Generators and Automorphism Bases of the Computably Enumerable Degrees

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CONTENTS

JOINS AND MEETS IN THE C.E. DEGREES

$(\mathbf{R}, \leq, \vee)$ is an upper semi-lattice with least and greatest elements (Post 1944).

- join operator $\approx$ effective disjoint union:

$$deg(A) \vee deg(B) = deg(A \oplus B)$$

- the question of meets is less obvious:

  $deg(A) \wedge deg(B) \approx$ 'the common information content' of A and B

  Does this (always) exist?

Early results on the structure of $\mathbf{R}$ are on ordering and joins:

- Every countable u.s.l. is embeddable into $(\mathbf{R}, \leq, \vee)$ (Friedberg 1956, Muchnik 1956, Sacks 1963).

- $(\mathbf{R}, \leq)$ is a dense partial ordering (Sacks 1964).

- Every nonzero c.e. degree splits (is join-reducible), i.e., is the join of two lesser ones (Sacks 1963).

Shoenfield's Conjecture (1965): $(\mathbf{R}, \leq, \vee)$ is a countable homogeneous u.s.l. with least and greatest elements. In particular, no nontrivial meets exist:

$$\forall\, \mathbf{a}, \mathbf{b} \in \mathbf{R}\; (\mathbf{a}|\mathbf{b} \Rightarrow \mathbf{a} \wedge \mathbf{b} \uparrow)$$

Lachlan and Yates (1966) refuted Shoenfield's conjecture by showing the existence of nontrivial meets:

- There are minimal pairs $\mathbf{a}$, $\mathbf{b}$, i.e., incomparable c.e. degrees $\mathbf{a}$, $\mathbf{b}$ such that $\mathbf{a} \wedge \mathbf{b} = \mathbf{0}$.

- There are incomparable c.e. degrees $\mathbf{a}$, $\mathbf{b}$ such that $\mathbf{a} \wedge \mathbf{b} \downarrow\neq \mathbf{0}$.

- There are incomparable c.e. degrees $\mathbf{a}$, $\mathbf{b}$ such that $\mathbf{a} \wedge \mathbf{b} \uparrow$.

- There are incomplete nonbranching c.e. degrees, i.e., c.e. degrees $\mathbf{a} < \mathbf{0}'$ which are not the meet of two greater c.e. degrees. (Note that, by the above, $\mathbf{0}$ is branching and there are nonzero branching degrees.)

- For c.e. degrees $\mathbf{a}$, $\mathbf{b}$, $\mathbf{a} \wedge \mathbf{b}$ exists in $\mathbf{R}$ iff $\mathbf{a} \wedge \mathbf{b}$ exists in $\mathbf{D}$, and if the meet of $\mathbf{a}$, $\mathbf{b}$ exists then it agrees in $\mathbf{R}$ and $\mathbf{D}$.

Joins vs. Meets: Domains of the operators

- $\mathbf{a} \vee \mathbf{b}$ always exists.

- $\mathbf{a} \wedge \mathbf{b}$ may exist or may not exist:

  - In every interval of $\mathbf{R}$ there are c.e. degrees $\mathbf{a}|\mathbf{b}$ such that
    - $\mathbf{a} \wedge \mathbf{b}$ exists (Slaman 1991)
    - $\mathbf{a} \wedge \mathbf{b}$ does not exist (Ambos-Spies 1984)

  - For every c.e. degree $\mathbf{a} \neq \mathbf{0}, \mathbf{0}'$ there is a c.e. degree $\mathbf{b}$ such that $\mathbf{a} \wedge \mathbf{b} \uparrow$; and there are c.e. degrees $\mathbf{a} \neq \mathbf{0}, \mathbf{0}'$ such that, for all c.e. degrees $\mathbf{b}$ incomparable with $\mathbf{a}$, $\mathbf{a} \wedge \mathbf{b} \uparrow$ (Ambos-Spies 1984, Harrington).

Joins vs. Meets: Decomposability

- Every nonzero c.e. degree splits, i.e., is join-reducible.
- An incomplete c.e. degree may be branching (meet-reducible) or may be nonbranching (meet-irreducible):
  - In every interval of $\mathbf{R}$ there is a branching c.e. degree, i.e., the branching degrees are dense (Slaman 1991).
  - In every interval of $\mathbf{R}$ there is a nonbranching c.e. degree, i.e., the nonbranching degrees are dense (Fejer 1983).

