Approximation power of refinable spaces

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Outline

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Refinement equations

Vector refinement equation

$$\widehat{\Phi}(2\cdot) = P\widehat{\Phi}.$$

- P, a square matrix-valued 2π-periodic measurable function, is a refinement (matrix) mask,
- Φ, a solution, is a refinable vector.

The space of all tempered distributional solutions Φ is generally infinite-dimensional, but

$R(P) :=$ the space of compactly supported solutions

is always finite-dimensional.

Compactly supported solutions

Result [Jia, Jiang, Shen]. P a trig. polynomial,

$$N := \max\{n : 2^n \in \sigma(P(0))\},$$
$$\mathcal{Z}_N := \{\alpha \in \mathbb{Z}_+^d : |\alpha| \le N\}.$$

The map

$$\Phi \mapsto ((D^\alpha \widehat{\Phi})(0))_{\alpha \in \mathcal{Z}_N}$$

is then a bijection between the space $R(P)$ and the kernel $\ker L$ of the map

$$L : \mathbb{C}^r \times \mathcal{Z}_N \to \mathbb{C}^r \times \mathcal{Z}_N :$$
$$(w_\alpha) \mapsto (2^{|\alpha|} w_\alpha - \sum_{0 \le \beta \le \alpha} (D^{\alpha-\beta} P)(0)\, w_\beta),\ \alpha \in \mathcal{Z}_N.$$

More about the space $R(P)$

Theorem. Suppose there are matrices T and $\tilde{P}$ s.t.

(i) T is analytic and invertible around the origin,

(ii) $\tilde{P}$ is a trig. polynomial,

(iii) $T(2\cdot)P - \tilde{P}T = O(|\cdot|^{N+1})$,

(iv) $\tilde{P}$ is block diagonal to order $N+1$ around 0 and the spectrum of each block evaluated at zero intersects the set $\{2^j : \ j = 0, \ldots, N\}$ at ≤ 1 point.

Let Φ be in $R(P)$, and assume that each entry of $\widehat{\Phi}$ has a zero of order l at the origin. Then

$$\Phi = \sum_{j=l}^{N} p_j(D)\Phi_j, \quad \Phi_j \in R(P/2^j), \ \widehat{\Phi}_j(0) \neq 0,$$

and p_j a homogeneous polynomial of degree j, $j = l, \ldots, N$.

More about the space $R(P)$

Fact. The ‘layer’ decomposition of the previous theorem may not be possible.

Example. Let $d = 2$ and let P be s.t.

$$P(0) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & -1 & 4 \end{bmatrix},$$

$$(D^{(0,1)}P)(0) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$(D^{(1,0)}P)(0) = 0.$$

SI spaces

F a space of functions over $\mathbb{R}^d$. $S \subset F$ is a shift-invariant (SI) space if

$$f \in S \Longrightarrow f(\cdot - \alpha) \in S, \quad \text{all } \alpha \in (h)\mathbb{Z}^d.$$

- A principal shift-invariant (PSI) space S_ϕ is the closure of

$$\text{span}[\phi(\cdot - j) : j \in \mathbb{Z}^d]$$

in the topology of F.

- A finitely generated shift-invariant (FSI) space S_Φ is the closure of

$$\sum_{\phi\in\Phi} S_\phi$$

in F, with Φ a finite subset of F.

Approximation order

Sobolev space $W_2^s(\mathbb{R}^d)$: tempered distributions f with $\widehat{f}$ locally in $L_2(\mathbb{R}^d)$ and

$$\|f\|^2_{W_2^s(\mathbb{R}^d)} := \int_{\mathbb{R}^d} (1+|\cdot|)^{2s} |\widehat{f}|^2 < \infty.$$

A ladder $\mathcal{S} := (S^h := S^h(W_2^s))_{h>0}$ of SI spaces provides approximation order k, $k > s$, in $W_2^s(\mathbb{R}^d)$ if, for every $f \in W_2^k(\mathbb{R}^d)$,

$$\operatorname{dist}_s(f, S^h) := \inf_{g \in S^h} \|f - g\|_{W_2^s(\mathbb{R}^d)} \leq C h^{k-s} \|f\|_{W_2^k(\mathbb{R}^d)},$$

with constant C independent of f and h.

Characterization of appr. order

Theorem. An FSI stationary ladder $(S^h := S^h(W_2^s))$, with $S^h = S_\Phi(\cdot/h)$, $\Phi \subset W_2^s$, provides approximation order $k > 0$ if and only if there exists a neighborhood Ω of 0 such that the function

$$\mathcal{M}_{\Phi,s} : \omega \mapsto \frac{1}{|\omega|^{2k-2s}} \inf_{v \in \mathbb{C}^\Phi} \frac{v^* G^0_{\Phi,s}(\omega) v}{v^* G_{\Phi,s}(\omega) v}$$

lies in $L_\infty(\Omega)$. Here
$G_{\Phi,s} := \sum_{\alpha \in 2\pi\mathbb{Z}^d} \widehat{\Phi}(\cdot + \alpha)\widehat{\Phi}^*(\cdot + \alpha)|\cdot + \alpha|^{2s}$,
$G^0_{\Phi,s} := \sum_{\alpha \in 2\pi\mathbb{Z}^d \backslash 0} \widehat{\Phi}(\cdot + \alpha)\widehat{\Phi}^*(\cdot + \alpha)|\cdot + \alpha|^{2s}$
(the Gramian and the truncated Gramian).

