Wavelets:
Motivation, Construction, & Application

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Overview

- Seeking the simple codes of complex images
- From vision, neurons, to wavelets
- Multiresolution framework of Mallat and Meyer
- Two key equations for Shape Function & Wavelet
- The fundamental theorem of Multiresolution
- 2-channel orthogonal & biorthogonal filter banks
- Application I. Sparse representation and compression
- Application II. Variational denoising of Besov images
- Some new trends of wavelets theory.

Behind complexity is simplicity

Examples:

- The universal path to chaos is *period doubling*.
- (Biology) ACTG encode the complexity of life.
- (Computer) “0” and “1” (or spin up and down for *Quantum Computers*) are the digital “seeds.”
- (Physics) The complexity of the material world is based on the limited number of basic particles.
- (Fractals) Simple algebraic rules hidden in complex shapes.

Conclusion:

Hidden in a complex phenomenon, is its simple evolutionary codes or building blocks.

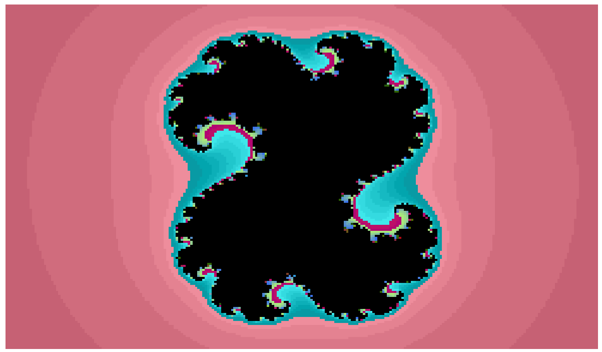
The complexity of image signals

Images:

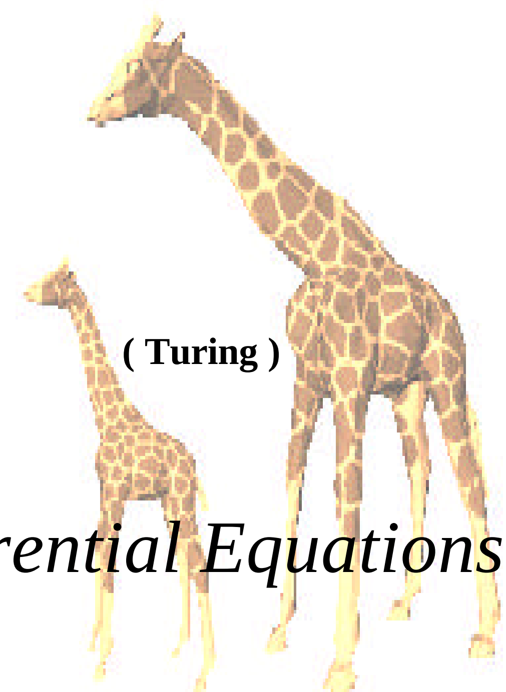
- Large dynamic range of scales.
- Often no good regularity as functions.
- Rich variations in intensity and color.
- Complex shapes and boundaries of “objects.”
- Noisy or blurred (astronomical or medical image).
- “The lost dimension” --- range is lost but depth is still crucial for image interpretation.

Searching for the hidden code of images (I)

- Fractals: by *Iterated Function Systems.*



(Mandelbrot)



(Turing)

- Pattern formation: via *Differential Equations*.

Searching for the hidden code of images (II)

Statistical modeling (Geman's, Mumford, Zhu, Yuille…):

- Image prior models (edge, regularity,...).
- Image data models (noise, blurring,...).
- Synthetic/generative models.

- Parametric methods, lattice models, Gibbs fields.
- Non-parametric methods & learning via the maximum entropy principle.

A representation, not an interpretation...

- Benoit Mandelbrot (interview on France-Culture):
  "The world around us is very complicated. The tools at our disposal to <u>*describe*</u> it are very weak."

- Yves Meyer (1993):
  "Wavelets, whether they are …, will not help us to *explain scientific facts*, but they serve to <u>*describe*</u> the reality around us, whether or not it is scientific."

- Thus, to represent a signal, is to find a good way to <u>*describe*</u> it, not to *explain* the underlying physical process that generates it.

General images

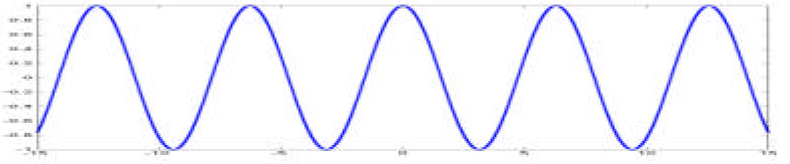
- Mostly no rigorous multi-scale self-similarity.
- Contain both man-made and natural “objects”
- Mostly no simple and universal underlying physical or biological processes that generate the patterns in a generic image.
- Thus, representation tools have to be universal.
- Then, how about *Fourier spectral representation*?

Fourier was born too early…

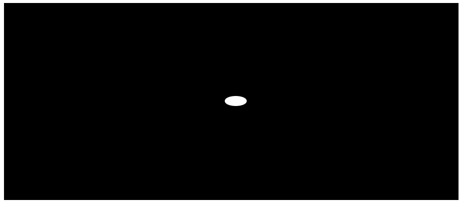
<u>Claim</u>: Harmonic waves are **bad** *vision neurons*…

Proof.

– A typical Fourier neuron is $f = \exp(iax)$.



– To "see" a simple bright spot $d(x)$ in the visual field,



all such neurons have to fire since

$$\langle d, f \rangle \equiv 1 .$$

Efficiency of representation

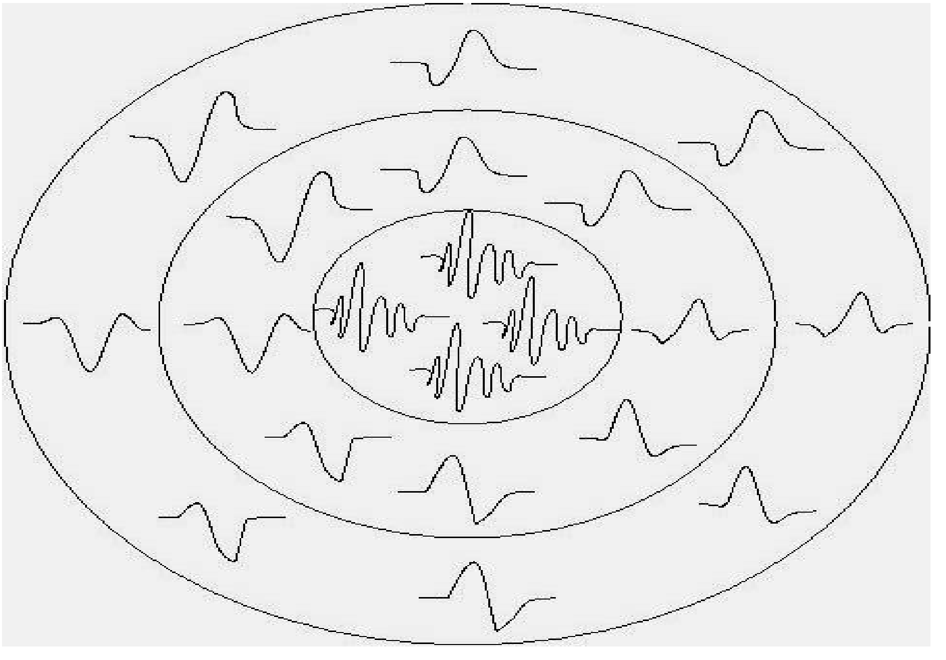
- Thus harmonic waves are not so efficient in coding visual information.

- David Field (Cornell U, Vision psychologist):

  "To discriminate between objects, an effective transform (representation) encodes each image using the *smallest* possible number of neurons, chosen from a *large pool*."

Asking our own "headtop"…

- Psychologists show that visual neurons are *spatially* organized, and each behaves like a small sensor (receptor) that can respond strongly to spatial changes such as edge contours of objects (*Fields*, *1990*).