GENERATORS OF $\mathbf{R}$: DEFINITIONS AND BASIC FACTS

The closure $\mathsf{CL}(\mathbf{A})$ of a class $\mathbf{A}$ of c.e. degrees is the least class $\mathbf{B} \subseteq \mathbf{R}$ such that

(1) $\mathbf{A} \subseteq \mathbf{B}$
(2) $\forall\, n \geq 0\, \forall\, \mathbf{a}_0, ..., \mathbf{a}_n \in \mathbf{B}\ (\mathbf{a}_0 \vee \cdots \vee \mathbf{a}_n \in \mathbf{B})$
(3) $\forall\, n \geq 0\, \forall\, \mathbf{a}_0, ..., \mathbf{a}_n \in \mathbf{B}\ (\mathbf{a}_0 \wedge \cdots \wedge \mathbf{a}_n \Rightarrow \mathbf{a}_0 \wedge \cdots \wedge \mathbf{a}_n \in \mathbf{B})$

(1) + (2) $\Rightarrow$ $\mathsf{CL}_j(\mathbf{A})$ closure of $\mathbf{A}$ under join
(1) + (3) $\Rightarrow$ $\mathsf{CL}_m(\mathbf{A})$ closure of $\mathbf{A}$ under meet

$\mathbf{A}$ generates $\mathbf{B}$ [$\mathbf{A}$ generates $\mathbf{B}$ under join; $\mathbf{A}$ generates $\mathbf{B}$ under meet] if $\mathbf{B} \subseteq \mathsf{CL}(\mathbf{A})$ [$\mathbf{B} \subseteq \mathsf{CL}_j(\mathbf{A})$; $\mathbf{B} \subseteq \mathsf{CL}_m(\mathbf{A})$].

$\mathbf{A}$ is a generator if $\mathbf{A}$ generates $\mathbf{R}_- = \mathbf{R} \setminus \{\mathbf{0}, \mathbf{0}'\}$.
$\mathbf{A}$ is a join-generator if $\mathbf{A}$ generates $\mathbf{R}_-$ under join.
$\mathbf{A}$ is a meet-generator if $\mathbf{A}$ generates $\mathbf{R}_-$ under meet.

Nontrivial (join-)generators are provided by Sacks' splitting theorem and more general splitting theorems like Robinson's splitting theorem (= Sacks' splitting above a low degree).

Sacks' Splitting Theorem. For any nonzero c.e. degree $\mathbf{a}$ and for any recursive class of nonzero c.e. degrees $\mathbf{B}$ there are c.e. degrees $\mathbf{a}_0$ and $\mathbf{a}_1$ such that $\mathbf{a} = \mathbf{a}_0 \vee \mathbf{a}_1$ and

(1) $\quad \mathbf{a}_0, \mathbf{a}_1 \in \mathbf{L}$

(2) $\quad \forall\, \mathbf{b} \in \mathbf{B}\, (\mathbf{b} \not\leq \mathbf{a}_0, \mathbf{a}_1)$

hold.

Robinson's Splitting Theorem. For any nonzero c.e. degree $\mathbf{a}$ and for any low c.e. degree $\mathbf{c}$ and any recursive class of nonzero c.e. degrees $\mathbf{B}$ such that $\mathbf{c} < \mathbf{a}$ and $\forall\, \mathbf{b} \in \mathbf{B}\, (\mathbf{b} \not\leq \mathbf{c})$ there are c.e. degrees $\mathbf{a}_0$ and $\mathbf{a}_1$ such that $\mathbf{a} = \mathbf{a}_0 \vee \mathbf{a}_1$ and (1) and (2) and

(3) $\quad \mathbf{c} \leq \mathbf{a}_0, \mathbf{a}_1$

hold.

Applications of Sacks' Splitting Theorem:

- $\mathbf{L}$ is a (join-)generator.

- For any nonzero c.e. degree $\mathbf{a}$, $\mathbf{R}(\not\geq \mathbf{a})$ is a (join-)generator.

- In fact, for any recursive $\mathbf{A} \subset \mathbf{R}_-$,

$$\bigcap_{\mathbf{a}\in\mathbf{A}} \mathbf{R}(\not\geq \mathbf{a}) \cap \mathbf{L}$$

  is a (join-)generator.

