The Major Sub-degree Theorem

I: The First Strategy and the Permitting Rules

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The Plan

Lecture I The First Strategy, and the Permitting Rules

Lecture II The Second Strategy and the General Methods

The Major Sub-degree Problem

The Problem (Communicated by Lachlan in 1967, see JS (1983), and ALS (1993))

- Let $\mathbf{b} < \mathbf{a}$.
  We say that **b** is a *major sub-degree (*MSD*) of* **a**, if for any c.e. x,

$$\mathbf{b} \vee \mathbf{x} = \mathbf{0}' \longleftrightarrow \mathbf{a} \vee \mathbf{x} = \mathbf{0}'.$$

- Does every c.e. $\mathbf{b} \neq \mathbf{0}, \mathbf{0}'$ have a MSD?

The Progress

- Cooper 1974, $\exists \mathbf{a} \neq \mathbf{0}\ \forall \mathbf{x}$
$$\mathbf{a} \vee \mathbf{x} = \mathbf{0}' \leftrightarrow \mathbf{x} = \mathbf{0}'.$$
Every such $\mathbf{a}$, called *noncuppable*, has a MSD.
- Jockusch, Shore, 1983: $\exists$ a high cuppable deg has a low MSD. (A corollary of the results there)
- Ambos-Spies, Lachlan, Soare, 1993: For any splitting $\mathbf{a}, \mathbf{b}$ of $\mathbf{0}'$, there is a $\mathbf{c} < \mathbf{a}$ such that $\mathbf{c} \vee \mathbf{b} = \mathbf{0}'$.
– A non-uniform instance of the MSD problem, providing the first idea
- Seetapun, 1993: Every nonzero low2 c.e. deg has a MSD.
- Seetapun, 1993: A note for: for any b, $\exists a \not\geq b$, $\forall x$, if $\mathbf{b} \vee \mathbf{x} = \mathbf{0}'$, then $\mathbf{a} \vee \mathbf{x} = \mathbf{0}'$.

The Theorem

Theorem

(With Cooper)
Every nontrivial c.e. degree has a MSD, i.e., for every c.e. $\mathbf{b} \neq \mathbf{0}, \mathbf{0}'$, *there is a c.e.* $\mathbf{a}$ *such that* $\mathbf{a} < \mathbf{b}$, *and for any c.e.* $\mathbf{x}$,

$$\mathbf{b} \vee \mathbf{x} = \mathbf{0}' \iff \mathbf{a} \vee \mathbf{x} = \mathbf{0}'.$$

In fact, the proof is uniform, giving a Turing index of **a** uniformly form one of **b**.

The Requirements

We construct A uniformly from B, to satisfy:

$\mathcal{T} : A \leq_{\mathrm{T}} B$

$\mathcal{R}_e$: $D = \Phi_e(B, X_e) \rightarrow B \leq_{\mathrm{T}} X_e \oplus A$

$\mathcal{S}_e$: $B = \Theta_e(A) \rightarrow B \leq_{\mathrm{T}} \emptyset$

Convention. All use functionals of given Turing functionals are:

– increasing in arguments,

– nondecreasing in stages,

– bounded by stages, and

– dominate the identity function.

The $\mathcal{T}$-Strategy

- The *permitting method*.
- For any x, s:
  If $x \in A_{s+1} - A_s$, then there is a $y \leq x$, $y \in B_{s+1} - B_s$.
- So A is computable in B.

An $\mathcal{R}$-Module

Given $\mathcal{R}$:

$$D = \Phi(B, X) \rightarrow B \leq_{\mathrm{T}} X \oplus A$$

- Define the *length of agreement function* l as usual. That is: $l = \max\{x \mid (\forall y < x)[\Phi(B, X; y) \downarrow = D(y)]\}$.
- Say s *$\mathcal{R}$-expansionary*, if $l[s] > l[v]$, for all $v < s$.
- At $\mathcal{R}$-expansionary stages, we build Γ via a sequence of cycles k.

Cycle k

Let τ be the $\mathcal{R}$-strategy.