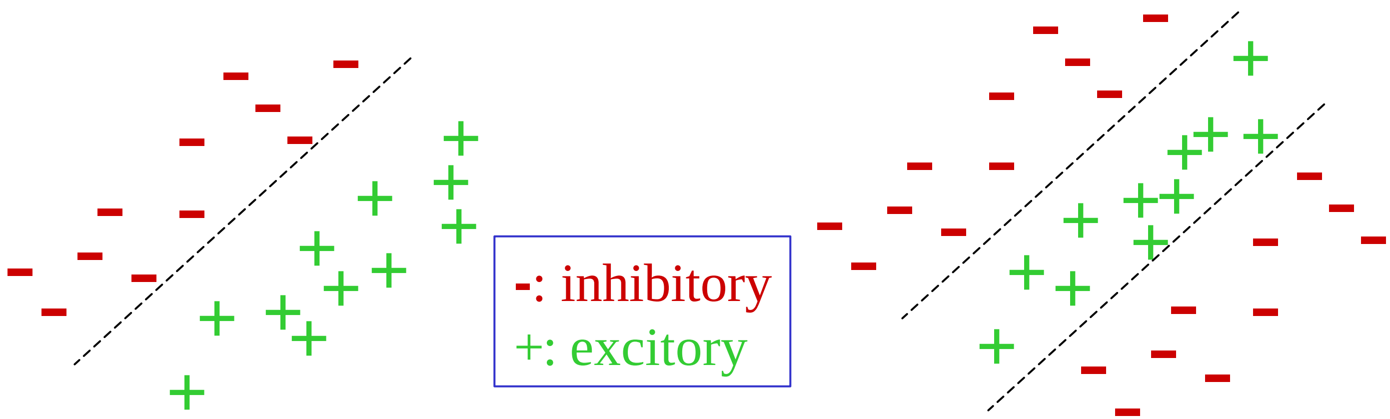
The discovery by Nobel Laureates

- Torsten Wiesel and David Hubel

  (**Nobel Prize in Physiology or Medicine, 1981**)

  for their major discoveries on the structure and functions of the visual system and pathways of vision neurons.

- Major discovery: simple cells and complex cells



A typical simple cell   complex cell

The Marr's edge neuron model

- Detection of edge contours is a critical ability of human vision (Marr, 1982).

- Marr and Hildreth (1980) proposed a model for human detection of edges at all scales. This is Marr's *Theory of Zero-Crossings*:

$$G_s = \exp\left(-\frac{x^2+y^2}{2s^2}\right),$$

$$\Psi_s = \Delta G_s = -\frac{2}{s^2}\left(1-\frac{x^2+y^2}{2s^2}\right)\exp\left(-\frac{x^2+y^2}{2s^2}\right),$$

Edge occurs in I where $(\Psi_s * I) = 0.$

Haar's average-difference coding

- Marr's *edge detector* is to use second derivative to *locate* the maxima of the first derivative (which the edge contours pass through).

- *Haar Basis* (1909) encodes (modern language :-) the edges into image representation via the first derivative operator (i.e. moving difference):

$$(\cdots x_{2n}, x_{2n+1}, \ldots) \leftrightarrow \left(\cdots a_n = \frac{x_{2n} + x_{2n+1}}{2}, d_n = \frac{x_{2n} - x_{2n+1}}{2}, \ldots \right)$$

A good representation should *respect* edges

- Edge is so important a feature in image and vision analysis.
- A good image representation (or basis) should be capable of providing the *edge* information easily.
- Edge is a *local* feature. Local operators like differentiation must be incorporated into the representation, as in the coding by the Haar basis.
- Wavelets improve Haar, while respecting the above principle of edge representation.

What to expect from a good representation?

- Mathematically *rigorous* (i.e. a clean and stable program exists for analysis/synthesis. FT & IFT…).
- Having nice *digital formulation* and computationally *efficient* (FFT, FWT…).
- Capturing the *characteristics* of the input signals, and thus many existing processing operators (e.g. image indexing, image searching …) are *directly* permitted on such representation.

Understanding images mathematically

- Let $\sum$ denote the collection of "all" images. What is the mathematical structure of Σ? Suppose that $f \in \Sigma$ is captured by a camera. Then Σ should be invariant under
  - Euclidean motion of the camera:

    $$f(x) \to f(Qx+a), \quad Q \in O(2), a \in R^2.$$

  - Flashing:

    $$f(x) \to mf(x), \qquad m \in R^+,$$

    or, more generally, a *morphological transform* ---

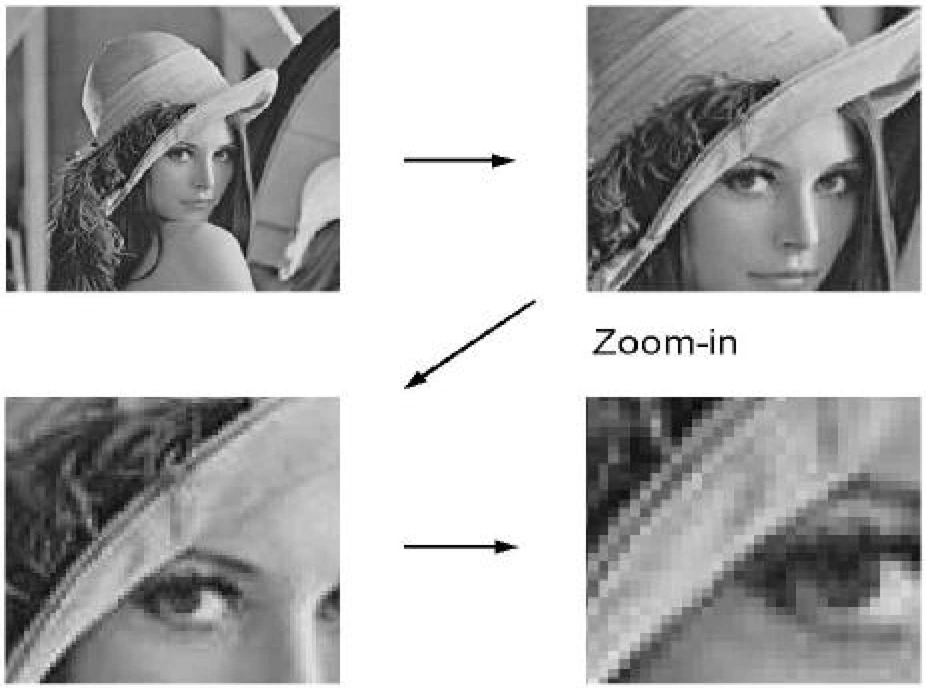
    $$f(x) \to h(f(x)), \quad h: R \to R, h' > 0.$$

  - Zooming:

    $$f(x) \to f(lx), \quad l \in R^+.$$

Let us focus on zooming

Zooming in 2-D



What is zooming?

- Zooming (aiming) center: *a*.

- Zooming scale: *h*.

- Zoom into the *h*-neighborhood at *a* in a given image *I*:

$$I_{a,h}(x) = I(a + h \cdot x), \quad x \in \Omega, \text{ the visual field;}$$

$$I_{a,h}\left(\tfrac{y-a}{h}\right) = I(y) \cdot 1_{\Omega}\left(\tfrac{y-a}{h}\right), \text{ the aperture.}$$

- Zooming is the most fundamental and characteristic operator for image analysis and visual communication. It reflects the ***multi-scale nature*** of images and vision.

The zooming neuron representation

- The zooming "**neuron**": $\psi(x)$.

- aiming (a) and zooming-in-or-out (h):

$$\psi_{a,h}(x) = \frac{1}{\sqrt{h}}\psi\left(\frac{x-a}{h}\right).$$

- Generating response (or neuron firing):

$$I_{a,h} = \langle I, \psi_{a,h} \rangle = \int I(x)\psi_{a,h}(x)\,dx.$$

- The zooming space: $(a, h) \in R \times R^{+}$.

A “good” neuron must be differentiating

- A *good* neuron should fire *strongly* to abrupt changes, and *weakly* to smooth domains (for purposes like efficient memory, object recognition, and so on).