- Any dense subset $\mathbf{A}$ of $\mathbf{R}$ (or even of $\mathbf{L}$) is a (join-)generator.

Examples of dense classes of c.e. degrees, hence (join-)generators:

- As mentioned before, the branching degrees are dense (Slaman) and the nonbranching degrees are dense (Fejer). So there is a partition of $\mathbf{R}$ into two definable generators.

- By density of $\mathbf{R}$ there are mutually disjoint dense subclasses $\mathbf{A}_n$ of $\mathbf{R}$ ($n \geq 0$). So there is a partition of $\mathbf{R}$ into infinitely many disjoint generators. (But the classes $\mathbf{A}_n$ may not be definable.)

- Ambos-Spies, Hirschfeldt and Shore 2000 give a partition of $\mathbf{R}$ into infinitely many definable generators by showing that, for any $k \geq 2$, the class of the k-maximal-branching c.e. degrees is dense. Here a c.e. degree $\mathbf{a}$ is k-maximal-branching if there are k pairwise incomparable c.e. degrees $\mathbf{b}_0, \ldots, \mathbf{b}_{k-1} > \mathbf{a}$ such that

$$\forall\, i < k\ \exists\, \mathbf{c}_i > \mathbf{a}\ (\mathbf{b}_i \wedge \mathbf{c}_i = \mathbf{a}) \text{ and}$$

$$\forall\, \mathbf{c}, \mathbf{d} > \mathbf{a}\ (\mathbf{c} \wedge \mathbf{d} = \mathbf{a} \Rightarrow \exists\, i, j < k\ (\mathbf{c} \leq \mathbf{b}_i\ \&\ \mathbf{d} \leq \mathbf{b}_j))$$

Do the duals of the above basic results on (join-)generators hold?

- Is $\mathbf{H}$ a (meet-)generator?

- For any incomplete c.e. degree $\mathbf{a}$, is $\mathbf{R}(\not\leq \mathbf{a})$ a (meet-)generator?

- Is any dense subset $\mathbf{A}$ of $\mathbf{R}$ (or even of $\mathbf{H}$) a (meet-)generator?

Results on other degree structures corresponding to the first question:

Every c.e. weak truth-table degree is the meet of two high c.e. wtt-degrees. So, for $\mathbf{R}_{wtt}$, the answer is positive.

Jockusch and Posner 1981 have shown that any jump class generates the class $\mathbf{D}(\leq \mathbf{0}')$ under join and meet.

For meet-generators the answer to the preceding questions is negative. This is immediate by the following characterization of meet-generators.

FACT. For any class $\mathbf{A}$ of c.e. degrees the following are equivalent.

(1) $\mathbf{A}$ is a meet-generator.

(2) The class of the nonbranching c.e. degrees is contained in $\mathbf{A}$.

COROLLARY. The (dense) class of the nonbranching degrees is the least meet-generator.

A LIMITING RESULT ON GENERATORS

THEOREM (Ambos-Spies 1985). For any c.e. degree $\mathbf{a} > \mathbf{0}$, $\mathbf{R}(\not\leq \mathbf{a})$ is not a generator.

COROLLARY (Ambos-Spies and Lerman 1984). The class $\overline{\mathbf{L}}$ of the nonlow c.e. degrees does not generate $\mathbf{R}$. Hence the class of the low degrees is the only jump class which is a generator.

The proof of the theorem uses the following notion: A c.e. degree $\mathbf{a}$ is *meet-inaccessible (m.i.)* if $\mathbf{a} \notin \mathrm{CL}(\mathbf{R}(\not\leq \mathbf{a}))$.

FACT. If $\mathbf{a} \leq \mathbf{b}$ and $\mathbf{a}$ is m.i. then $\mathbf{a} \notin \mathrm{CL}(\mathbf{R}(\not\leq \mathbf{b}))$, hence $\mathbf{R}(\not\leq \mathbf{b})$ is not a generator. So, for a proof of the theorem, it suffices to show that any nonzero c.e. degree bounds an m.i. degree. (For a proof of the corollary, Ambos-Spies and Lerman had shown that the top of any embedding of the $1-3-1$ lattice is m.i.)