1. Define $d(k)$, an *agitator* of τ.
2. Wait for stage v at which $l(D, \Phi(B, X)) > d(k)$. Then:
   – define $\Gamma(X, A; k) \downarrow = B(k)$, with $\gamma(k)$ *fresh*.
3. If X has changed below $\phi(d(k))[v]$ since stage v, then
   – $\Gamma(X, A; k') \uparrow$, all $k' \geq k$, automatically whenever it occurs.
4. O.W. and B changes below $\phi(d(k))[v]$, then
   – enumerate $\gamma(k)$ into A.

The Honestification

1. $d(k)$ is used to measure the building of Γ.
2. $d(k)$ is *honest*, if $\gamma(k) > \phi(d(k))$, and *dishonest*, O.W.
3. The honestification of τ's agitators as described in steps 3 and 4 of the module ensures that
   - If $\Gamma(X, A)$ is total, then $\Gamma(X, A) = B$, and
   - $\Gamma(X, A)$ is total unless $\Phi(B, X)$ is partial.

The $\mathcal{R}$-Principle

From the module above, we ensure that the following property will be satisfied:

Definition

The $\mathcal{R}$-Principle. At every $\mathcal{R}$-expansionary stage, *s* say, every agitator of the $\mathcal{R}$-strategy is honest.

Definition

The *possible outcomes*:

$$0 <_{\mathrm{L}} 1$$

denoting infinite and finite actions respectively.

An $\mathcal{S}$-Module

- An $\mathcal{S}$-module in isolation is the *Sacks preservation method*.
- Preserve A, so that if $B = \Theta(A)$, then $\Theta(A)$ is computable, and so is B.
- However an $\mathcal{S}$-strategy below an $\mathcal{R}$-strategy is the main topic of this lecture.

The Goal of an $\mathcal{S}$-Strategy

We satisfy the following:

$$\mathcal{R}, \mathcal{S}_0, \mathcal{S}_1, \cdots$$

Examine a general $\mathcal{S}$-strategy below the $\mathcal{R}$-strategy:

1. τ, and α are the $\mathcal{R}$-, and $\mathcal{S}$-strategies respectively. Let $\tau\hat{}\langle 0\rangle \subseteq \alpha$.
2. α will try to satisfy its requirement while dealing with the injury from the Γ built by τ.
3. It will build f, and Δ, such that one of the following holds,
   - α verifies that $\Phi(B, X)$ is partial,
   - f is computable and $f = B$,
   - $\Delta(B)$ is total and $=^* K$, and
   - $B \neq \Theta(A)$.

Base Point and Base Markers

- For each k, τ defines $d_\tau^\tau(k)$, to measure $\Gamma(k)$.
- Define a *base point* of α, $n(\alpha)$ (or n) say, for Γ.
- For each $k > n$, define
  – $d_\tau^\alpha(k)$, the agitator of α for $\Gamma(k)$.

Definition

Define the *base marker* $bm(\alpha)$ of α:

$$bm(\alpha) = \phi(d_\tau^{<\alpha}(\leq n))$$

If $bm(\alpha)[s]$ unbounded, then $\Phi(B, X)$ is partial.

The Rough Outcomes

1. The *rough outcomes* of α by

$$b <_{\mathrm{L}} -1 <_{\mathrm{L}} \omega <_{\mathrm{L}} 2$$

2. The intuition:
   - b: $bm(\alpha)[s]$ unbounded,
   - -1: f is computable, and $f = B$,
   - 2: $l(B, \Theta(A))[s]$ are bounded, and
   - ω: Otherwise.

The $\mathcal{S}$-Principle

Definition

1. Given $\mathcal{S}$, define $l = l(B, \Theta(A))$ as usual.
2. Call a stage s $\mathcal{S}$-*expansionary*, if $l[s] > l[v]$ for all $v < s$.
3. For $k > n$, the agitator $d_\tau^\alpha(k)$ is *honest*, if $\gamma(k) > \phi(d_\tau^\alpha(k))$, and *dishonest*, otherwise.
4. The $\mathcal{S}$-*Principle*:
   At any $\mathcal{S}$-expansionary stage s, all agitators of α are honest.

