Generalized tabular reducibilities in infinite levels of the Ershov difference hierarchy.

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It is known that finite and ω levels of the Ershov difference hierarchy are connected with bounded truth table and truth table reducibilities accordingly. In my talk I consider a collection of reducibilities which are intermediate between Turing and truth table reducibilities and have similar properties relatively to infinite levels of the Ershov hierarchy which are defined by means of limit constructive ordinals.

Theorem 1. *(Shoenfield, Ershov) A set A is T-reducible to $\emptyset'$ if and only if there exists a uniformly computable sequence of c.e. sets $\{R_x\}_{x\in\omega}$ such that*

$$R_0 \supseteq R_1 \supseteq \dots , \qquad \bigcap_{x=0}^{\infty} R_x = \emptyset,$$

and

$$A = \bigcup_{x=0}^{\infty} (R_{2x} - R_{2x+1}).$$

Let $\{R_x\}_{x\in\omega}$ be a uniformly computable sequence of c.e. sets such that $R_0 \subseteq R_1 \subseteq ...$, and let

$$A = \bigcup_{x=0}^{\infty} (R_{2x} - R_{2x+1}).$$

Definition 1. *A set A is n-computably enumerable (n-c.e. set), if either $n = 0$ and $A = \emptyset$, or $n > 0$ and there exist c.e. sets*

$$R_0 \subseteq R_1 \subseteq R_2 \subseteq ... \subseteq R_{n-1}$$

such that

$$A = \bigcup_{i=0}^{\left[\frac{n-1}{2}\right]} \{(R_{2i+1} - R_{2i}) \cup (R_{2i} - R_{2i+1})\}.$$

(Here if n an odd number then $R_n = \emptyset$.)

Definition 2. *A set A belong to the level Σ_n^{-1} of Ershov's hierarchy (A is Σ_n^{-1}-set), if it is n-c.e. set. A set A belong to level Π_n^{-1} of the hierarchy (A is Π_n^{-1}-set), if $\overline{A} \in \Sigma_n^{-1}$ and A is Δ_n^{-1}-set, if A and $\overline{A}$ both are Σ_n^{-1}-sets, i.e.*

$$\Delta_n^{-1} = \Sigma_n^{-1} \cap \Pi_n^{-1}.$$

Theorem 2. *(Ershov; Epstein, Haas, Kramer)*
a) A set A is n-c.e. set for some $n \geq 0$ iff there is a computable function g of two variables s and x such that for all x $A(x) = lim_s g(s,x)$, $g(0,x) = 0$, and

$$|\{s|g(s+1,x) \neq g(s,x)\}| \leq n.$$

b) A set A is Δ_{n+1}^{-1}-set for some n, $1 \leq n < \omega$, iff there is a partial-computable function ψ such that for all x

$$A(x) = \psi(\mu t_{\leq n}(\psi(t,x) \downarrow), x).$$

Definition 3. *(Ershov; Epstein,Haas,Kramer)* *A set* $A \subseteq \omega$ *belong to the level* Σ^{-1}_{ω} *of Ershov's hierarchy (*A *is* Σ^{-1}_{ω}*-set), if there exists a partial-computable function* ψ *such that for all* x*,*

$x \in A \rightarrow \exists s(\psi(s,x)\downarrow$
and $A(x) = \psi(\mu s(\psi(s,x)\downarrow), x)$*;*

$x \notin A \rightarrow$ *either* $\forall s(\psi(s,x)\uparrow$*, or*
$\exists s(\psi(s,x)\downarrow) \& A(x) = \psi(\mu s(\psi(s,x)\downarrow), x)$.

(In other words, $A \subseteq dom(\psi(\mu s(\psi(s,x)\downarrow), x)$, and
for any $x \in dom(\psi(\mu s(\psi(s,x)\downarrow), x))$
we have $A(x) = \psi(\mu s(\psi(s,x)\downarrow), x))$.

A set A belong to level Π^{-1}_{ω} of the hierarchy (A is Π^{-1}_{ω}-set), if $\overline{A} \in \Sigma^{-1}_{\omega}$. At last, A is Δ^{-1}_{ω}-set, if A and $\overline{A}$ both are Σ^{-1}_{ω}-sets,
i.e. $\Delta^{-1}_{\omega} = \Sigma^{-1}_{\omega} \cap \Pi^{-1}_{\omega}$.