The proof uses the following sufficient condition for meet inaccessibility:

Assume that, for any $n \geq 0$ and for any degrees $\mathbf{a}_i, \mathbf{b}_i, \mathbf{c}_i$ $(i \leq n)$ such that

$$\mathbf{a} = \mathbf{a}_0 \vee \cdots \vee \mathbf{a}_n$$

and

$$\mathbf{a}_i \leq \mathbf{b}_i, \mathbf{c}_i \ \ \& \ \ \mathbf{b}_i, \mathbf{c}_i \not\leq \mathbf{a}$$

for all numbers $i \leq n$, there is a number $i \leq n$ and a c.e. degree $\mathbf{d}_i$ such that

$$\mathbf{d}_i \leq \mathbf{b}_i, \mathbf{c}_i \ \ \& \ \ \mathbf{d}_i \not\leq \mathbf{a}.$$

Then $\mathbf{a}$ is meet-inaccessible.

Construction of a meet-inaccessible degree $\mathbf{a}$ (for the case $n = 1$):

Assume

- $A = \Phi^{X_0 \oplus X_1}$
- $Y_{00}, Y_{01}, Y_{10}, Y_{11} \not\leq_T A$

Build B_0, B_1, Γ_{00}, Γ_{01}, Γ_{10}, Γ_{11} such that

- $B_i = \Gamma_{ij}^{X_i \oplus Y_{ij}}$ (for $i, j \leq 1$)
- $\mathcal{R} : B_0 \neq \Psi_0^A$ *or* $B_1 \neq \Psi_1^A$ (for all reductions Ψ_0 and Ψ_1)

Strategy for meeting $\mathcal{R}$:

Reserve an infinite recursive set of $\mathcal{R}$-numbers (targeted for B_0, B_1) and for each $\mathcal{R}$-number x reserve an infinite set of x-numbers (targeted for A). Initially set $\Gamma_{ij}^{X_i \oplus Y_{ij}}(x) = 0$ with use $\gamma_{ij}(x) = x$ for all $\mathcal{R}$-numbers x.

Step 1. Pick a fresh $\mathcal{R}$-number x.

Step 2. Wait for a stage s such that

- $\Psi_0^A(x) = \Psi_1^A(x) = 0$
- $A(y) = \Phi^{X_0 \oplus X_1}(y) = 0$ for the least x-number y with $y \geq \psi_0(x), \psi_1(x)$
- $\gamma_{ij}(x) > \varphi(y)$ for $i, j \leq 1$

While waiting for s start new attack with bigger $\mathcal{R}$-number x'. Moreover, if $Y_{ij} \upharpoonright x$ changes, lift $\gamma_{ij}(x)$ to current stage.

Step 3. At stage $s+1$ put y into A, restrain $A \upharpoonright y$ (in order to preserve $\Psi_0^A(x) = \Psi_1^A(x) = 0$) and wait for a stage $t > s$ such that, for some $i \leq 1$, X_i changes below $\varphi_s(y)$ (which will allow us to change $\Gamma_{ij}^{X_i \oplus Y_{ij}}(x)$).

Step 4. If $X_0 \upharpoonright \varphi_s(y)$ has changed, then at stage $t+1$ put x into B_0 and correct the values of $\Gamma_{0j}^{X_i \oplus Y_{0j}}(x)$ for $j \leq 1$. Otherwise, put x into B_1 and correct $\Gamma_{1j}^{X_i \oplus Y_{1j}}(x)$.

The finite-injury construction of an m.i. degree - which is a variant of the nonbranching degree construction - can be easily combined with other techniques. For instance the following stronger existence results for meet-inaccessible degrees have been obtained.

- Every nonzero c.e. degree splits into two m.i. degrees (Ambos-Spies 1985).

- Every P-generic degree (in the sense of Ingrassia - Jockusch; hence every e-generic degree in the sense of Jockusch) is m.i. (Ding 1993).

- The m.i. degrees are dense (Ding 1993 and Zhang 1992).

OPEN PROBLEM. Is every nonbranching degree meet-inaccessible?

REMARK. Another limitation on generators gives the following partition of $\mathbf{R}$ (Ambos-Spies, Jockusch, Shore, Soare 1984):

- The class $\mathbf{NCa}$ of the noncappable (= promptly simple = low-cuppable) degrees is a filter.

- The class $\mathbf{Ca}$ of the cappable degrees is an ideal.

Hence any generator $\mathbf{G}$ can be partitioned into an generator $\mathbf{G}_0 \subseteq \mathbf{NCa}$ of $\mathbf{NCa}$ and an generator $\mathbf{G}_1 \subseteq \mathbf{Ca}$ of $\mathbf{Ca}$.