Preuse of Agitators

Definition

1. Define functions g, and h by

$$g^\alpha(k) = \max\{\phi(d_\tau^{<\alpha}(\leq k)), \phi(d_\tau^\alpha(< k))\}$$

   In this simple case, h^α is the same as g^α.
2. Call $g^\alpha(k)$ the *preuse* of $d_\tau^\alpha(k)$.

Well Ordering of Agitators

1. We have:
   If $g^\alpha(k)[s]$ unbounded, by induction, Φ partial.
2. We need:
   If $g^\alpha(k)[s]$ are bounded, so are $d^\alpha_\tau(k)[s]$.
   [One of the crucial point in this proof.]

Definition

(*Well ordering*) For 2, we define $\Delta(B; k)$, only if

$$g^\alpha(k) < \phi(d^\alpha_\tau(k))$$

Markers below $\alpha\hat{}\langle b\rangle$

Definition

- Define:

$$b^\alpha = \max\{\gamma_{\tau'}(y') \mid \tau' \supseteq \alpha\hat{}\langle b\rangle\}$$

α will deal with injury from markers $\gamma_{\tau'}(y')$ for $\tau' \supseteq \alpha\hat{}\langle b\rangle$.

Defining $\Delta(B;k)$

For $k > n$, wait for v, at which:

- $b^\alpha < \phi(d^\alpha_\tau(k))$
- $g^\alpha(k) < \phi(d^\alpha_\tau(k))$
- $l(D, \Phi(B, X)) > d^\alpha_\tau(k)$
- $l(B, \Theta(A)) > \phi(d^\alpha_\tau(k))$, then

— define $\Delta(B;k) \downarrow = K(k)$ with $\delta(k) = \theta(\phi(d^\alpha_\tau(k)))$, and
– define *the valid use* $\delta^*(k)$ of $\Delta(B;k)$ to be the same as $\delta(k)$.

Δ-Rules

- The valid use may decrease in the construction.
- $\Delta(B;k)$ becomes undefined iff B changes below the valid use $\delta^*(k)$ of $\Delta(B;k)$.

Opening a Procedure for Rectifying Δ

1. If $\exists\, x$ such that $\Delta(B; x)\downarrow = 0 \neq 1 = K(x)$, then let k be the least such x, and we say that a cycle of rectification is started.
2. Once a cycle of rectification of $\Delta(B; k)\downarrow \neq K(k)$ is started, keep it until $\Delta(B; k)$ is rectified or Δ is *reset*.

The Choice of Agitator Algorithm

- Suppose we are rectifying $\Delta(B;k)\downarrow = 0 \neq 1 = B(k)$.
- Let v be the stage at which the $\Delta(B;k)$ was created.

Definition

The *Choice of Agitator Algorithm*:
Let m be the greatest $y > n$ such that $\gamma(y) \leq \delta(k)[v]$ holds at the current stage.

Waiting for B-change

1. Let $p(\alpha) = \max\{\phi(d^{\alpha}_{\tau}(y))[v] \mid \gamma(y) \leq \delta(k)[v]\}$.
2. Let $q(\alpha) = \delta(k)[v]$.
3. Create a *conditional restraint* $\vec{r}(\alpha) = (p(\alpha), q(\alpha))$.
4. We are going to enumerate $d^{\alpha}_{\tau}(m)$ into D.
5. Wait for B changes below $p(\alpha)$.

The Choice Lemma

Our approach works, because:

Lemma

Let $d^{\alpha}_{\tau}(m)$ be defined by the CA.

1. *For any $y > n$, if $d^{\alpha}_{\tau}(y)\downarrow$ and $\gamma(y) \leq \delta(k)[v]$, then*

$$g^{\alpha}(y)[v] \leq g^{\alpha}(m)[v]$$

2. *If $m \leq k$, then for any $y > n$, if $d^{\alpha}_{\tau}(y)$ defined, and $\gamma(y) \leq \delta(k)[v]$, then*

$$\phi(d^{\leq\alpha}_{\tau}(y))[v] \leq \phi(d^{\alpha}_{\tau}(k))[v].$$

Proof of the Choice Lemma

1. by definition of g^{α}.