- That means, for an uninteresting constant image I=c, the responses $I_{a,h}$ are all zeros:

$$I_{a,h} = \left\langle I, \psi_{a,h} \right\rangle \equiv 0 .$$

This is the “**differentiating**” property of the neuron, just like “*d/dx*”:

$$\int_R \psi(x) = 0 .$$

The continuous wavelet representation

Definition: (**Wavelets in broad sense**)

A *differentiating* zooming neuron $y(x)$ is said to be a (continuous) *wavelet*. Representing a given image $I(x)$ by *all* the neuron responses $I_{a,h} = \langle I, y_{a,h} \rangle$ is the corresponding *wavelet representation*.

Questions:

- Does there exist a "best" wavelet neuron $y(x)$?
- Does a wavelet representation allow **perfect reconstruction**?

Synthesizing a wavelet representation

- Goal: to recover *perfectly* an image signal I from its wavelet representation $I(a,h)$.

- (Continuous) Wavelet synthesis:

$$I(a,h) = \left\langle I, y_{a,h} \right\rangle = \left\langle \hat{I}, \hat{y}_{a,h} \right\rangle = \left\langle \hat{I}\,\overline{\hat{y}}(hx),\, e^{-iax} \right\rangle,$$

which is in the form of IFT. Thus $J(x,h) = \hat{I}\,\overline{\hat{y}}(hx)$ can be perfectly recovered via the a-FT of $I(a,h)$.

Then $\hat{I}$ can be perfectly recovered from J via

$$\hat{I}(x) = \int_{[0,\infty)} J(x,h)\hat{y}(hx)\,\frac{dh}{h}, \quad \int_0^\infty |\hat{y}(h)|^2\,\frac{dh}{h} = 1.$$

The admissibility condition & differentiation

- The *admissibility condition* of a continuous wavelet:

$$\int_0^{\infty} |\hat{y}(h)|^2 \frac{dh}{h} < \infty.$$

- A differentiating zooming neuron satisfies the AC since:

$$\hat{y}(0) = \int_R y(x)dx = 0, \text{ and } \hat{y}(h) = ch + o(h).$$

- Examples:
  - The Marr wavelet (Mexican-hat): second derivative of Gaussian.
  - The Shannon wavelet: $y(x) = 2\mathrm{sinc}(2x) - \mathrm{sinc}(x).$

The discrete set of zooming neurons

- Make a log-linear discretization to the scale parameter h:

$$j \to j = -\log_2 h_j = 0, \pm 1, \pm 2, \cdots .$$

- Make a *scale-adaptive* discretization of the zooming centers:

$$\text{at scale } h_j = 2^{-j} : k \to a_k = kh_j = k/2^j,$$

$$k = 0, \pm 1, \pm 2, \cdots .$$

- The discrete set of zooming neurons:

$$y_{j,k}(x) = \frac{1}{\sqrt{h_j}} y\left(\frac{x - kh_j}{h_j}\right) = 2^{j/2} y(2^j x - k).$$

The discrete wavelet representation

- The wavelet coefficients:

$$d_{j,k} = \left\langle I, y_{j,k} \right\rangle = 2^{j/2} \int_R I(x) y(2^j x - k) dx.$$

$$d_{j,k} = I_{2^{-j}k, 2^{-j}}, \text{ in terms of the continuous WT.}$$

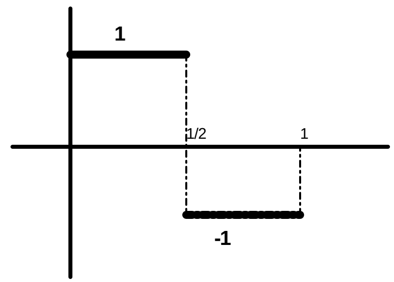
- <u>Questions:</u>
  - Does the set of all wavelet coefficients still encode the *complete* information of each input image I? Or equivalently,
  - Is the set of wavelets $\{y_{j,k}(x) : j, k \in Z\}$ a basis?

We don't know. But let's check out some examples...

Example 1: Haar wavelet

- The Haar "aperture" function is

$$\boxed{y^{\text{harr}}(x) = 1_{0 \le x < 1/2}(x) - 1_{1/2 \le x < 1}(x).}$$



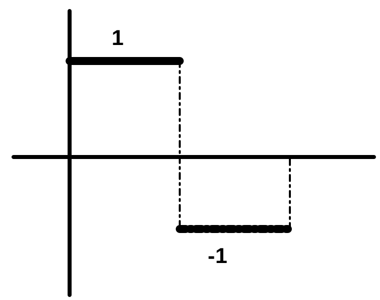
- Haar's theorem (1905):

All Haar wavelets $y_{j,k}^{\text{haar}}$ *, together with the constant function 1, consist into an* ***orthonormal basis*** *for the Hilbert space of all square integrable functions on* [*0, 1*].

Haar wavelets (cont'd)

- Haar's mother wavelet:

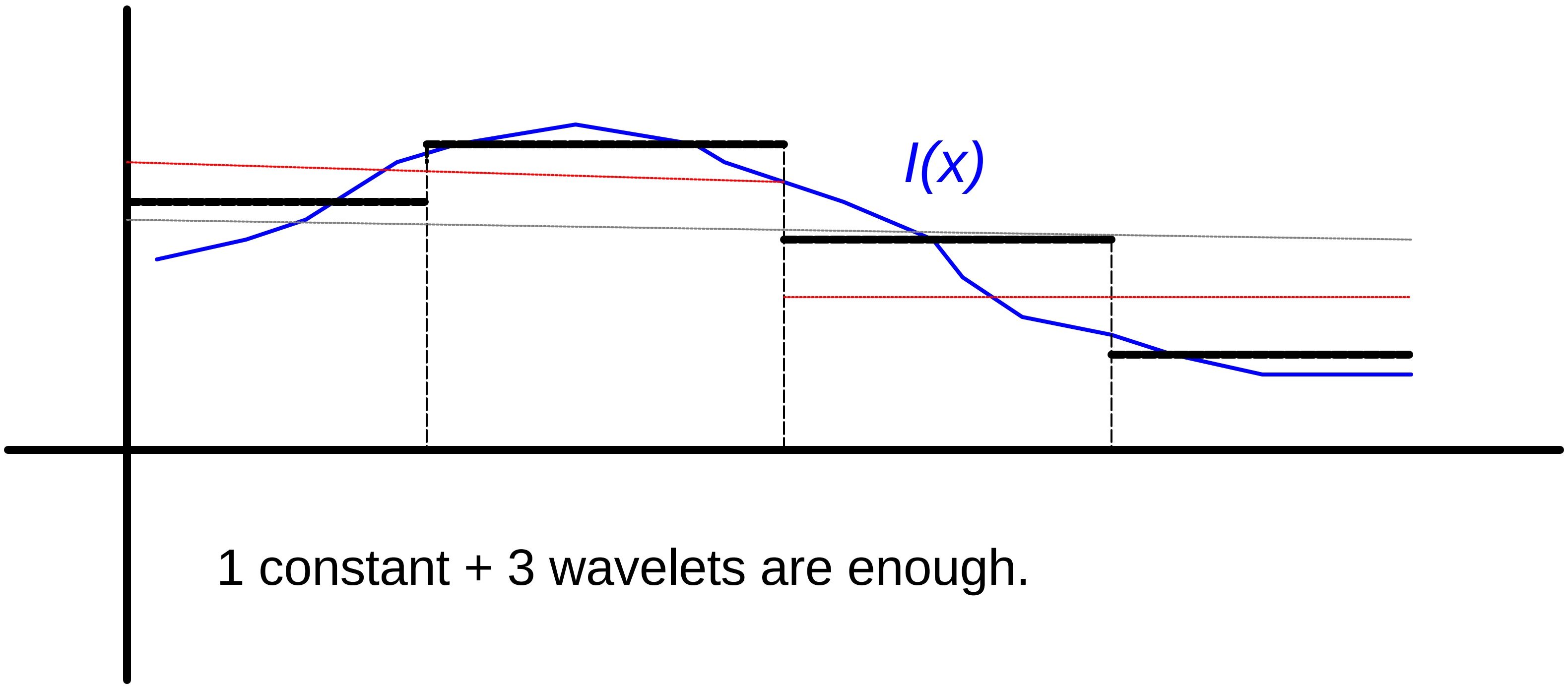
$$\boxed{y^{\text{Harr}}(x) = 1_{0 \le x < 1/2}(x) - 1_{1/2 \le x < 1}(x).}$$