Superfunctions

$\psi \in S$ is a superfunction for S if

$$\text{appr.order}\,(S_\psi) = \text{appr.order}\,(S).$$

Theorem. Any FSI space $S_\Phi \subset W_2^s(\mathbb{R}^d)$ contains a superfunction.

A superfunction for an FSI space S_Φ is good if it is nondegenerate:

$$|\widehat{\psi}| \text{ is bounded away from } 0 \text{ in a nbhd of } 0,$$

and finitely spanned by the shifts of Φ:

$$\widehat{\psi} = \tau^* \widehat{\Phi}, \text{ with } \tau \text{ a trigonometric polynomial.}$$

Strang-Fix conditions

Theorem. If $S_\phi \subset W_2^s(\mathbb{R}^d)$ provides approximation order k, then

$$\widehat{\phi}(\cdot + \alpha) = O(|\cdot|^k), \quad \text{all} \quad \alpha \in 2\pi\mathbb{Z}^d \backslash 0.$$

Polynomial reproduction

Theorem. If $S_\phi \subset W_2^s(\mathbb{R}^d)$ provides appr. order k and ϕ is compactly supported, then

$$\phi *' \Pi_{<k} \subseteq \Pi_{<k}.$$

Here $*'$ is the semi-discrete convolution

$$g*' : f \mapsto \sum_{j \in \mathbb{Z}^d} g(\cdot - j) f(j),$$

$\Pi := \Pi(\mathbb{R}^d)$ is all d-variate polynomials, and $\Pi_{<k} := \{p \in \Pi : \deg p < k\}$.
Moreover, if $\widehat{\phi}(0) \neq 0$, then $\phi *' \Pi_{<k} = \Pi_{<k}$.

Smoothness+refinability⇒appr.order

Theorem. Suppose $s \leq 0$, $\Phi \subset W_2^s$ is refi nable, there exists a compact set A s.t. $A \cap 2A$ has measure zero and $\cup_{m=-\infty}^{0} A/2^m$ contains a nbhd of 0, and some function $f \in S_\Phi(W_2^s)$ satisfi es

- $|\widehat{f}|$ is bounded above and away from zero on A;
- the numbers $\lambda_m := \| \sum_{\alpha \in 2^m (2\pi \mathbb{Z}^d \backslash 0)} |\widehat{f}(\cdot + \alpha)|^2 \|_{L_\infty(A)}$, $m \in \mathbb{Z}_+$, decay as $\lambda_m = O(2^{-2mk})$, for some positive k.

Then $S_\Phi(W_2^s)$ provides approximation order k.

Coherent appr. orders

To analize:

$$\omega \mapsto \inf_{v \in \mathbb{C}^\Phi} \frac{v^* G^0_{\Phi,s}(\omega) v}{v^* G_{\Phi,s}(\omega) v}.$$

Difficult already in L_2 since the Gramian G_Φ of each solution Φ is not invertible at zero if the refinement equation has multiple solutions.

Result [Jiang, Shen]. Let $\Phi \subset L_2$ be a compactly supported refinable vector with Gramian G_Φ. If $G_\Phi(0)$ is invertible, then the spectral radius $\varrho(P(0))$ of $P(0)$ is equal to 1, 1 is the only eigenvalue on the unit circle, and 1 is a simple eigenvalue.

Coherent appr. orders

Instead, analize this:

$$\omega \mapsto \inf_{v \in \mathbb{C}^{\Phi}} \frac{v^* G^0_{R(P),s} v}{v^* G_{R(P),s}(\omega) v},$$

where $G_{R(P),s} := \sum_{j=1}^{n} G_{\Phi_j,s}$ (combined Gramian), $G^0_{R(P),s} := \sum_{j=1}^{n} G^0_{\Phi_j,s}$ (truncated combined Gramian), (Φ_j) is a basis for $R(P)$. We say that $R(P)$ provides coherent appr. order k if there exists a nbhd Ω of 0 such that the function $\mathcal{M}_{P,s,k}$:

$$\omega \mapsto \frac{1}{|\omega|^{2k-2s}} \inf_{v} \frac{v^* G^0_{R(P),s}(\omega) v}{v^* G_{R(P),s}(\omega) v} \quad \text{belongs to } L_\infty(\Omega).$$

Universal supervectors

A vector v that realizes coherent appr. order is a universal supervector. It is a regular universal supervector if

$$\frac{v^* G_{R(P),s} v}{v^* v} \sim |\cdot|^{2s}, \qquad \frac{v^* G^0_{R(P),s} v}{v^* v} = O(|\cdot|^{2k}).$$

Theorem. Let $R(P)$ provide coherent appr. order k in $W_2^s(\mathbb{R}^d)$.