Definition 4. *A set A is ω-c.e. set if and only if there are computable function g of two variables s and x and a computable function f such that for all x $A(x) = lim_s g(s, x)$, $g(0, x) = 0$, and*

$$|\{s|g(s+1, x) \neq g(s, x)\}| \leq f(x).$$

Theorem 3. *(Ershov) A set A is ω-c.e. iff*
– it is a Δ^{-1}_{ω}-set iff
– there is a partial-computable function ψ such that for all x,
$A(x) = \psi(\mu t(\psi(t, x)\downarrow), x)$, iff

– there is a uniformly c.e. sequence of c.e. sets $\{R_x\}_{x\in\omega}$, such that $\bigcup_{x\in\omega} R_x = \omega$, $R_0 \subseteq R_1 \subseteq \ldots$, and $A = \bigcup_{n=0}^{\infty}(R_{2n+1} - R_{2n})$.

Theorem 4. *(Ershov)* $A \in \Sigma^{-1}_{\omega}$ *iff there is a uniformly computable sequence of c.e. sets* $\{R_x\}_{x\in\omega}$ *such that* $R_0 \subseteq R_1 \subseteq \ldots$ *(ω-sequence of c.e. sets), and*
$A = \bigcup_{x=0}^{\infty}(R_{2x} - R_{2x+1})$

Theorem 5. *(H.G. Carstens, 1976) a) A set* A *is* Δ^{-1}_{ω}*-set if and only if it is tt-reducible to* $\emptyset'$*;*

b) For any $n \geq 1$ *a set* A *is* Δ^{-1}_{n+1}*-set if and only if it is btt-reducible to* $\emptyset'$ *with norm* n*.*

Definition 5. *Let* $P(x, y)$ *be a computable predicate which on* ω *defines a partial ordering. (If* $P(x, y)$ *we write* $x \leq_P y$*.) A uniformly c.e. sequence* $\{R_x\}$ *of c.e. sets is* P*- (or* $\leq_P$*-) sequence, if for all* x, y*,* $x \leq_P y$ *implies* $R_x \subseteq R_y$*.*

Definition 6. *Hereinafter we will use the Kleene system of notation (*$\mathcal{O}, <_0$*). For* $a \in \mathcal{O}$ *we denote by* $|a|_0$ *ordinal* α*, which have* $\mathcal{O}$*-notation* a*. Therefore* $|a|_0$ *have the order type* $\langle \{x | x <_{\mathcal{O}} a\}, <_0 \rangle$*, and words "*$a$*-sequence of c.e. sets* $\{R_x\}$*" for* $a \in \mathcal{O}$ *have usual sense.*

Definition 7. *An ordinal is even, if it is either* 0*, or a limit ordinal, or a follower of an odd ordinal. Otherwise the ordinal is odd. Therefore, if* α *is even, then* α' *(follower of* α*) is odd and vise versa.*

For system of notation $\mathcal{O}$ the parity function $e(x)$ is defined as follows: Let $n \in \mathcal{O}$. Then $e(n) = 1$, if ordinal $|n|_{\mathcal{O}}$ is odd, and $e(n) = 0$, if $|n|_{\mathcal{O}}$ is even.

For any $a \in \mathcal{O}$ we define operations S_a and P_a, which map a-sequences $\{R_x\}_{x<_{\mathcal{O}}a}$ to subsets of ω, as follows:

$S_a(R) =$

$$\{z | \exists x <_{\mathcal{O}} a(z \in R_x \& e(x) \neq e(a) \& \forall y <_{\mathcal{O}} x(z \notin R_y))\}.$$

$$P_a(R) = \{z | \exists x <_{\mathcal{O}} a(z \in R_x \& e(x) = e(a) \& \\ \& \forall y <_{\mathcal{O}} x(z \notin R_y))\} \cup \{\omega - \bigcup_{x<_{\mathcal{O}}a} R_x\}.$$

It follows from these definitions that $P_a(R) = \overline{S_a(R)}$ for all $a \in \mathcal{O}$ and all a-sequences R.

Class Σ_a^{-1} (Π_a^{-1}) for $a \in \mathcal{O}$ is the class of all sets $S_a(R)$ (accordingly all sets $P_a(R)$), where $R = \{R_x\}_{x<_{\mathcal{O}}a}$ all a-sequences of c.e. sets, $a \in \mathcal{O}$. Define $\Delta_a^{-1} = \Sigma_a^{-1} \cap \Pi_a^{-1}$.