- Why orthonormal basis?
  - Orthonormality is easy to see.
  - Completeness is due to the fact that:

    <u>All dyadically piecewise constant functions are dense in $L_2(0,1)$.</u>

Haar wavelets (cont'd)

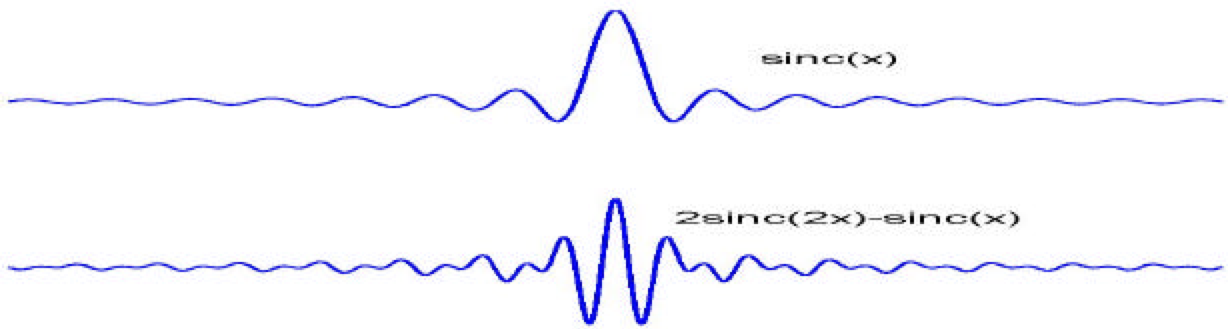


- Three Haar wavelets and the mean (constant) encode *all* the information of the piecewise constant approximation (i.e., 4 darker line segments).

Example 2: The Shannon wavelets

- The Shannon's "aperture" function is:

$$y^{\text{Shannon}}(x) = 2\,\text{sinc}\,(2x) - \text{sinc}\,(x).$$



- <u>Theorem</u>:

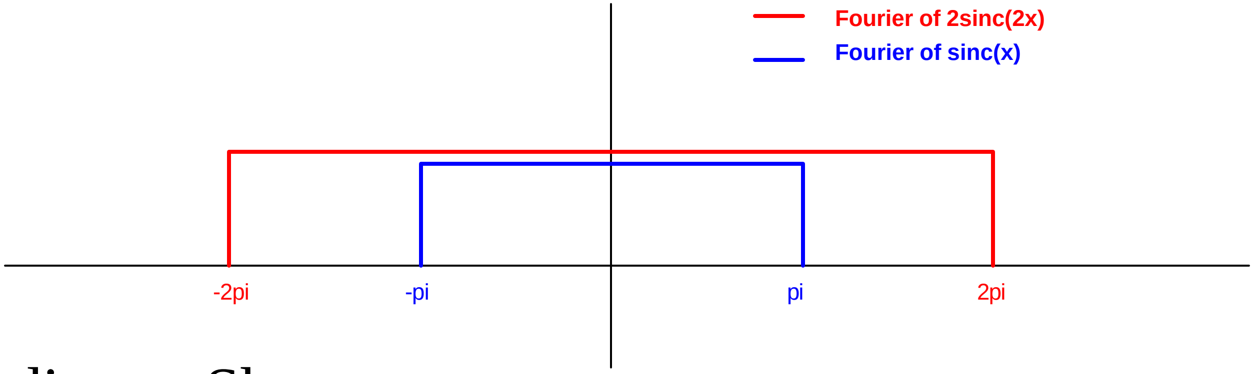
$\{y^{\text{Shannon}}_{j,k}(x) : j, k \in Z\}$ is an orthonormal basis of $L_2(\text{R})$.

Shannon wavelets (cont'd)

- How to visualize the orthonormal basis ?

  Answer: go to the Fourier domain !

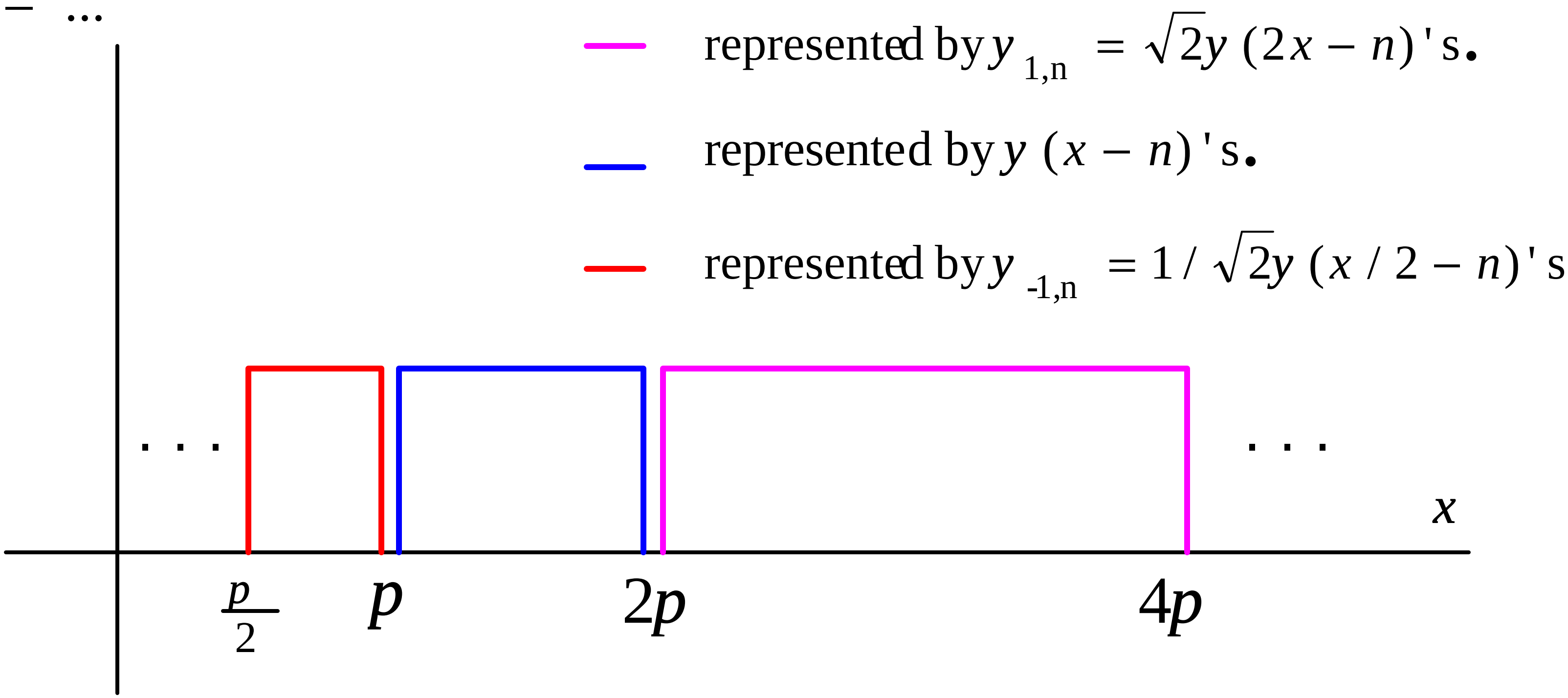
$$\psi^{\text{shannon}}(x) = 2\,\text{sinc}\,(2x) - \text{sinc}\,(x).$$



- According to Shannon:
  - All signals bandlimited to $(-\pi, \pi)$ can be represented by sinc(***x-n***)…
  - those bandlimited to $(-2\pi, \pi) \cup (\pi, 2\pi)$, by $\psi(x-n)$.
  - those bandlimited to $(-4\pi, 2\pi) \cup (2\pi, 4\pi)$, by $\psi_{1,n} = \sqrt{2}\psi(2x-n)$.
  - …

