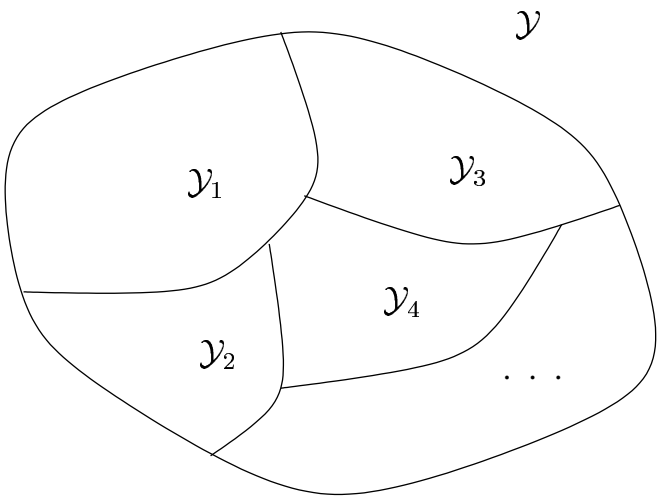
## SESSION 3: PERFORMANCE ANALYSIS: $P_e$

- Probability of Error Analysis for SSM and QIM
- Binary Detection, One sample
- Binary Detection, $N$ samples
- Multiple Codewords