- Let $S_P \subset W_2^s(\mathbb{R}^d)$ be the SI space generated by $R(P)$. Then S_P is an FSI space and provides appr. order k.

Universal supervectors

- Let v be a regular universal supervector of order k bounded in a nbhd of 0. Then, for any $\Phi \in R(P)$,

  (i) $v^* G^0_{\Phi,s} v = O(|\cdot|^{2k})$ around 0. The function ψ defined by $\widehat{\psi} := v^* \widehat{\Phi}$ satisfies the Strang-Fix conditions of order k.

  (ii) If $|v^* \widehat{\Phi}| \geq c > 0$ a.e. in a nbhd of 0, then S_Φ provides appr. order k. Moreover, with $\psi \in S_\Phi$ defined by $\widehat{\psi} := v^* \widehat{\Phi}$, the PSI space S_ψ already provides that appr. order.

Condition Z_k and sum rules

Condition Z_k. $v^*(2\cdot)P - \delta_{l,0}v^*$ has a zero or order k at each πl, $l \in \{0,1\}^d$, while $v(0) \neq 0$.

Sum rules (variant I).

$$\sum_{\sigma\in\mathbb{Z}^d}\sum_{\gamma\in\mathbb{Z}^d} v^*_{\sigma-\gamma} P_{l+2\sigma} q(l+2\gamma) = 2^{-d} \sum_{\gamma\in\mathbb{Z}^d} v^*_{-\gamma} q(\gamma), \; l\in E, \; q\in\Pi_{<k}.$$

Sum rules (variant II).

$$\sum_{\beta\leq\alpha} 2^{|\alpha-\beta|}(v^{\alpha-\beta})^*(D^\beta P)(\pi l) = \delta_{l,0}(v^\alpha)^*, \quad l \in E, \; |\alpha| < k.$$

Result. Condition $Z_k \iff$ either of the sum rules.

Condition $Z_k \Longrightarrow$ coherent a.o. k

Theorem. Let P be a trig. polynomial mask, let $R(P) \subset W_2^s$, and let v be a trig. polynomial vector. Suppose that P and v satisfy Condition Z_k for some $k > 0$. Then:

- For each $\Phi \in R(P)$, the function ψ defined by $\widehat{\psi} := v^* \widehat{\Phi}$ satisfies the Strang-Fix conditions of order k, and $\widehat{\psi} - \widehat{\psi}(0) = O(|\cdot|^k)$. Consequently, if $v^*(0)\widehat{\Phi}(0) \neq 0$, then $S_\Phi(W_2^s)$ provides appr. order k, and $\psi \in S_\Phi(W_2^s)$ is a corresponding superfunction.

- If $v^*(0)\widehat{\Phi}(0) \neq 0$ for some $\Phi \in R(P)$, then $R(P)$ provides coherent appr. order k, and v is a corresponding universal regular supervector.

Condition $Z_k \overset{?}{\Longleftarrow}$ coherent a.o. k

Theorem. Let P be an $r \times r$ trig. polynomial mask. Suppose that $R(P) \subset L_2$ and that the combined Gramian $G_{R(P)}$ satisfies some technical assumptions, is smooth around each $l \in E$ and is boundly invertible around each $l \in E$. If $P(0) = I$, TFAE:

(a) P satisfies Condition Z_k with some vector v.

(b) There exists a regular universal supervector v of order k for the space $R(P)$.

In addition, a regular universal supervector v of order k can be always chosen so that, for every $\Phi \in R(P)$,

$$v^* \widehat{\Phi} - (v^* \widehat{\Phi})(0) = O(|\cdot|^k).$$

Coherent polynom. reproduction

Theorem. Let P be a refinement mask such that $R(P) \subset W_2^s(\mathbb{R}^d)$. Let $k > 0$ and let v be a trig. polynomial such that one of the following holds:

- v satisfies Condition Z_k.
- v and P satisfy either version of the sum rules
- v is a regular universal supervector of order k.

Let $\tilde{v} =: (\tilde{v}_1, \ldots, \tilde{v}_r)$ be the sequence of the Fourier coefficients of v^*, and let $\Phi =: (\phi_1, \ldots, \phi_r)' \in R(P)$. Then the map T_Φ maps $\Pi_{<k}$ into itself:
$T_\Phi : q \mapsto \sum_{i=1}^r \phi_i *' (\tilde{v}_i *' q) =: \Phi *' (\tilde{v} *' q)$
The map is surjective iff $v^*(0)\widehat{\Phi}(0) \neq 0$.

Post Scriptum

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Thanks

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attention!