GENERATORS VS. JOIN-GENERATORS

In the following we will deal with

QUESTION (Ambos-Spies 1985) Is every generator a join-generator?

More generally: Which parts of $\mathbf{R}$ are generated under join by *all* generators.

THEOREM (Ambos-Spies, Ding, Fejer) For any generator A, A generates the high c.e. degrees under join.

The proof is based on a strengthening of meet - inaccessibility.

A c.e. degree $\mathbf{a}$ is *strongly meet-inaccessible (s.m.i.)* if

$$\forall\, \mathbf{C} \subseteq \mathbf{R}\ (\mathbf{a} \in \mathsf{CL}(\mathbf{C}) \Rightarrow \mathbf{a} \in \mathsf{CL}_j(\mathbf{C}))$$

FACT. For any generator $\mathbf{G}$, $\mathsf{CL}_j(\mathbf{SMI}) \subseteq \mathsf{CL}_j(\mathbf{G})$.

So, for a proof of the theorem, it suffices to show that any high c.e. degree can be split into two s.m.i. degrees (and to show that any generator is a join-generator it would suffice to prove that any nonzero c.e. degree can be split into two s.m.i. degrees).

The proof of the theorem uses the following sufficient condition for strong meet-inaccessibility.

LEMMA. Let $\mathbf{a}$ be a c.e. degree such that

- $\mathbf{a} > \mathbf{0}$
- $\forall\, \mathbf{b} \in \mathbf{Cu(a)}\ \forall\, \mathbf{c}, \mathbf{d} > \mathbf{b}\ (\mathbf{c}, \mathbf{d} \not\leq \mathbf{a} \Rightarrow \exists\, \mathbf{e} \leq \mathbf{c}, \mathbf{d}\ (\mathbf{e} \not\leq \mathbf{a}))$.

Then $\mathbf{a}$ is strongly meet-inaccessible.

Here $\mathbf{Cu(a)} = \{\mathbf{b} \leq \mathbf{a} : \exists\, \mathbf{c} < \mathbf{a}\ (\mathbf{a} = \mathbf{b} \vee \mathbf{c})\}$ is the class of the $\mathbf{a}$-cuppable degrees.

Construction of a strongly meet-inaccessible degree $\mathbf{a}$:

The construction resembles the construction of a meet-inaccessible degree. There we had symmetric requirements for any splitting $deg(A) = \mathbf{a} = deg(X_0 \oplus X_1)$. Now the requirements are asymmetric: the requirements for the 0-side are unchanged. If the 0-side fails, however, we have to argue that $A \leq_T X_1$.

Assume

- $A = \Phi^{X_0 \oplus X_1}$
- $Y_{00}, Y_{01} \not\leq_T A$

Build B_0, Γ_{00}, Γ_{01} such that

- $B_0 = \Gamma_{0j}^{X_0 \oplus Y_{0j}}$ (for $j \leq 1$)
- $\mathcal{R} : B_0 \neq \Psi_0^A \; or \; A \leq_T X_1$ (for all reductions Ψ_0)

The basic strategy for meeting a single $\mathcal{R}$ is the same as the one for meeting the corresponding requirement $\mathcal{R}$ in the construction of a meet-inaccessible degree; only the references to the 1-side are dropped.

Step 1. Pick a fresh $\mathcal{R}$-number x.

Step 2. Wait for a stage s such that

- $\Psi_0^A(x) = 0$
- $A(y) = \Phi^{X_0 \oplus X_1}(y) = 0$ for the least x-number y with $y \geq \psi_0(x)$
- $\gamma_{0j}(x) > \varphi(y)$ for $j \leq 1$

While waiting for s start new attack with bigger $\mathcal{R}$-number x'. Moreover, if $Y_{0j} \restriction x$ changes, lift $\gamma_{0j}(x)$ to current stage.

Step 3a. At stage $s+1$ call y $\mathcal{R}$-*certified*. (y will stay certified as long as $A \restriction y$ (hence $\Psi_0^A(x) = 0$) and the computation $\Phi^{X_0 \oplus X_1}(y)$ will not change.)