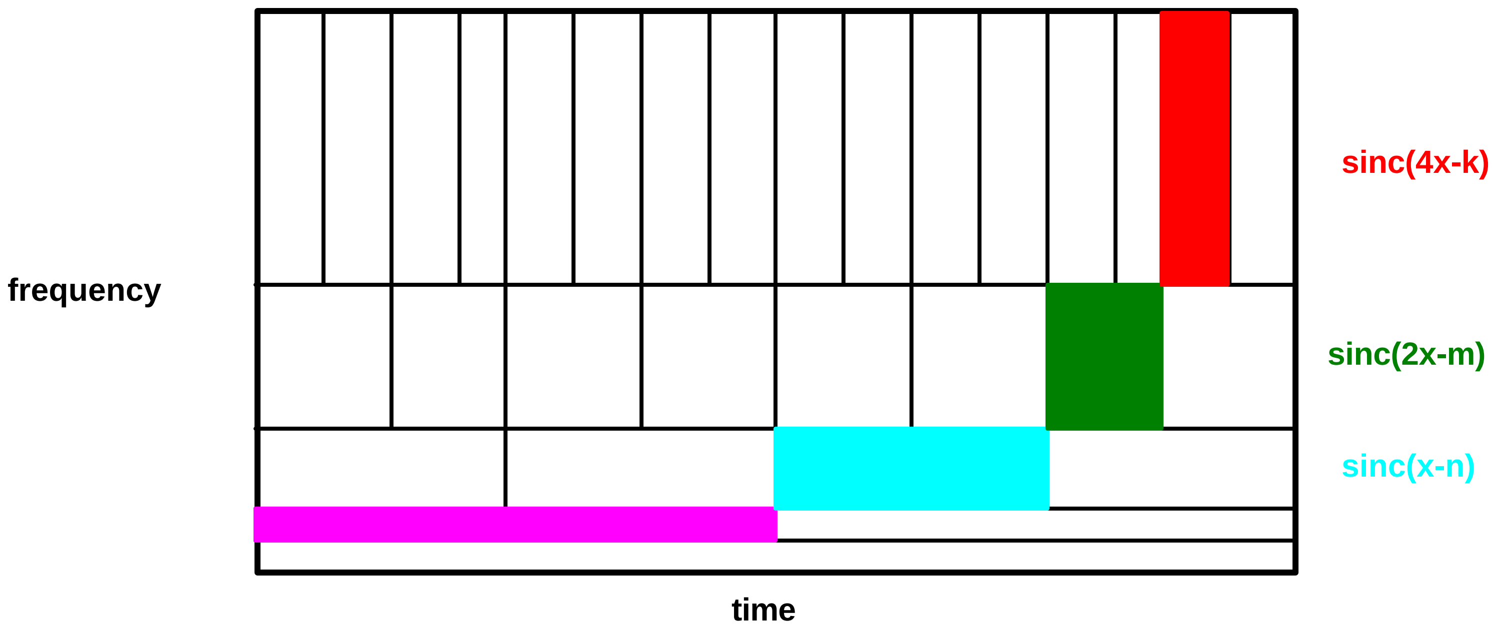
Shannon wavelets (cont'd)

- According to Shannon:
  - All signals bandlimited to $(-p\ ,p)$ can be represented by sinc(***x-n***)…
  - those bandlimited to $(-2p, -p)$ U $(p, 2p)$, by $y\ (x-n)$.
  - those bandlimited to $(-4p,-2p)$U$(2p,\ 4p)$, by $y_{1,n} = \sqrt{2}y\ (2x-n)$.
  - …



Partition of the time-frequency plane

- Heisenberg's uncertainty principle requires that each TF (time-frequency) atom must have: $\Delta t \cdot \Delta x \geq 2p$.
- Thus, for an *optimal* localization, the "life time" of an atom must influence its scale or frequency content.



General way of construction

Multiresolution analysis

Mallat and Meyer (1986):

An (orthogonal) multiresolution of $L_2(R)$ is a chain of closed subspaces indexed by all integers:

$$\cdots V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \cdots$$

subject to the following three conditions:

- (**completeness**) $\overline{\lim_{n\to\infty} V_n} = L_2(R), \qquad \lim_{n\to-\infty} V_n = \{0\}.$
- (**scale similarity**) $f(x) \in V_n \Leftrightarrow f(2x) \in V_{n+1}.$
- (**translation seed**) V_0 has an orthonormal basis consisting of all integral translates of a single function $f(x): \{f(x-n): n \in Z\}.$

Equations for designing MRA

- The refinement (dilation) equation for the “seed” function:

$$\boxed{f(x) = 2\sum_n h_n f(2x - n), \text{ for a suitable set of } h_n\text{'s.}}$$

This seed function is called: scaling function, shape fcn…

- Where is the wavelet?

Let W_0 denote the orthogonal complement of V_0 in V_1. Then W_0 is also orthogonally spanned by the integer translates of a single translation seed $y(x)$, the **wavelet**!

$$\boxed{y(x) = 2\sum_n g_n f(2x - n), \text{ for a suitable set of } g_n\text{'s.}}$$

Wavelets representation

Theorem:

$\{y_{j,k} = 2^{j/2} y\,(2^j x - k) : j,k \in Z\}$ is an orthonormal basis for L_2.

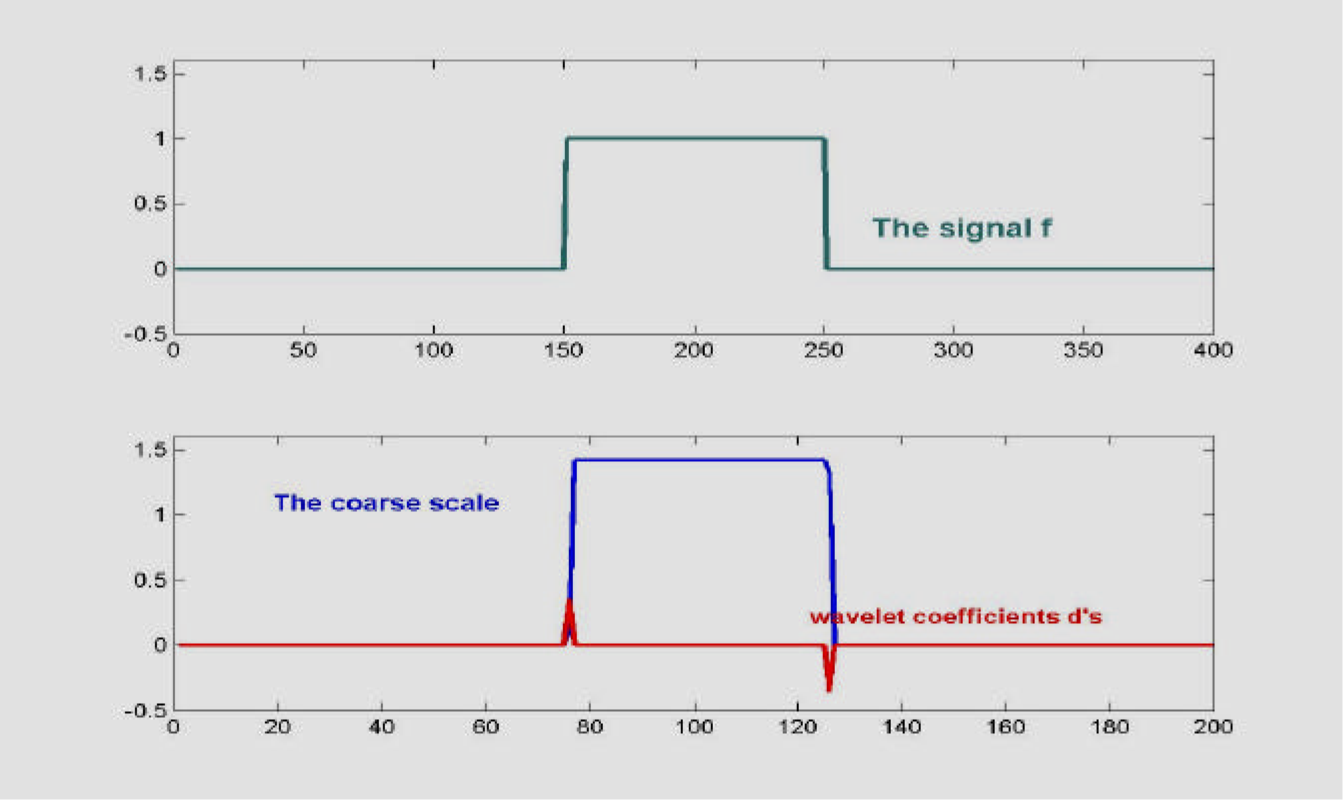
Wavelets representation of a signal:

$$\begin{tikzcd} I \sim I_j \in V_j \arrow[r, dash] \arrow[dr, dash, "+"] & I_{j-1} \in V_{j-1} \arrow[r, dash] \arrow[dr, dash, "+"] & {} & {} \arrow[r, dash] \arrow[dr, dash, "+"] & I_0 \in V_{0.} \\ & d_{j-1} \in W_{j-1} & d_{j-2} \cdots & & d_0 \in W_0. \end{tikzcd}$$

$$I_j = d_{j-1} + d_{j-2} + \cdots + d_0 + I_0 .$$

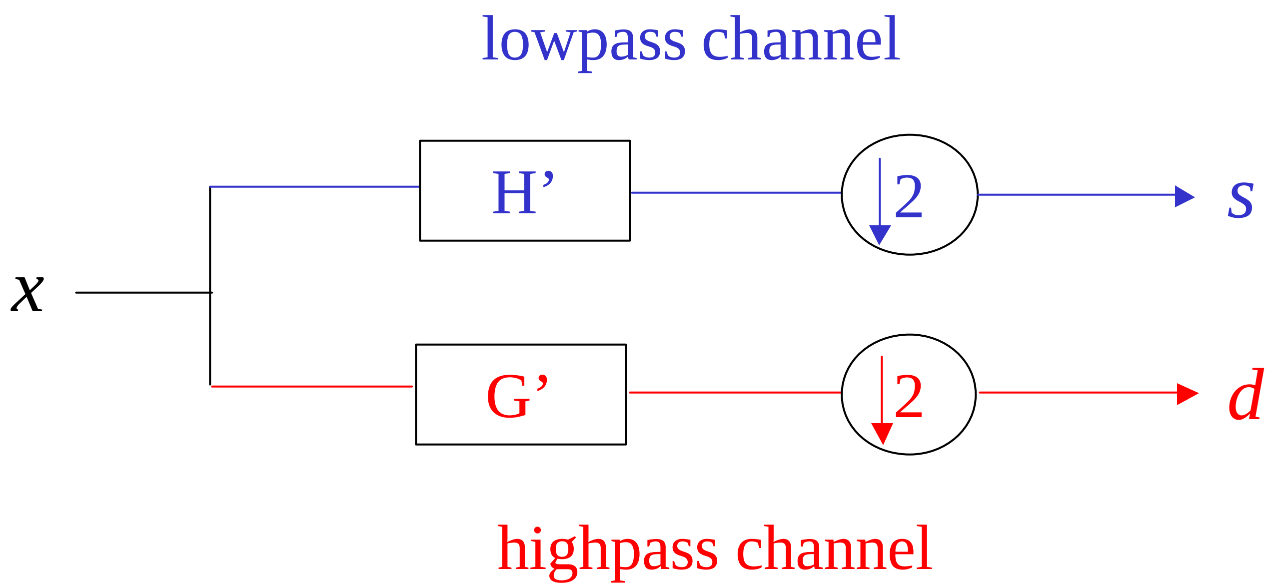
An example of wavelet decomposition

One level wavelet decomposition of a 1-D signal



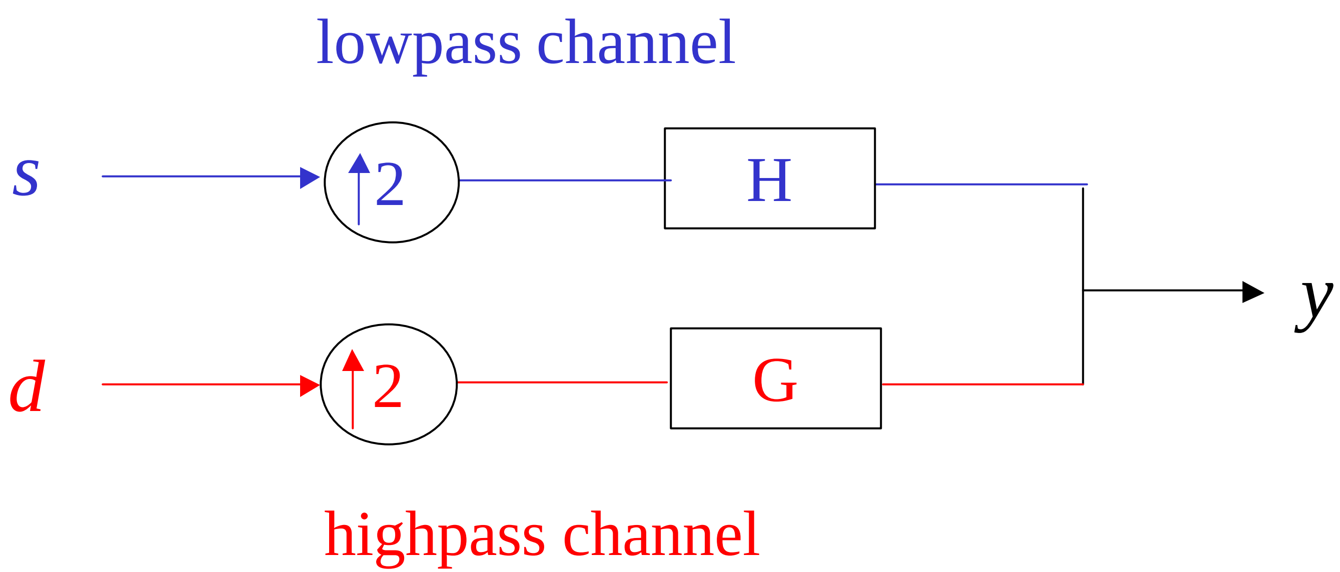
2-channel filter bank: Analysis bank

- H' is the lowpass filter and G' is the highpass filter.
- $\downarrow$2 is the downsampling operator: (1 3 4 6 5) $\longrightarrow$ (1 4 5).