Theorem 6. *(Epstein, Haas, Kramer; Selivanov) Let $A \subseteq \omega$ and α be a limit ordinal which obtains a notation a in $\mathcal{O}$. Following three statements are equivalent:*

a) $A \in \Delta_a^{-1}$;

b) For some partial-computable function Ψ and any x, $A(x) = \Psi((\mu\lambda < \alpha)_{\mathcal{O}}(\Psi((\lambda)_s, x) \downarrow, x)$;

c) There is an a-sequence $\mathcal{R} = \{R_x\}_{x<_0 a}$ such that $A = S_a(\mathcal{R})$ and $\bigcup_{b<_{\mathcal{O}}a} R_b = \omega$.

Ershov Theorems on the hierarchy

Theorem 7. *Let* $a, b \in \mathcal{O}$ *and* $a <_{\mathcal{O}} b$.
Then $\Sigma_a^{-1} \cup \Pi_a^{-1} \subsetneq \Sigma_b^{-1} \cap \Pi_b^{-1}$.

Corollary 8. *For any* $a \in \mathcal{O}$, $\Sigma_a^{-1} \subsetneq \Sigma_2^0 \cap \Pi_2^0$.

Theorem 9.

$$\bigcup_{a \in \mathcal{O}} \Sigma_a^{-1} = \bigcup_{a \in \mathcal{O}, |a|_{\mathcal{O}} = \omega^2} \Sigma_a^{-1} = \Sigma_2^0 \cap \Pi_2^0.$$

It follows from theorem 10 that theorem 9 cannot be strengthen:

Theorem 10. $\bigcup_{a \in \mathcal{O}, |a|_{\mathcal{O}} < \omega^2} \Sigma_a^{-1} \neq \Sigma_2^0 \cap \Pi_2^0$.

Theorem 11. *a) For any $a \in \mathcal{O}$ there is a path T_0 in $\mathcal{O}$ through a such that*

$$\bigcup_{b \in T_0} \Sigma_b^{-1} = \Sigma_2^0 \cap \Pi_2^0.$$

b) There is a path T in $\mathcal{O}$ such that $|T|_{\mathcal{O}} = \omega^3$ and $\bigcup_{a \in T} \Sigma_a^{-1} = \Sigma_2^0 \cap \Pi_2^0$.

Theorem 12. *If a path T in $\mathcal{O}$ such that $|T|_{\mathcal{O}} < \omega^3$, then $\bigcup_{a \in T} \Sigma_a^{-1} \neq \Sigma_2^0 \cap \Pi_2^0$.*

For convenience we will consider only constructive ordinals $\leq \omega^{\omega}$.

It follows from the universality properties of $(\mathrm{O}, <_o)$, that for $\alpha < \omega^{\omega}$ for simplicity instead notations from O we may use ordinals meaning their representation in normal form

$$\alpha = \omega^m \cdot n_0 + \ldots + \omega \cdot n_{m-1} + n_m.$$

We first define the following classes of formulas $\mathcal{B}_\alpha$, $\alpha \le \omega^\omega$.

$\alpha = n > 1$: $\mathcal{B}_\alpha$ consists from all tt -conditions with norm $< n$;

$\alpha = \omega$: $\mathcal{B}_\alpha$ consists from all tt-conditions;

$\alpha = \omega^m \cdot n + \beta$, $\beta < \omega^n$
($n > 1$, if $n = 1$ then $\beta > 0$):
$\mathcal{B}_\alpha$ consists from all formulas such that

$$\sigma_1 \& \tau_1 \vee ... \vee \sigma_n \& \tau_n \vee \rho, \text{ or}$$
$$\neg[\sigma_1 \& \tau_1 \vee ... \vee \sigma_n \& \tau_n \vee \rho],$$

where $\sigma_i \in \mathcal{B}_\omega$, $\tau_i \in B_{\omega^m}$, $\rho \in \mathcal{B}_\beta$;

$\alpha = \omega^{m+1}$: $\mathcal{B}_\alpha = \bigcup_n \mathcal{B}_{\omega^m \cdot n}$;

$\alpha = \omega^\omega$: $\mathcal{B}_\alpha = \bigcup_n \mathcal{B}_{\omega^n}$.

An enumeration $\{\sigma^\alpha_n\}_{n\in\omega}$ *(by induction on* α*) of formulas from* $\mathcal{B}_\alpha$*.*
We denote by σ^ω_n the n-th tt-formula with norm n (which is a formula of propositional logic constructed from atomic propositions $\langle k \in X\rangle$ for several $k \in \omega$. The norm of the tt-formula is the number of its atomic propositions).