# Probability of Error

- Decoding regions $\mathcal{Y}_m, m \in \mathcal{M}$

  $\Rightarrow$ decoder outputs $m$ for all $y \in \mathcal{Y}_m$



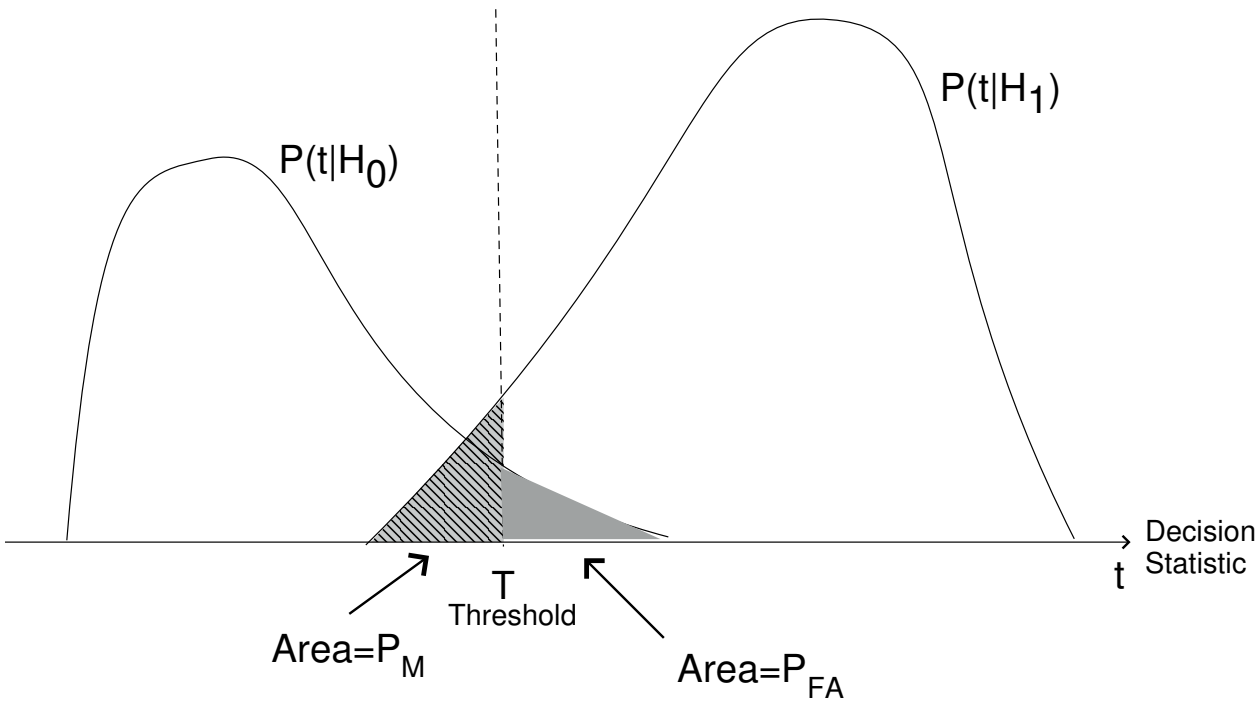

- Conditional error probability: $P_{e|m} = Pr[Y \notin \mathcal{Y}_m \mid m]$

## Binary Detection: $\mathcal{M} = \{0, 1\}$

- Binary hypothesis testing:

$$\begin{cases} H_0 \ : \ Y \sim p_0 \\ H_1 \ : \ Y \sim p_1 \end{cases}$$

- Decision rule: $t(y) \underset{<}{\overset{>}{}} T$ (decide $H_1$ if $t(y) > T$, $H_0$ if $t(y) < T$)




- Two types of error : $P_{FA}$ and $P_M$
- $P_e = \frac{1}{2}(P_{FA} + P_M) = \frac{1}{2}\int \min(p_0(y), p_1(y))\, dy$

## SSM, One Sample

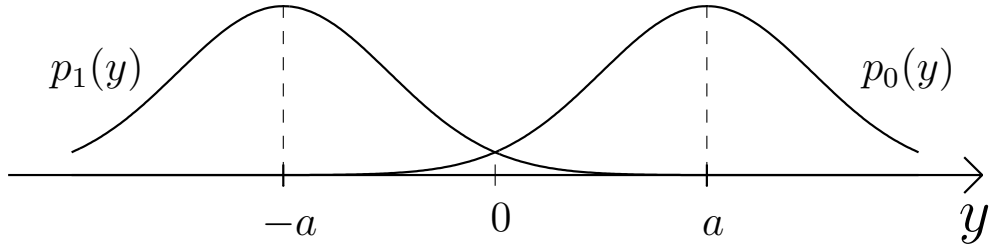
- Embedding & Detection:

$$x = \begin{cases} s + a & : \ m = 0 \\ s - a & : \ m = 1 \end{cases}$$

- Attack: $y = x + w$
- Statistical model: $S \sim \mathcal{N}(0, \sigma_s^2)$ and $W \sim \mathcal{N}(0, \sigma_w^2)$
- $WNR = \frac{a^2}{\sigma_w^2}$

# SSM (Cont'd)

- Rival pdf's for $y$:



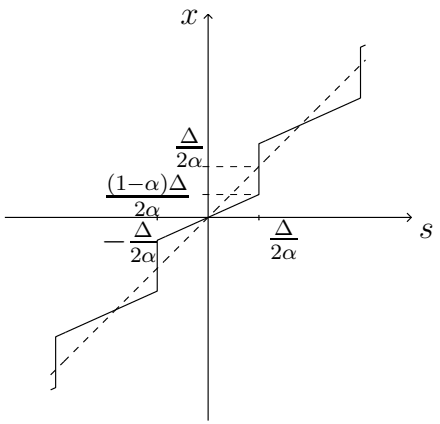

$$p_0 = \mathcal{N}(a, \sigma^2_{noise}), \quad p_1 = \mathcal{N}(-a, \sigma^2_{noise})$$

$$\text{where } \sigma^2_{noise} = \begin{cases} \sigma_w^2 & : \text{ private WM} \\ \sigma_s^2 + \sigma_w^2 & : \text{ blind WM} \end{cases}$$

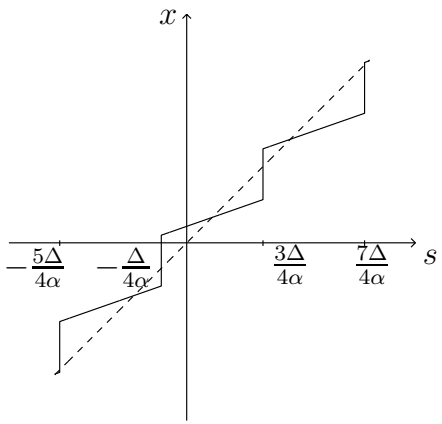
- $P_e = Q(d/2)$ where $d = \frac{2a}{\sigma_{noise}}$
- Performance is typically **much worse** for blind WM.

