives the following expressions for the support function of M^* :

$$h^*(y) = \left(\sum_{i=1}^n y_i^2/a_i^2 \right)^{1/2}, \quad y \in \mathbb{R}^n \setminus \{0\}$$

and

$$h^*_{M^*}(x) = \frac{\sum_{i=1}^n a_i^2 x_i^2}{(\sum_{i=1}^n a_i^4 x_i^2)^{1/2}}, \quad x \in \mathbb{R}^n \setminus \{0\}\ .$$

Also the curvature of M^*,

$$K^*(x) = \frac{\prod_{i=1}^n a_i^2}{(\sum_{i=1}^n a_i^4 x_i^2)^{(n+1)/2}}, \quad x \in M\ .$$

As a homogeneous function of degree 0,

$$K^*(x) = \prod_{i=1}^n a_i^2\ \left\{ \frac{\sum_{i=1}^n a_i^2 x_i^2}{\sum_{i=1}^n a_i^4 x_i^2} \right\}^{(n+1)/2}, \quad x \in \mathbb{R}^n \setminus \{0\}$$

and

$$K^* \circ (G^*)^{-1}(y) = \frac{\prod_{i=1}^n a_i^2\ \ (\sum_{i=1}^n y_i^2/a_i^2)^{(n+1)/2}}{||y||^{n+1}}, \quad y \in \mathbb{R}^n \setminus \{0\}\ .$$

Note that $h^*(y) = 1$ is the implicit equation of M and

$$\frac{K^*(x)}{h^*_{M^*}(x)^{n+1}} = \frac{\prod_{i=1}^n a_i^2}{(\sum_{i=1}^n a_i^2 x_i^2)^{(n+1)/2}} = \frac{1}{||x||^{n+1}\ K \circ G^{-1}(x)}$$

by (4.13) and

$$\frac{K(y)}{h_M(y)^{n+1}} = \frac{1}{\prod_{i=1}^n a_i^2\ \ (\sum_{i=1}^n y_i^2/a_i^2)^{(n+1)/2}} = \frac{1}{||y||^{n+1}\ K^* \circ (G^*)^{-1}(y)}$$

by duality .

4.4. **Discrete representations and generalized barycentric coordinates.** Since mean value representations hold for any piecewise smooth hypersurface in $\mathbb{R}^n$, they hold for any $(n-1)$-dimensional polyhedron, from which discrete representations and the corresponding generalized barycentric coordinates for points $v \in \mathbb{R}^n$ can be derived. Floater's coordinates for very general configurations of points have been constructed in [11] from vector fields defined by the curvature of the unit sphere. Here we give another illustration of the construction of barycentric type coordinates from the mean value representations by vector fields defined by the curvature of the ellipse $M : x^2/a^2+y^2/b^2 = 1$, $\mathbf{r} := (x, y) \in \mathbb{R}^2$. For notational convenience and consistency with many applications, here we use coordinate functions (x, y) for points and vectors in $\mathbb{R}^2$ and boldface letters to denote vectors. Further, it is also convenient to consider conservative fields, instead of divergence free ones.

The support function h_M and curvature K of M are

$$h_M(x,y) = \frac{b^2x^2 + a^2y^2}{(b^4x^2 + a^4y^2)^{1/2}}, \quad (x,y) \in \mathbb{R}^2 \setminus (0,0)$$

and

$$K(x,y) = \frac{ab(b^2x^2 + a^2y^2)^{3/2}}{(b^4x^2 + a^4y^2)^{3/2}}, \quad (x,y) \in \mathbb{R}^2 \setminus (0,0) \ ,$$

respectively. Hence,

$$\frac{K(x,y)}{h_M(x,y)^3} = \frac{ab}{(b^2x^2 + a^2y^2)^{3/2}} \ .$$

Instead of considering the divergence free vector fields $xK(x,y)\ (x,y)/h_M(x,y)^3$ and $yK(x,y)\ (x,y)/h_M(x,y)^3$, we consider the conservative fields

$$\begin{aligned} x\,\mathbf{F}(x,y) &:= \frac{x\ K(x,y)\ (-y,x)}{h_M(x,y)^3} = \frac{ab(-xy,x^2)}{(b^2x^2 + a^2y^2)^{3/2}}, \\ y\,\mathbf{F}(x,y) &:= \frac{y\ K(x,y)\ (-y,x)}{h_M(x,y)^3} = \frac{ab(-y^2,xy)}{(b^2x^2 + a^2y^2)^{3/2}} \end{aligned}$$

and the corresponding line integrals. The potential functions of $x\mathbf{F}(x,y)$ and $y\mathbf{F}(x,y)$ are

$$\begin{aligned} \phi_1(x,y) &:= \frac{ay}{b(b^2x + a^2y^2)^{1/2}} = \frac{ay}{b||\tilde{\mathbf{r}}||} \ , \\ \phi_2(x,y) &:= \frac{-bx}{a(b^2x + a^2y^2)^{1/2}} = \frac{-bx}{a||\tilde{\mathbf{r}}||} \ , \end{aligned}$$

respectively, where $||\tilde{\mathbf{r}}|| := (x^2/a^2 + y^2/b^2)^{1/2}$.