For $\alpha = \omega^m \cdot n + \delta$, $m \geq 1, n \geq 1, \delta < \omega^m$ the formula $\sigma^\alpha_{\langle l,n,p,q,r\rangle}$ with number $\langle l, n, p, q, r\rangle$ is the formula

$$(\neg)[\sigma^\beta_{\Phi_p(0)} \& \sigma^\gamma_{\Phi_q(0)} \vee \ \ldots \ \vee \sigma^\beta_{\Phi_p(n-1)} \& \sigma^\gamma_{\Phi_q(n-1)} \vee \sigma^\delta_r],$$

where $l = 1$ ($l = 0$) means the presence (accordingly absence) of the negation in the beginning of the formula. Ordinals β, γ and δ from the definition of $\mathcal{B}_\alpha$, $\Phi_p(i)$ is the partial-computable function with index p, defined for all $i \leq n-1$, $\Phi_q(i)$ is the partial-computable function with number q.

$\sigma \in \mathcal{B}_\alpha$ means that $\sigma = \sigma_i^\alpha$ for some i.

Therefore, a number $\langle l, n, p, q, r \rangle$ is an index for some formula $\sigma \in \mathcal{B}_\alpha$, $\alpha = \omega^m \cdot n + \delta$, $m \geq 1, n \geq 1, \delta < \omega^m$ iff $l \leq 1$, $\Phi_p(x) \downarrow$ for all $x \leq n-1$ and r is an index for some formula from $\mathcal{B}_\delta$.

For $\alpha = \omega^{m+1}$ and $\alpha = \omega^\omega$ the numbering of formulas $\{\sigma^\alpha\}$ is defined using a fixed effective enumeration of all formulas from $\bigcup_n \mathcal{B}_{\omega^m \cdot n}$ (accordingly from $\bigcup_n \mathcal{B}_{\omega^n}$).

For the convenience we add to integers two additional objects $true$ and $false$, for which σ^α_{true} is a tt-formula which is identically truth and σ^α_{false} is an inconsistent tt-formula.

Definition 8. *We say that a formula σ from $\mathcal{B}_\alpha$ converges on a set $A \subseteq \omega$, if*

- $\alpha \leq \omega$, *i.e. any formula from $\mathcal{B}_\alpha$, $\alpha \leq \omega$, converges on any set $A \subset \omega$, or*

- σ *is equal to* $(\neg)[(\bigvee_{i \leqslant m} \sigma^{\beta}_{f(i)} \& \sigma^{\gamma}_{g(i)}) \vee \sigma^{\delta}_{j}]$ *, and for any $i \leq m$ if $A \models \sigma^{\beta}_{f(i)}$, then $g(i) \downarrow$ and $\sigma^{\gamma}_{g(i)}$ converges on A.*

Definition 9. *A formula σ from $\mathcal{B}_\alpha$ is realizable on a set $A \subseteq \omega$ (we write $A \models \sigma$), if σ converges on A and*

- *If $\sigma \in \mathcal{B}_\omega$, then A satisfies to the tt-condition σ,*

- *If σ is equal to $(\bigvee\limits_{i \leqslant m} \sigma_i \& \tau_i) \vee \rho$, then $A \models \rho$ or there is an $i \leq m$ such that $A \models \sigma_i$ and $A \models \tau_i$,*

- *If σ is equal to $\neg[(\bigvee\limits_{i \leqslant m} \sigma_i \& \tau_i) \vee \rho]$, then $A \not\models \rho$ and for all $i \leq m$, if $A \models \sigma_i$ then $A \not\models \tau_i$.*

$A \not\models \sigma$ means $A \models \neg\sigma$

Definition 10. *A set* A $gtt(\alpha)$*-reducible to a set B (we write* $A \leq_{gtt(\alpha)} B$*), if there is a computable function* f *such that for any* x *we have* $\sigma^{\alpha}_{f(x)}$ *converges on* B *and* $x \in A \leftrightarrow B \models \sigma^{\alpha}_{f(x)}$*.*

Corollary 13.
(i) *For* $\alpha < \omega$ $A \in \Delta^{-1}_{\alpha+1} \leftrightarrow A \leq_{gtt(\alpha)} K$*.*
(ii) $A \in \Delta^{-1}_{\omega} \leftrightarrow A \leq_{gtt(\omega)} K$*.*

