

# The Bramble-Hilbert Lemma and Whitney estimates for Convex Domains: Applications to multivariate piecewise polynomial approximation

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## Abstract

The Bramble-Hilbert lemma is a fundamental result on multivariate polynomial approximation. It is frequently applied in the analysis of Finite Element Methods (FEM) used for numerical solutions of PDEs. However, this classical estimate depends on the geometry of the domain and may ‘blow-up’ for simple examples such as a sequence of triangles of equivalent diameter that become thinner and thinner. Thus, in FEM applications one usually requires that the mesh has ‘quasi-uniform’ geometry. This assumption is too restrictive when one tries to obtain estimates of nonlinear approximation methods that use piecewise polynomials. We show that it is possible to obtain estimates where the constant is independent of the geometry of the domain.

We also prove the following Whitney estimate. Given  $0 < p \leq \infty$ ,  $r \in \mathbb{N}$ , and  $d \geq 1$ , there exists a constant  $C(d, r, p)$ , depending only on the three parameters, such that for every bounded convex domain  $\Omega \subset \mathbb{R}^d$ , and each function  $f \in L_p(\Omega)$ ,

$$E_{r-1}(f, \Omega)_p \leq C(d, r, p)\omega_r(f, \text{diam}(\Omega))_p,$$

where  $E_{r-1}(f, \Omega)_p$  is the degree of approximation by polynomials of total degree  $r - 1$  and  $\omega_r(f, \cdot)_p$  is the modulus of smoothness of order  $r$ . again, estimates like this can be found in the literature, but with constants that depend in an essential way on the geometry of the domain, in particular the domain is assumed to be a Lipschitz domain and the above constant  $C$  depends on the minimal head-angle of the cones associated with the boundary.

The estimates we obtain allow us to characterize nonlinear multivariate approximation by piecewise polynomials on families of nested triangulations of  $\mathbb{R}^d$  into simplices without having to pay attention to how slim they may become. We also apply our results to a new family of what we call geometric wavelets.

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