Step 3b. Put y into A and restrain $A \restriction y$ (in order to preserve $\Psi_0^A(x) = 0$). Wait for a stage $t > s$ such that, for some $i \leq 1$, X_i changes below $\varphi_s(y)$ (which, in case that X_0 changes, will allow us to change $\Gamma_{0j}^{X_i \oplus Y_{0j}}(x)$).

Step 4. If $X_0 \restriction \varphi_s(y)$ has changed, then at stage $t+1$ put x into B_0 and correct the values of $\Gamma_{0j}^{X_0 \oplus Y_{0j}}(x)$ for $j \leq 1$.

NOTE. Note that if X_0 does not have changed, then X_1 has changed, hence 'knows' that y has entered A. So if (up to a recursive set) only $\mathcal{R}$-*certified* numbers enter A then $\mathcal{R}$ is met. Moreover, if the hypothesis of $\mathcal{R}$ is correct then there are infinitely many (permanently) $\mathcal{R}$-*certified* numbers.

So $\mathcal{R}$ will be met, if all requirements are A-finitary, and followers targeted for A of lower priority requirements are subject to $\mathcal{R}$-certification, i.e., will be replaced by an available $\mathcal{R}$-certified number if they are not $\mathcal{R}$-certified themselves.

This certfication process is compatible with the standard nonrecursiveness requirements and permitting. So, for instance, below any nonzero c.e. degree there is an s.m.i. degree.

In general, however, certification is not compatible with splitting, i.e., we cannot split every nonzero degree into two s.m.i. degrees using this technique. If we split the degree of a c.e. set A into two degrees $deg(A_0)$ and $deg(A_1)$ then for any number x we have a trace $t(x)$ such that $x \in A$ will be witnessed by $t(x) \in A_0 \cup A_1$. Now if $\mathcal{R}$-certification requests that the current value of $t(x)$ is raised to a bigger $\mathcal{R}$-certified number, we have to put the old position of $t(x)$ into A_0 or A_1. But the latter has to be permitted by A (in order to keep A_0 and A_1 below A). This requires an 'almost everytime' permitting which we only get for high sets A.

THEOREM (Ambos-Spies, Lempp, Slaman) There is a generator A which is not a join-generator.

The proof is based on the following technical result.

LEMMA. There is a c.e. degree $\mathbf{a} > \mathbf{0}$ such that for any splitting $\mathbf{a} = \mathbf{x} \vee \mathbf{y}$ of $\mathbf{a}$ into c.e. dgerees $\mathbf{x}$ and $\mathbf{y}$ one of the following conditions holds.

1. There are c.e. degrees $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$, and $\mathbf{e}$ such that
   (a) $\mathbf{x} = \mathbf{b} \vee \mathbf{c}$,
   (b) $\mathbf{b} = \mathbf{d} \wedge \mathbf{e}$ and $\mathbf{d}, \mathbf{e} \not\leq \mathbf{a}$, and
   (c) $\mathbf{a} \not\leq \mathbf{c} \vee \mathbf{y}$.
2. $\mathbf{a} \leq \mathbf{y}$.

This lemma is proved by an extension of the branching degree technique in the style of Fejer and Slaman.

PROOF OF THE THEOREM BASED ON LEMMA. For $\mathbf{a}$ as in the lemma let

$$\mathbf{G} = \mathbf{R}(\not\leq \mathbf{a}) \cup \mathbf{I}$$

where

$$\mathbf{I} = \{\mathbf{y} : \exists n\ (\mathbf{y} \leq \mathbf{y}_n)\} \subseteq \mathbf{R}(< \mathbf{a})$$

is an ideal defined by an appropriate nondecreasing sequence of degrees $\mathbf{y}_0 \leq \mathbf{y}_1 \leq \mathbf{y}_2 \leq \ldots$ less than $\mathbf{a}$.

By definition, $\mathrm{CL}_j(\mathbf{G}) = \mathbf{G}$ and $\mathbf{a} \notin \mathbf{G}$. So $\mathbf{G}$ is not a join-generator.

In order to make $\mathbf{G}$ a generator, it suffices to ensure $\mathbf{x} \in \mathrm{CL}(\mathbf{G})$ for all c.e. degrees $\mathbf{x} < \mathbf{a}$.

Let $\langle \mathbf{x}_n : n \geq 0 \rangle$ be a (noneffective) enumeration of $\mathbf{R}(< \mathbf{a})$. The following choice of $\mathbf{y}_n$ will ensure that $\mathbf{x}_n \in \mathrm{CL}(\mathbf{G})$.