Theorem 14. *For* $\alpha = \omega, \omega^2, ..., \omega^{\omega}$ *reducibilities* $\leq_{gtt(\alpha)}$ *are reducibilities which are intermediate between* tt*- and* T*-reducibilities, and for different* α *all* $\leq_{gtt(\alpha)}$ *are different.*

Sketch of proof. By indexes i and j of formulas σ_i^α, σ_j^β we can effectively compute an index k of formula $\sigma_k^{\alpha\cdot\beta}$, which is obtained by substitution of σ_i^α into σ_j^β, which means that for $\alpha = 1, 2, ..., \omega, \omega^2, ..., \omega^\omega$ the set $\mathcal{B}_\alpha$ effectively closed on substitutions, and the relation $\leqslant_{gtt(\alpha)}$ in this case transitive. It is also clear that the relation $\leqslant_{gtt(\alpha)}$ is reflexive and it is intermediate between Turing and truth-table reducibilities.

It is known that for all $a <_{\mathcal{O}} b$ the set of T-degrees of Δ_a^{-1}-sets is proper subset of the set Δ_b^{-1}-sets and, therefore, it follows from theorem 15 that at least degrees of creative sets for nthese reducibilities are different.

Theorem 15. *For any* $\alpha \leq \omega^\omega,\ \omega \leq \alpha,$

$$A \in \Delta_\alpha^{-1} \Leftrightarrow A \leq_{gtt(\alpha)} K.$$

The following theorem shows that the Turing reducibility is not exhausted by any collection of $gtt(\alpha)$-reducibilities.

Theorem 16. *There is a set $A \leq_T \emptyset''$ such that for all α $A \not\leq_{gtt(\alpha)} \emptyset''$.*

The proof is based on the following
Lemma. If $B \leqslant_{gtt(\alpha)} C$ for some α, then there is a computable in $\emptyset'$ function $\Phi_e^{\emptyset'}$ such that
$(\forall x)[\ x \in B \ \leftrightarrow \ C \models \sigma^{\omega}_{\Phi_e^{\emptyset'}(x)}]$

Let $B = \{x | (\exists y)[\varphi_x^{\emptyset'}(x) = y \& \emptyset'' \models \sigma_y^{\omega}]\}$ and let $A = \omega - B$.

The reducibility $A \leq_T \emptyset''$ is obvious. If $A \leqslant_{gtt(\alpha)} \emptyset''$ for some α, then there exists $\Phi_e^{\emptyset'}(x)$ from the lemma. Then

$$e \in A \ \leftrightarrow \ \emptyset'' \models \sigma^{\omega}_{\Phi_e^{\emptyset'}(x)} \ \leftrightarrow \ e \in \ B \leftrightarrow \ e \notin \ A \ \blacksquare$$

From other side, the weak truth-table reducibility is a special case of the $gtt(\omega^2)$-reducibility.

Definition 11. *$A \leq_{wtt} B$, if $A = \Phi_e^B$ for some e and for all x $\varphi_e^B(x) \leq f(x)$ for some computable function f.*

Theorem 17. *If $A \leq_{wtt} B$, then $A \leq_{gtt(\omega^2)} B$.*

Proof. Let $A = \Phi^B$ and g a computable function such that $\varphi^B(x) < g(x)$ for all x. There are $2^{g(x)}$ subsets $X_i \subseteq \{0, 1, ..., g(x) - 1\}$. For each of them we compose a tt-formula $\sigma^{\omega}_{p(i)}$, $i \leq 2^{g(x)}$, as follows:

$$X \models \sigma^{\omega}_{p(i)} \ \leftrightarrow \ X \upharpoonright g(x) = \ X_i.$$

Now consider the formula

$$\sigma^{\omega^2}_{f(x)} \fallingdotseq \sigma^{\omega}_{p(1)} \& \sigma^{\omega}_{q(1)} \ \vee ... \ \vee \ \sigma^{\omega}_{p(2^{g(x)})} \& \sigma^{\omega}_{q(2^{g(x)})},$$

where $\sigma^{\omega}_{p(i)}$ from above and $q(x)$ is defined as follows:

$$q(x) = \begin{cases} true, & \text{if} \quad \Phi^{X_i}(x) \downarrow = 1; \\ false, & \text{if} \quad \Phi^{X_i}(x) \downarrow = 0; \\ \uparrow, & \text{if} \quad \Phi^{X_i}(x) \uparrow . \end{cases}$$

Now $\Phi^B(x) = 1 \leftrightarrow \ B \models \sigma^{\omega^2}_{f(x)}$, i.e. $A \leqslant_{gtt(\omega^2)} B$ by function $f(x)$. ∎