# Scalar QIM, One Sample

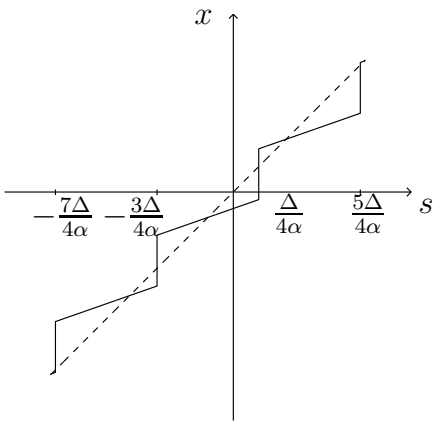
- Blind watermarking, 1-bit embedding:




Prototype $X_{sym}(s)$




$m = 0$

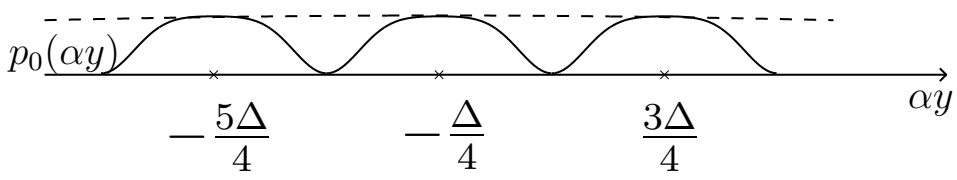


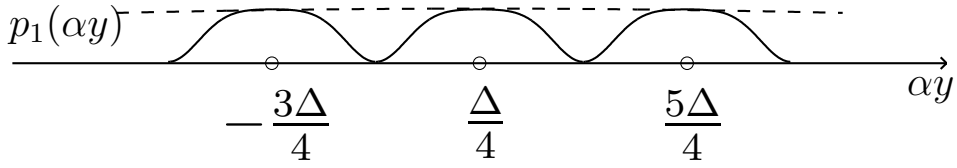

$m = 1$

- Can make quantization noise $E \sim \mathbb{U}\left[-\frac{(1-\alpha)\Delta}{2\alpha}, \frac{(1-\alpha)\Delta}{2\alpha}\right]$

# Scalar QIM (Cont'd)

- Rival pdf's are quasi-periodic, with period $\frac{\Delta}{\alpha}$:

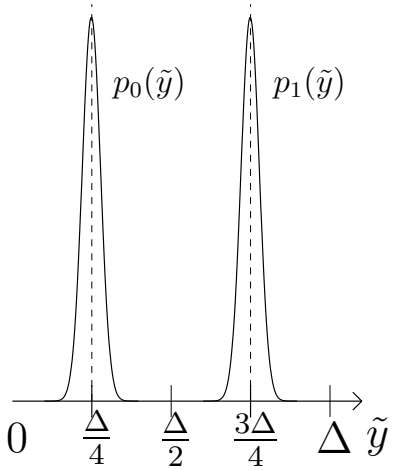


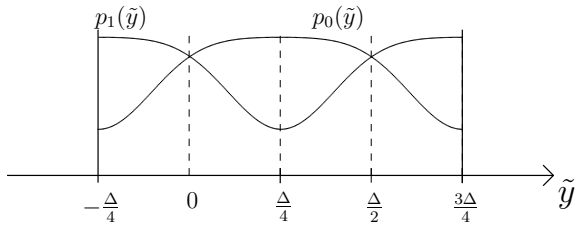




- Pulse = convolution of $\mathbb{U}\left[-\frac{(1-\alpha)\Delta}{2\alpha}, \frac{(1-\alpha)\Delta}{2\alpha}\right]$ with $\mathcal{N}(0, \sigma_w^2)$