$n = 0$. Let $\mathbf{y}_0 = \mathbf{x}_\mathbf{0}$.

$n \to n+1$. If $\mathbf{y}_n \vee \mathbf{x}_{n+1} < \mathbf{a}$ then let $\mathbf{y}_{n+1} = \mathbf{y}_n \vee \mathbf{x}_{n+1}$. Otherwise apply the lemma to $\mathbf{x} = \mathbf{x}_{n+1}$ and $\mathbf{y} = \mathbf{y}_n$, and let $\mathbf{y}_{n+1} = \mathbf{y}_n \vee \mathbf{c}$.

AUTOMORPHISM BASES FOR $\mathbf{R}$

An *automorphism* (of $\mathbf{R}$) is a bijective function $f : \mathbf{R} \to \mathbf{R}$ preserving ordering, i.e.,

$$\forall \mathbf{a}, \mathbf{b} \in \mathbf{R}(\mathbf{a} \leq \mathbf{b} \iff f(\mathbf{a}) \leq f(\mathbf{b})).$$

As one can easily show, automorphisms also preserve joins and meets (if exist).

An *automorphism base* (of $\mathbf{R}$) is a subclass $\mathbf{A}$ of $\mathbf{R}$ such that any automorphism is determined by its behaviour on $\mathbf{A}$: If f and g are automorphisms such that $f \restriction \mathbf{A} = g \restriction \mathbf{A}$ then $f = g$. Alternatively, $\mathbf{A}$ is an automorphism base, if for any automorphism f, $f \restriction \mathbf{A} = id \restriction \mathbf{A}$ implies that $f = id$. So, to show that a class $\mathbf{A}$ is an automorphism base, it suffices to show that any nontrivial automorphism moves some element of $\mathbf{A}$.

Obviously, any generator is an automorphism base, but there also are automorphism bases which do not generate $\mathbf{R}$:

THEOREM (Lerman 1977) For any c.e. degree $\mathbf{a}$, the jump class $[\mathbf{a}] = \{\mathbf{b} : \mathbf{b}' = \mathbf{a}'\}$ is an automorphism base.

PROOF. Let f be a nontrivial automorphism. Then $f(\mathbf{c}) \neq \mathbf{c}$ for some low degree $\mathbf{c}$ (since $\mathbf{L}$ is a generator, hence an automorphism base). W.l.o.g. $f(\mathbf{c}) \not\leq \mathbf{c}$. By Robinson's jump interpolation theorem, there is a degree $\mathbf{b}$ such that $\mathbf{c} \leq \mathbf{b}$, $f(\mathbf{c}) \not\leq \mathbf{b}$ and $\mathbf{b}' = \mathbf{a}'$.

THEOREM (Ambos-Spies 1982/ta) For any c.e. degree $\mathbf{c} > \mathbf{0}$, $\mathbf{R}(\leq \mathbf{c})$ is an automorphism base.

The proof follows from the following two lemmas.

LEMMA 1. For any c.e. degree $\mathbf{c} > \mathbf{0}$ and for any nontrivial automorphism f there is a promptly simple degree $\mathbf{a}$ such that $f(\mathbf{a}) \neq \mathbf{a}$ and $\mathbf{a} \vee f(\mathbf{a}) \not\geq \mathbf{c}$.

DEFINITION. A degree $\mathbf{a}$ has the *downward separation property* if, for any degree $\mathbf{b} \not\leq \mathbf{a}$ and for any degree $\mathbf{c} \not\leq \mathbf{a} \vee \mathbf{b}$, $\mathbf{R}(\mathbf{a}, \mathbf{c}) \neq \mathbf{R}(\mathbf{b}, \mathbf{c})$.

LEMMA 2. Every promptly simple degree is the join of two degrees with the downward separation property.

PROOF OF THE THEOREM (WITH LEMMAS 1 AND 2).

Fix $\mathbf{c} > \mathbf{0}$ and let f be a nontrivial automorphism. We have to show, that for some degree $\mathbf{d} \leq \mathbf{c}$, $f(\mathbf{d}) \neq \mathbf{d}$. If $f(\mathbf{c}) \neq \mathbf{c}$ then this is obvious. So in the following w.l.o.g. we may assume that $f(\mathbf{c}) = \mathbf{c}$.

