

Eigenvalues and Biorthogonal Eigensystems of Scaling Operators

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Abstract

A finitely supported sequence a that sums to 2 defines a scaling operator $T_a f = \sum_{k \in \mathbb{Z}} a(k) f(2 \cdot -k)$ on functions f , a transition operator $S_a v = \sum_{k \in \mathbb{Z}} a(k) v(2 \cdot -k)$ on sequences v , and a unique compactly supported scaling function ϕ that satisfies $\phi = T_a \phi$ with $\widehat{\phi}(0) = 1$. The transition operator S_a is a discrete analogue of T_a . We show that the eigenvalues of T_a on the space of compactly supported square-integrable functions are identical with the nonzero eigenvalues of S_a on the space of finitely supported sequences if and only if the corresponding scaling function is a uniform B -spline. It is also shown that on spaces where T_a and its adjoint share the common set of eigenvalues $\{2^{-n} : n \in \mathbb{Z}_+\}$, the corresponding eigenfunctions form a biorthogonal system comprising the distributional derivatives of the scaling function ϕ and an Appell sequence of polynomials generated by $e^{xz}/\widehat{\phi}(iz)$. Interestingly the Appell polynomials generated by the uniform B -spline of order N are the classical Bernoulli polynomials of order N , and when suitably normalized they converge to the Hermite polynomials as $N \rightarrow \infty$. In this talk we develop these two observations and study the eigenvalues of the scaling and transition operators and their adjoints in full generality. We also formulate and unify both the scaling operator and its adjoint as a conditional expectation of functions of two independent random variables X and T , and study its action on related Appell polynomials.