# Scalar QIM (Cont'd)

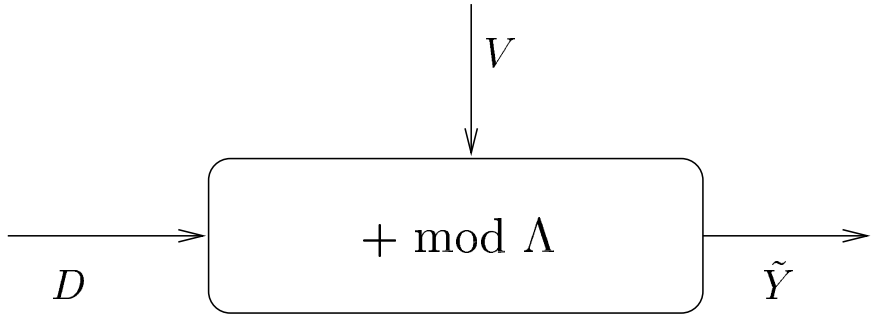
- Use test statistic $\tilde{Y} := \alpha Y \bmod \Delta$



# Scalar QIM (Cont'd)

- Communication model using **Modulo Additive Noise** (MAN) channel:




  where $d_0 = -\frac{\Delta}{4} = -d_1$, and $V = E + W \bmod \Delta$

- Equivalent hypothesis test:

$$\begin{cases} H_0 \ : \ \tilde{Y} = d_0 + V \\ H_1 \ : \ \tilde{Y} = d_1 + V \end{cases}$$

# Scalar QIM (Cont'd)

- ML Detector: $\frac{p_1(\tilde{y})}{p_0(\tilde{y})} = \frac{p_V(\tilde{y}-d_1)}{p_V(\tilde{y}-d_0)} \begin{array}{c} H_1 \\ > \\ < \\ H_0 \end{array} 1$
- Probability of error: $\tilde{P}_e = \frac{1}{2}\int \min(p_0(\tilde{y}), p_1(\tilde{y}))\, d\tilde{y}$
- The rival pdf's $p_0(\tilde{y})$ and $p_1(\tilde{y})$ have means $d_0$ and $d_1$, resp., and common variance

$$\sigma_v^2 = (1-\alpha)^2 \frac{\Delta^2}{12} + \alpha^2 \sigma_{\tilde{w}}^2$$

# Scalar QIM (Cont'd)

- The "generalized SNR" for detection

$$GSNR := \frac{(d_1 - d_0)^2}{\sigma_v^2} = \frac{\frac{1}{4}\Delta^2}{(1-\alpha)^2 \frac{\Delta^2}{12} + \alpha^2 \sigma_{\tilde{w}}^2}$$