Let C be a piecewise smooth closed curve in $\mathbb{R}^2$. In particular, C may be the ellipse M. Let $\mathbf{v}_j = (x_j, y_j)$, $j = 0, 1, \ldots, n-1$, be points on C arranged in the positive direction,

$\tilde{\mathbf{v}}_j := (x_j/a, y_j/b)$ and let C_j, $j = 0, \ldots, n-1$, be the arc of C from $\mathbf{v}_j$ to $\mathbf{v}_{j+1}$. Then

$$\begin{aligned} (0,0) &= \int_C \mathbf{r}\ \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} \\ &= \sum_{j=0}^{n-1} \int_{C_j} \mathbf{r}\ \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} \\ &= \sum_{j=0}^{n-1} \left(\int_{C_j} x\ \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}, \int_{C_j} y\ \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} \right) \\ &= \sum_{j=0}^{n-1} (\phi_1(\mathbf{v}_{j+1}) - \phi_1(\mathbf{v}_j), \phi_2(\mathbf{v}_{j+1}) - \phi_2(\mathbf{v}_j)) \\ &= \sum_{j=0}^{n-1} \frac{(y_{j+1}/b^2\ , -x_{j+1}/a^2)}{||\tilde{\mathbf{v}}_{j+1}||} - \frac{(y_j/b^2\ , -x_j/a^2)}{||\tilde{\mathbf{v}}_j||}\ . \end{aligned} \tag{4.14}$$

A straightforward computation gives

$$\frac{(y_{j+1}/b^2\ , -x_{j+1}/a^2)}{||\tilde{\mathbf{v}}_{j+1}||} - \frac{(y_j/b^2\ , -x_j/a^2)}{||\tilde{\mathbf{v}}_j||} = p_j \mathbf{v}_j + q_j \mathbf{v}_{j+1}\ , \tag{4.15}$$

where

$$p_j = \frac{||\tilde{\mathbf{v}}_{j+1}||\ (1 - \cos\alpha_j)}{ab\ ||\tilde{\mathbf{v}}_j \times \tilde{\mathbf{v}}_{j+1}||} = \frac{\tan(\alpha_j/2)}{ab\ ||\tilde{\mathbf{v}}_j||}\ , \tag{4.16}$$

$$q_j = \frac{||\tilde{\mathbf{v}}_j||\ (1 - \cos\alpha_j)}{ab\ ||\tilde{\mathbf{v}}_j \times \tilde{\mathbf{v}}_{j+1}||} = \frac{\tan(\alpha_j/2)}{ab\ ||\tilde{\mathbf{v}}_{j+1}||} \tag{4.17}$$

and α_j is the angle at $\mathbf{0}$, of magnitude $< \pi$, of the oriented triangle $[\mathbf{0}, \mathbf{v}_j, \mathbf{v}_{j+1}]$ taking a positive value if and only if the orientation of the triangle induced by the direction from $\mathbf{v}_j$ to $\mathbf{v}_{j+1}$ is anticlockwise.

Equations (4.14), (4.15), (4.16) (4.17) give

$$\begin{aligned} (0,0) &= \sum_{j=0}^{n-1} \frac{\tan(\alpha_j/2)}{ab\ ||\tilde{\mathbf{v}}_j||}\ \mathbf{v}_j + \frac{\tan(\alpha_j/2)}{ab\ ||\tilde{\mathbf{v}}_{j+1}||}\ \mathbf{v}_{j+1} \\ &= \sum_{j=0}^{n-1} \frac{\tan(\alpha_{j-1}/2) + \tan(\alpha_j/2)}{ab\ ||\tilde{\mathbf{v}}_j||}\ \mathbf{v}_j \end{aligned}$$

and for any $\mathbf{v} \in \mathbb{R}^2$,

$$\sum_{j=0}^{n-1} \frac{\tan(\alpha_{j-1}/2) + \tan(\alpha_j/2)}{ab\ ||\tilde{\mathbf{v}}_j - \tilde{\mathbf{v}}||}\ (\mathbf{v}_j - \mathbf{v}) = 0\ ,$$

so that

$$\mathbf{v} = \sum_{j=0}^{n-1} \lambda_j\ \mathbf{v}_j, \quad \text{where} \quad \lambda_j := \frac{w_j}{\sum_{i=0}^{n-1} w_i}$$

and

$$w_j := \frac{\tan(\alpha_{j-1}/2) + \tan(\alpha_j/2)}{ab\ ||\tilde{\mathbf{v}}_j - \tilde{\mathbf{v}}||}, \quad j = 1, \ldots, n-1\ .$$

Here, α_j is the angle at $\mathbf{v}$ of the oriented triangle $[\mathbf{v}, \mathbf{v}_j, \mathbf{v}_{j+1}]$.

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Department of Mathematics, National University of Singapore, 10 Kent Ridge Road, Singapore 119260

E-mail address: matleesl@nus.edu.sg