Theorem 18. *(Arslanov, LaForte, Slaman) Let B be an ω-c.e. set, C be a c.e. set, $A <_T C$ and $B \leq_T C$. Then there exists a d-c.e. set D such that $B \leq_T D \leq_T C$.*

Corollary 19. *Any 2-CEA and ω-c.e. degree is d-c.e. degree.*

Theorem 20. *(Batyrshin) For any $n > 1$ if $|a|_{\mathcal{O}} = \omega^{n+1}$, $B \in \Delta_a^{-1}$, A is c.e., $B \leq_T A \oplus W^A$, then there is a set $D \in \Sigma_b^{-1}$, $b <_{\mathcal{O}} a$ and $|b|_{\mathcal{O}} = \omega^n$, such that $B \leq_T D \leq_T A \oplus W^A$.*

Corollary 21. *Any 2-CEA, Δ_a^{-1}-degree is also Σ_b^{-1}-degree, where $|a|_{\mathcal{O}} = \omega^{n+1}$, $b <_{\mathcal{O}} a$ and $|b|_{\mathcal{O}} = \omega^n$ $(n > 1)$.*

A list of natural questions:

1) Does the last theorem holds if the class of sets Δ_a^{-1} changed by a broader class Σ_a^{-1}?

2) Does the last theorem holds if the set B in this theorem belongs to the class Σ_c^{-1} for some c such that $|c|_{\mathcal{O}} < \omega^n$?

3) Is it possible to generalize this theorem to higher levels of the Ershov hierarchy: to level Δ_a^{-1} for some a such that $|a|_{\mathcal{O}} = \omega^\omega$?

The following two theorems give negative answers to all these questions.

Theorem 22. *(Batyrshin) For any $n > 0$ if $|a|_{\mathcal{O}} = \omega^n$, then there exists a properly Σ_a^{-1}-set B which is c.e. in a c.e. set $A <_T B$.*

Corollary 23. *For any $n > 0$ if $|a|_{\mathcal{O}} = \omega^{n+1}$, then there exists a set $B \in \Delta_a^{-1}$ which is c.e. in a c.e. set $A <_T B$, such that $(\forall c <_{\mathcal{O}} b)(\forall C \in \Sigma_c^{-1})(B \not\equiv_T C)$. Here $b <_{\mathcal{O}} a$ and $|b|_{\mathcal{O}} = \omega^n$.*

Theorem 24. *(Batyrshin) Let $|v|_{\mathcal{O}} = \omega^\omega$. There exists a 2-CEA set $B \in \Delta_v^{-1}$ such that $(\forall b <_{\mathcal{O}} v)(\forall C \in \Sigma_b^{-1})(B \not\equiv_T C)$.*

OPEN PROBLEMS

Problem 25. *Define a hierarchy* $\mathcal{Q}$ *of classes of c.e. sets which are connected with the* $gtt(\alpha)$*-reducibilities as classes of simple and hypersimple sets are connected with* btt*- and* tt*-reducibilities.*

All so far known sentences in the language of partial ordering, which are true in the n-c.e. degrees and false in $(n+1)$-c.e. degrees for some $n \geq 1$, belong to the level $\forall\exists$ or higher. We (Arslanov, Kalimullin, Lempp) conjectured that for all $n \geq 1$ the $\exists\forall$-theory of the n-c.e. degrees is a subtheory of the $\exists\forall$-theory of the $(n+1)$-c.e. sets.

If this conjecture fails then some sentence $\exists\bar{x}\forall\bar{y}P(\bar{x},\bar{y})$ true in n-c.e. sets and false in $(n+1)$-c.e. sets for some $n \geq 1$. Let $\bar{a}$ be fixed n-c.e. sets such that the sentence $\forall\bar{y}P(\bar{a},\bar{y})$ true in n-c.e. sets and false in $(n+1)$-c.e. sets. Therefore, the Δ_0 formula $\neg P(\bar{a},\bar{y})$ with c.e. parameters satisfiable in $(n+1)$-c.e. degrees but having no solution in n-c.e. degrees is a witness for the claim that n-c.e. sets with $\leq_T$ is not Σ_1-substructure of the $(n+1)$-c.e. degrees. This observation allows us to formulate the following

Conjecture 26. *For any $n \geq 1$, the structure of n-c.e. sets with $\leq_T$ is a Σ_1-substructure of the $(n+1)$-c.e. degrees with $\leq_T$.*