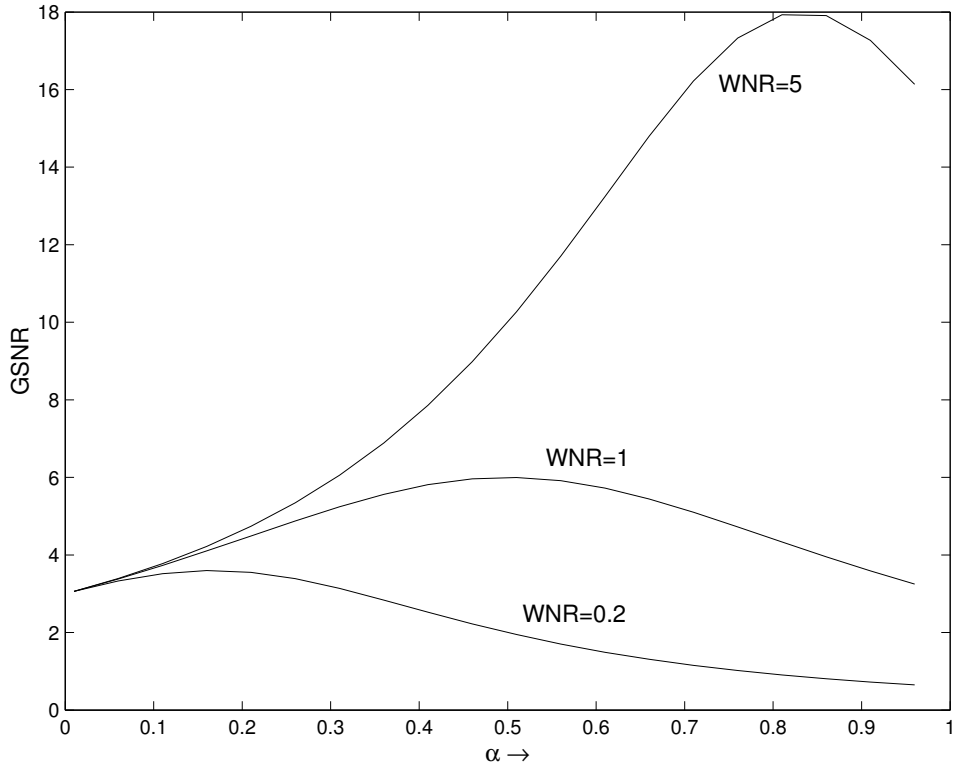
  is maximized by

$$\alpha \approx \frac{WNR}{WNR + 1}$$

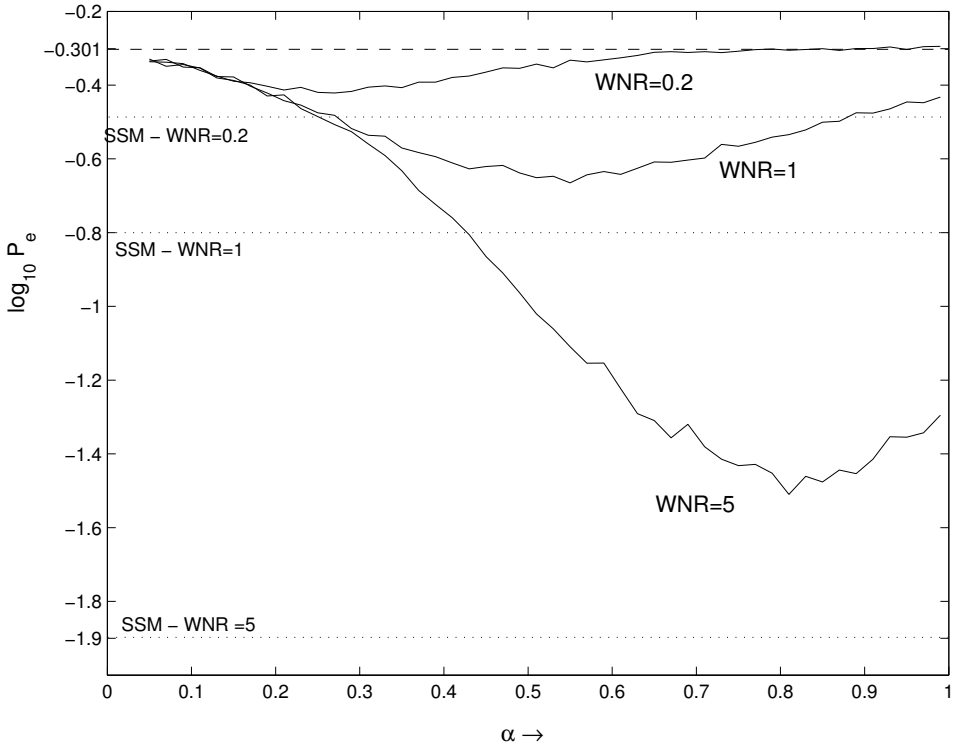
  where $WNR = \frac{\Delta^2/12}{\sigma_w^2}$