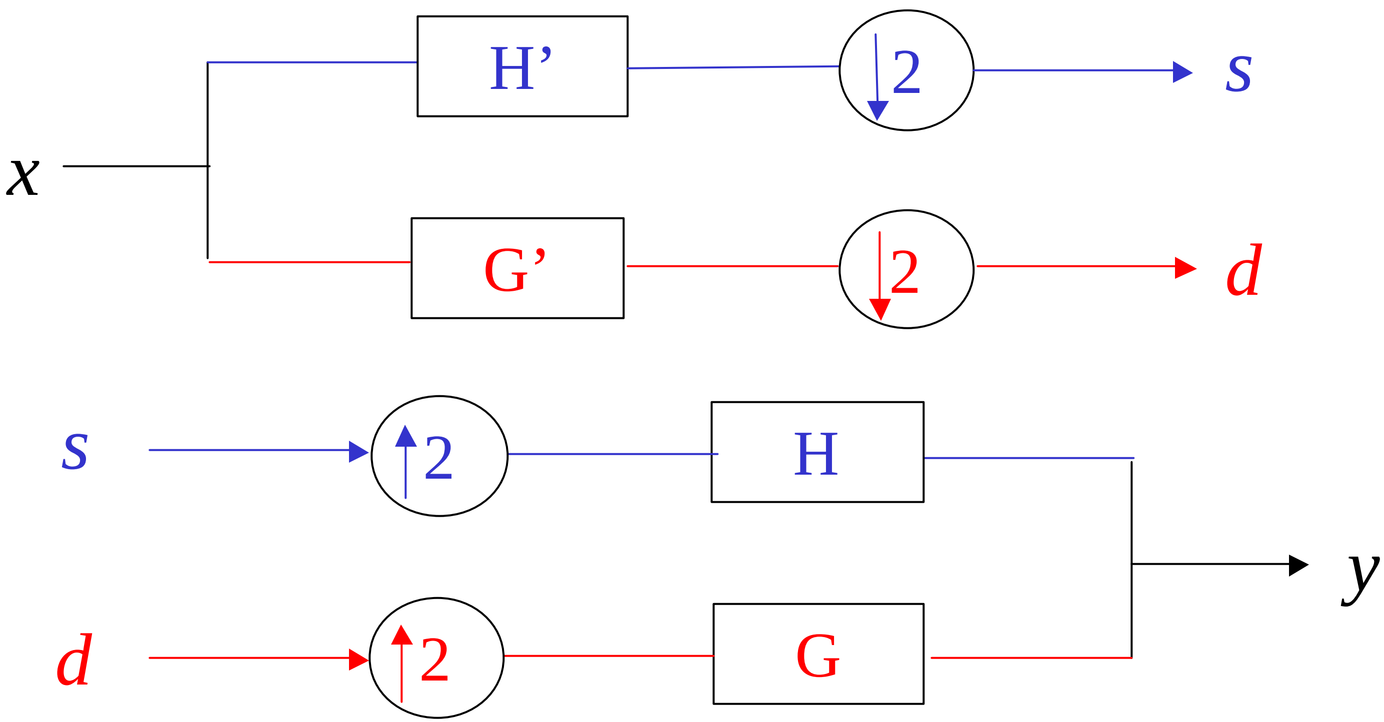


2-channel filter bank: Synthesis bank

- H is the lowpass filter and G is the highpass filter.
- $\uparrow$2 is the upsampling operator: (1 4 5) $\longrightarrow$ (1 0 4 0 5).

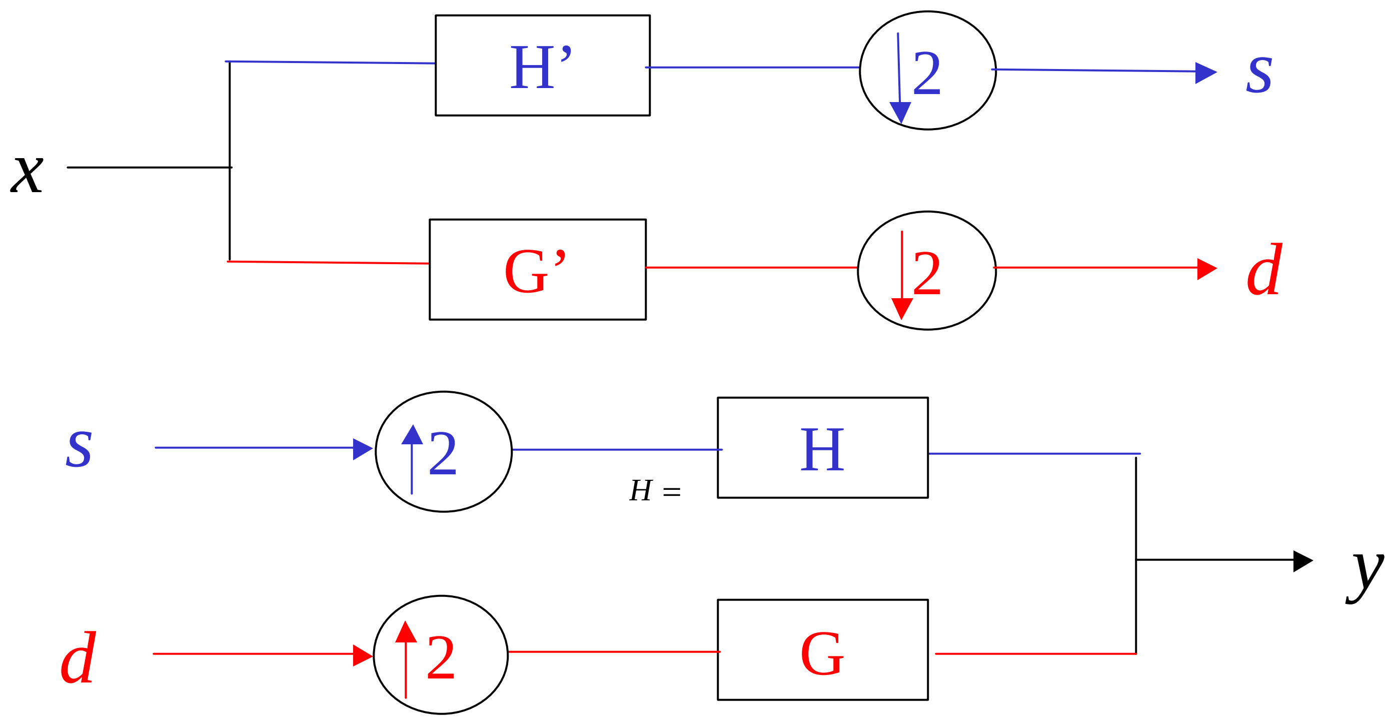


A *biorthogonal* filter bank



Biorthogonal (or perfect) filter bank: if $y=x$ for all inputs x .

An *orthogonal* filter bank



<u>Orthogonal filter bank</u>: if it is biorthogonal, and both *analysis* filters H' and G' are the *time reversals* of the *synthesis* filters H & G: H=(1, 2, 3) $\longrightarrow$ H'=(3, 2, 1).

The fundamental theorem of MRA

- An *orthogonal* Mallat-Meyer MRA corresponds to an *orthogonal* filter bank with the synthesis filters:

$$H = (h_n : n \in Z), \quad G = (g_n : n \in Z).$$

where, the h's and g's are the 2-scale *connection coefficients* in the dialation and wavelet equations:

$$f(x) = 2\sum_n h_n f(2x-n), \quad y(x) = 2\sum_n g_n f(2x-n).$$

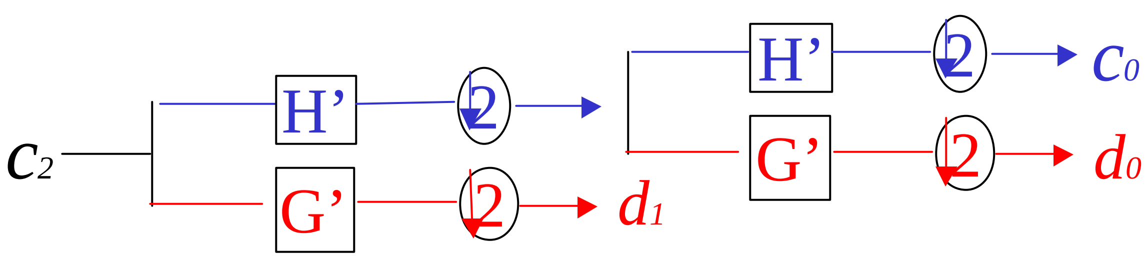
And, the *multiresolution* wavelet decomposition of f corresponds to the *iteration* of the analysis bank with the f-coefficients of f as the input digital data.