By Lemma 1 fix a promptly simple degree $\mathbf{a}$ such that $f(\mathbf{a}) \neq \mathbf{a}$ and $\mathbf{a} \vee f(\mathbf{a}) \not\geq \mathbf{c}$. Note that by the coincidence of the class of the promptly simple degrees with the definable class of the noncappable degrees, $f(\mathbf{a})$ is promptly simple too, whence by symmetry we may assume that $f(\mathbf{a}) \not\leq \mathbf{a}$. Moreover, since for any splitting of $\mathbf{a}$ one part of the splitting preserves the above properties, by Lemma 2 we may assume that $\mathbf{a}$ has the downward separation property. It follows (with $\mathbf{b} = f(\mathbf{c})$) that $\mathbf{R}(\mathbf{a}, \mathbf{c}) \neq \mathbf{R}(f(\mathbf{a}), \mathbf{c})$. Since $f(\mathbf{c}) = \mathbf{c}$ this implies

$$\mathbf{R}(\mathbf{a}, \mathbf{c}) \neq \mathbf{R}(f(\mathbf{a}), f(\mathbf{c})).$$

So f moves a degree which is below both $\mathbf{a}$ and $\mathbf{c}$.

LEMMA 1. For any c.e. degree $\mathbf{c} > \mathbf{0}$ and for any nontrivial automorphism f there is a promptly simple degree $\mathbf{a}$ such that $f(\mathbf{a}) \neq \mathbf{a}$ and $\mathbf{a} \vee f(\mathbf{a}) \not\geq \mathbf{c}$.

PROOF. Fix $\mathbf{c} > \mathbf{0}$, let f be a nontrivial automorphism and fix $\mathbf{a}$ such that $f(\mathbf{a}) \neq \mathbf{a}$. By symmetry, w.l.o.g. $f(\mathbf{a}) \not\leq \mathbf{a}$. By applying splitting theorems (if necessary) we may make the following additional assumptions (step by step):

- $\mathbf{a}$ is low and $\mathbf{c} \not\leq \mathbf{a}$. (If not, Sacks' split $\mathbf{a}$ avoiding $\mathbf{R}(\geq \mathbf{c})$.)

- $\mathbf{a} \vee f(\mathbf{a})$ is low and $\mathbf{c} \not\leq \mathbf{a} \vee f(\mathbf{a})$. (If not, Robinson split $f(\mathbf{a})$ w.r.t. the low degree $\mathbf{a}$ avoiding $\mathbf{R}(\geq \mathbf{c})$.)

- $\mathbf{a}$ is promptly simple. (If not, Robinson split $\mathbf{0}'$ over $\mathbf{a}$ w.r.t. $\mathbf{a} \vee f(\mathbf{a})$ avoiding $\mathbf{R}(\geq \mathbf{c})$. Then one part of the splitting inherits the properties of $\mathbf{a}$ and, by the coincidence of low cuppability and prompt simplicity, it is promptly simple.)

LEMMA 2. Every promptly simple degree is the join of two degrees with the downward separation property.

PROOF (IDEA). A set with the downward separation property can be constructed by a variant of the nonbranching degree technique (finite injury). The construction combines with prompt permitting and splitting a promptly simple degree. (The technique does not go along with plain permitting: degrees with the downward separation property are strongly nonbranching, hence promptly simple.)

Alternatively, one can show that e-generic degrees have the downward separation property and then apply Jockusch's result that any promptly simple degree can be split into two e-generic degrees.

AN APPLICATION OF AUTOMORPHISMBASES

THEOREM (Cholak, Downey, Walk 2002; Downey, Lempp 1997) Let f be a nontrivial automorphism of $\mathbf{R}$. Then there is a c.e. degree $\mathbf{a}$ such that $f(\mathbf{a})$ and $\mathbf{a}$ are incomparable.

THEOREM 1 (Downey, Lempp 1997). The contiguous degrees are definable.

THEOREM 2 (Cholak, Downey, Walk 2002). There are maximal contiguous degrees. In fact, the class of the maximal contiguous degrees is an automorphism base.

PROOF OF THEOREM. By Theorem 2, f has to move a maximal contiguous degree $\mathbf{a}$. Since, by Theorem 2, maximal contiguity is definable, $f(\mathbf{a})$ is maximal contiguous too. So $f(\mathbf{a})$ and $\mathbf{a}$ are incomparable.