- Note $GSNR$ is not necessarily an accurate predictor of detection performance

# Scalar QIM (Cont'd)



# Scalar QIM (Cont'd)




- $P_e$ is 2–3 times worse than $P_e$ for private SSM

## SSM, $N$ Samples

- Embedding & Detection:

$$x^N = \begin{cases} s^N + a^N & : \ m = 0 \\ s^N - a^N & : \ m = 1 \end{cases}$$

- Attack: $y^N = x^N + w^N$
- Statistical model: $S^N \sim \mathcal{N}(0, R_s)$ and $W^N \sim \mathcal{N}(0, \sigma_w^2 I_N)$
- $WNR = \frac{\|a^N\|^2}{\sigma_w^2}$
- $P_e = Q(d/2)$ where $d = 2\sqrt{WNR}$ for private WM

## Scalar QIM, $N$ Samples

- Apply scalar QIM to each sample, using dither vectors $d_0^N$ under $H_0$ and $d_1^N$ under $H_1$.
- W.l.o.g. use $d_{0,n} = -\frac{\Delta}{4}$ and $d_{1,n} = \frac{\Delta}{4}$ for $1 \leq n \leq N$
- Detector's problem: choose between two hypotheses

$$\begin{cases} H_0 \ : \ \tilde{Y}^N = d_0^N + V^N \\ H_1 \ : \ \tilde{Y}^N = d_1^N + V^N \end{cases}$$

# Scalar QIM (Cont'd)

- ML Detector: $\prod_{n=1}^{N} \frac{p_V(\tilde{y}_n - d_{1,n})}{p_V(\tilde{y}_n - d_{0,n})} \underset{H_0}{\overset{H_1}{\gtrless}} 1$
- Probability of error: $\tilde{P}_e = \frac{1}{2} \int \min(p_0(\tilde{y}^N), p_1(\tilde{y}^N))\, d\tilde{y}^N$
- Hard to evaluate! (integration over $[0, \Delta]^N$)

## Gaussian Approximation

- If noise $V^N$ was Gaussian, probability of error would be given by $\tilde{P}_e = Q(\frac{1}{2}\sqrt{GSNR})$ where

$$GSNR = \frac{N\Delta^2/4}{(1-\alpha)^2\frac{\Delta^2}{12} + \alpha^2\sigma_{\tilde{w}}^2}$$

- However, $V^N$ is non-Gaussian, and this is generally a *poor* approximation to $\tilde{P}_e$.

# Large Deviations

- Large $N \Rightarrow$ large $GSNR \Rightarrow$ rare events dominate $\tilde{P}_e$
- For all $N$ we have the large-deviations bound