The fundamental theorem (cont'd)

$$\begin{array}{ccccccc}
I_j \in V_j & \longrightarrow & I_{j-1} \in V_{j-1} & \longrightarrow & & \longrightarrow & I_0 \in V_{0.} \\
& \searrow + & & \searrow + & & \searrow + & \\
& & d_{j-1} \in W_{j-1} & & d_{j-2} \cdots & & d_0 \in W_0.
\end{array}$$

$$I_j = d_{j-1} + d_{j-2} + \cdots + d_0 + I_0 .$$

Suppose j=2, and $I_2 = \sum_k c_2(k) f_{2,k}(x).$



Besov images and multiscale control

- At each scale h, the p-modulus of continuity is

$$v_p(u,h) = \sup_{|a|\le h} \| u(x+a) - u(x) \|_{L^p}$$

- Cross-scale control via the homogeneous Besov norm

$$\|u\|_{\dot{B}^{\alpha}_{q}(L^p)} = \left(\int_0^\infty \left(\frac{v_p(u,h)}{h^\alpha} \right)^q \frac{dh}{h} \right)^{1/q}$$

- The meaning of α, p, q:
  - α : smoothness index (α <1, otherwise use high order FD)
  - p : intra-scale control index
  - q : inter-scale control index

Wavelets as building blocks of Besov Images

For an image u with wavelets representation

$$u = \sum_{j\ge 0,k} d_{j,k} y_{j,k} + \sum_{k} c_{0,k} j_{0,k} \Rightarrow \sum_{j\ge -1,k} d_{j,k} y_{j,k}$$

inhomogeneous Besov norm can be simply characterized via wavelets coeff.

$$\|u\|_{B^a_q(L^p)} \cong \left(\sum_{j\ge -1} h_j^{-q(a+1/2-1/p)} \|d_{j,\bullet}\|_{l^p}^q\right)^{1/q} \qquad h_j = 2^{-j}, \; h_{-1} = 1.$$

[in 2-D, replace (1/2-1/p) by 2(1/2-1/p)]

When $p=q$ (resonance), intra-scale correlation is decoupled

$$\|u\|^p_{B^a_p(L^p)} \cong \sum_{j\ge -1,k} h_j^{-p(a+1/2-1/p)} |d_{j,k}|^p$$

Linear compression: scale truncation

A linear compressed reconstruction T has to take the form of

$$T(u) = \sum_{j\geq -1,k} T(d_{j,k}\psi_{j,k}) \xrightarrow[\text{coeff .wise}]{} \sum_{j\geq -1,k} t_{j,k}(d_{j,k})\psi_{j,k}$$

where *t* is a univariate linear function [t..(d) =d t..(1)]

Compression via scale truncation is: $t_{..}(d) = 0$, if $j > J$; d, otherwise :

$$T(u) \Rightarrow u_J = E_J u = \text{projection of } u \text{ on } V_J$$

Suppose the target image u belongs to $B_2^a(L^2)$

$$\|u - u_J\|_{L^2}^2 = \sum_{j>J,k} |d_{j,k}|^2 \leq h_J^{2a} \sum_{j>J,k} h_j^{-2a} |d_{j,k}|^2$$
$$\leq h_J^{2a} \|u\|_{B_2^a(L^2)}^2, \qquad h_J = 2^{-J}.$$

Evaluation: Not so ideal if the image features concentrate on scales finer than J.

Keywords: Be adaptive, or data driven !!!

Nonlinear compression of images in $B_p^a(L^p)$ $\frac{a}{d}+\frac{1}{2}=\frac{1}{p}$

Step I. Order the wavelet coefficients by their significance (magnitude)

$$\{d_{j,k} : j \geq -1, k\} = \{a_1, a_2 \ldots\} \quad |a_1| \geq |a_2| \geq \ldots$$

Step II. Only keep the N largest terms, dump the rest, and reconstruct.

$$u^N(x) = \sum_{n=1}^{N} a_n g_n(x), \qquad g_n = \psi_{j,k} \text{ if } a_n = d_{j,k}.$$

Evaluation of reconstruction accuracy for images in $B_p^a(L^p)$ $\frac{a}{d}+\frac{1}{2}=\frac{1}{p}$

$$N\,|\,l_N\,|^p \leq \sum_{n=1}^{N} |a_n|^p \leq \sum_{j\geq -1,k} |d_{j,k}|^p = \|u\|^p_{B_p^a(L^p)}$$

$$\|u-u^N\|_2^2 = \sum_{n>N} |a_n|^2 \leq l_N^{2-p} \sum_{n>N} |a_n|^p$$
$$\leq l_N^{2-p} \|u\|^p_{B_p^a(L^p)} \leq N^{-2a} \|u\|^p_{B_p^a(L^p)}.$$

(in 2-D, d=2; then $-a$ instead of $-2a$)

Pro and Con: procedure is data driven, but *N* is still not. Remedy: Learning Theory

Signal and image denoising

Noising process u (clean image) → u_0 (noisy image)

$$\boxed{u_0 = u \otimes n \xrightarrow{\ additive\ } u_0 = u + n}$$

Why noise: (a) ubiquitous (thermal fluctuation/noise)
(b) $1/f$ noise in many areas (fractal, dynamic systems, etc)
(c) very useful (instead of being annoying) in EE/system/signal

Denoising process: $u_0 \to u$. Challenge and Approach:

- ill-posed inverse problem
- prior knowledge on u is crucial (Bayesian Methodology)
- Deterministic priors: Sobolev, BV, Besov…

Denoising of Besov images

(Chambolle, DeVore, Lee, Lucier, 1998)

Basic assumption: the target image u belongs to $B^a_q(L^p)$

Variational denoising scheme is to solve the optimization problem:

$$\boxed{\min \quad \|u_0\|^q_{B^a_q(L^p)} + \frac{l}{2}\|u - u_0\|^2_{L^2}}$$

Tikhonov Language: [Regularization] [Least Square Fitting]

Bayesian Language: [Prior Knowledge] [Gaussian Noise/Data Model]

The origin of soft thresholding

Consider for example, the denoising of Besov images in $\boxed{B_1^a(L^1)}$

The previous variational formulation allows clean wavelet representation

$$\min_{d_{j,k}\text{'s}} \underbrace{\tfrac{l}{2}\sum_{j\ge -1,k}(d^0_{j,k}-d_{j,k})^2}_{\text{least square fitting}} + \underbrace{\sum_{j\ge -1,k} h_j\,|d_{j,k}|}_{\text{Besov prior/regularity}}, \quad h_j = 2^{j(a-1/2)}$$

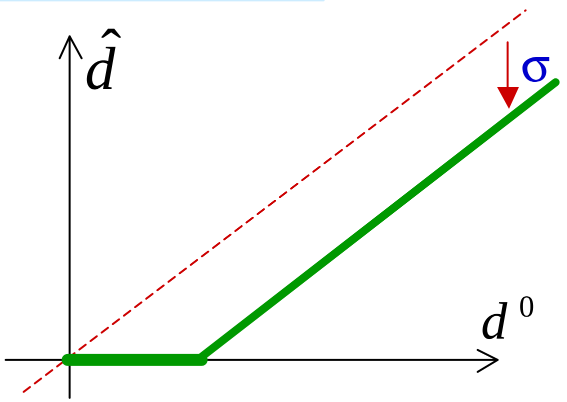
p=q=1 → allows a perfect decoupling: reduction to singleton optimization

$$\boxed{\hat{d} = \arg\min_d \tfrac{l}{2}(d^0-d)^2 + h\,|d|, \qquad d\in R.}$$

Leave you a simple homework assignment

$$\hat{d} = S_s(d^0) = \operatorname{sign}(d^0)(|d^0|-s)_+$$

$$x_+ = \frac{|x|+x}{2} \qquad s = \frac{h}{l}.$$



More about soft thresholding

- For Besov images, soft thresholding or hard truncation provides near optimal solutions to the variational cost function.
- The above variational approach to thresholding and truncation belongs to Ron DeVore's school **(Lucier, Jawerth, Lee, Chambolle, etc).**
- Soft thresholding technique was initially discovered and proposed by Donoho and Johnstone (**1994, 1995**), in the context of statistical estimation theory via wavelets (via oracles, uniform shrinkage, and near optimal minimax estimation, etc.).
- The above variational approach is convenient for this tutorial, and is directly connected to the two tutorial talks to come by Professor Tony Chan (in terms of framework and spirit).

More applications

- FBI fingerprints.
- JPEG2000.
- Image indexing and image search engines (for databank).
- Image modeling (such as MRF on the wavelets domain).
- Image restorations.
- Texture analysis and synthesis.
- Direct processing tools on the wavelets domain.
- Algorithm speeding up based on multi-resolution rep..
- Time series analysis.
- A lot of others ...

New trends of wavelets

- Random Wavelets Expansion (RWE) by Mumford-Gidas **[2001],** to model the scale-invariance of general images.

- Geometric Wavelets:

  - D. Donoho's school: ridgelets, wedgelets, beamlets, curvelets.

  - Mallat and Pennec [2000]: bandlets.

  - T. Chan & H.-M. Zhou [2000], A. Cohen [2002]: integrate computational PDE techniques such as the ENO scheme into wavelet transforms, to better capture shocks (discontinuities).

That is all, folks…

Thank you for your patience!