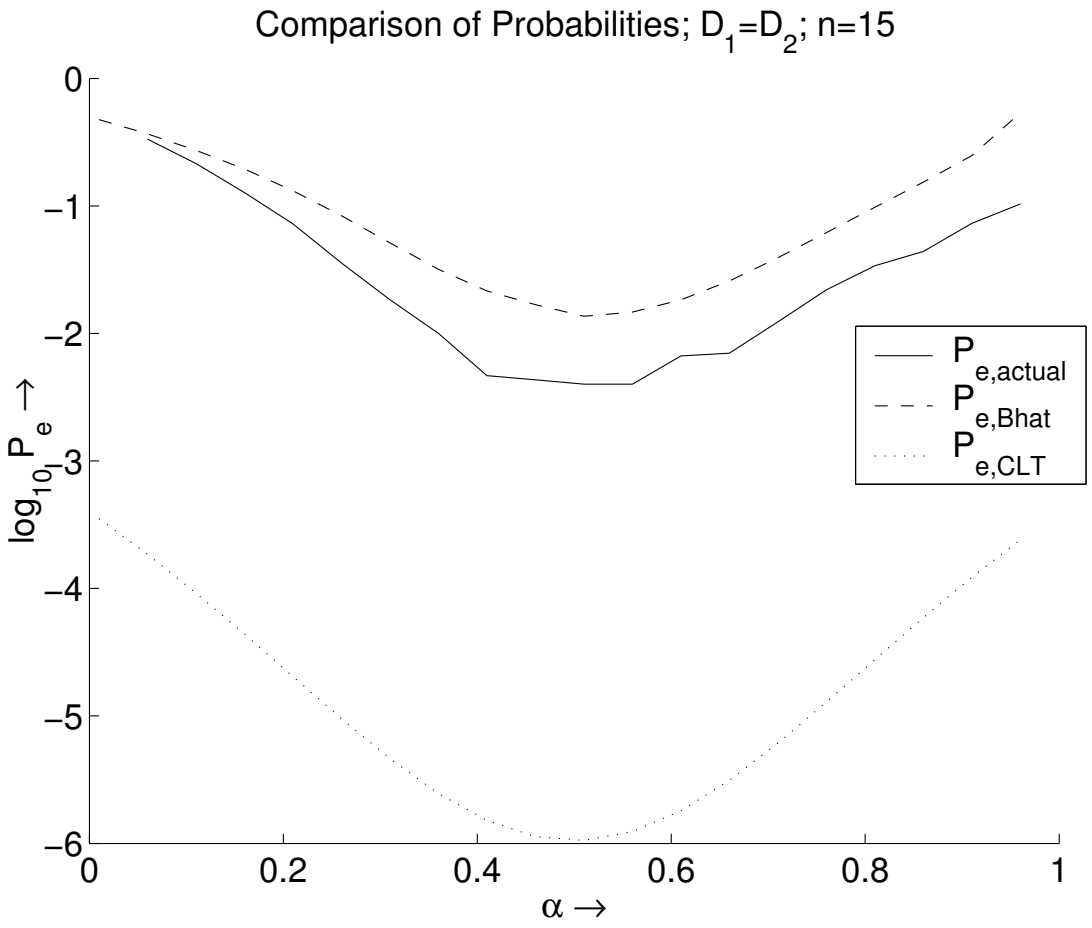
$$\tilde{P}_e \leq \tfrac{1}{2} e^{-NB(p_0,p_1)}$$

  where

$$B(p_0,p_1) = -\ln \int_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} \sqrt{p_0(\tilde{y})p_1(\tilde{y})}\, d\tilde{y}$$

  is the Bhattacharyya coefficient.

- Moreover, $\lim_{N\to\infty}[-\frac{1}{N}\ln \tilde{P}_e] = B(p_0,p_1)$
- Conclusion: $B(p_0,p_1)$ is an accurate performance predictor
- Approach is easily generalizable to lattice QIM


Comparison of Probabilities; D1=D2; n=15
0
−1
−2
−3
−4
−5
−6
log10 Pe →
0
0.2
0.4
0.6
0.8
1
α →
Pe,actual
Pe,Bhat
Pe,CLT

## Multiple Codewords: $|\mathcal{M}| > 2$

- Computation of $P_e = \frac{1}{|\mathcal{M}|} \sum_{m \in \mathcal{M}} Pr[y \notin \mathcal{Y}_m | m]$ is difficult
- For linear codes, we have $P_e = Pr[y \notin \mathcal{Y}_0 | m = 0]$
- Union bound:

$$P_e \leq (|\mathcal{M}| - 1) \max_{i \neq 0 \in \mathcal{M}} P_{e|i,0}$$

  which is tight at low rates.

- Let $d_H$ be minimum Hamming weight of code $\mathcal{C}$
- Using the Bhattacharyya bound, we obtain

$$P_e \leq (|\mathcal{M}| - 1) e^{-d_H B(p_0, p_1)}$$