By the choice of m, $y \leq m \leq k$, by the well ordering at v,
(1) $\phi(d^{\alpha}_{\tau}(k))[v] > g^{\alpha}(k)[v]$
By definition, if $y < k$, then
(2) $g^{\alpha}(k)[v] \geq \phi(d^{\leq\alpha}_{\tau}(y))[v]$, and
(3) $g^{\alpha}(k)[v] \geq \phi(d^{<\alpha}_{\tau}(k))[v]$.
By combining (1), (2) and (3):

$$\phi(d^{\leq\alpha}_{\tau}(y))[v] \leq \phi(d^{\alpha}_{\tau}(k))[v].$$

B Changes Above $g^\alpha(m)[v]$

From the choice lemma, if:

(i) $m \leq k$,

(ii) There is a b with $g^\alpha(m)[v] < b \leq \phi(d^\alpha_\tau(m))[v] \leq p(\alpha)$, which enters B after $d^\alpha_\tau(m)$ is enumerated, then

- this B-change is above the preuse of $d^\alpha_\tau(y)$ observed at stage v for any $y \leq m$,
- there is an obvious inequality

$$\Theta(A; b)[v] \downarrow = 0 \neq 1 = B(b).$$

How to preserve the Inequality?

To preserve this inequality, we just need a minimal cost of delaying the honestification of $d^{\alpha}_{\tau}(m)$, by requiring that

if B does not change below $g^{\alpha}(m)[v]$, the honestification of $d^{\alpha}_{\tau}(m)$ will not enumerate $\gamma(m) \leq \delta(k)[v]$ into A.

Special Action

More precisely:

1. Now we say that α receives *special attention* by redefining a *conditional restraint* by
$$\vec{r}(\alpha) = (g^{\alpha}(m)[v], \delta(k)[v]).$$
2. The choice lemma guarantees that the special action is of minimal cost, which delays no agitator higher than $d^{\alpha}_{\tau}(m)$, the lowest priority agitator of α.

Conditional Restraint $\vec{r}(\alpha)$

1. A *conditional restraint* $\vec{r}(\alpha)$ is a vector (p, q) for some p, q.
2. The intuition is as follows.
   If $\vec{r}(\alpha) = (p, q)$, then
   - there is no agitator belongs to any $\beta \supseteq \alpha$ can enumerate any γ-use less than or equal to q into A, unless B changes below p.

Valid Use $\phi^*(d_\tau^\beta(y))$

To realize the conditional restraint $\vec{r}(\alpha)$:

1. Define the *valid use* $\phi^*(d_\tau^\beta(y))$ for an agitator $d_\tau^\beta(y)$, any β, any y.
2. The valid use $\phi^*(d_\tau^\beta(y))$ is defined at the stage we create $\gamma_\tau(y)$, to be the ϕ-use $\phi(d_\tau^\beta(y))$.

Updating Valid Use

Definition

Let $\vec{r}(\alpha)\downarrow = (p, q)$. Then
– for any β, any y, if $\alpha \subseteq \beta$, $d_\tau^\beta(y)\downarrow$, and $\gamma_\tau(y) \leq q$, then define

$$\phi^*(d_\tau^\beta(y)) = \min\{p, \ \text{old } \phi^*(d_\tau^\beta(y))\}.$$

Enumeration of A

To better understand the conditional restraint, look at the enumeration of A.

1. A, B enumerated at odd stages, while the tree construction at even stages.
2. $d^{\beta}_{\tau}(y)$ *requires* to enumerate $\gamma_{\tau}(y)$ into A if B changes below $\phi^{*}(d^{\beta}_{\tau}(y))$.
3. A γ-marker $x = \gamma_{\tau}(y)$ is enumerated into A at stage $s+1$ iff there is an agitator $d^{\beta}_{\tau}(y)$ for some β and y, which requires to enumerate $\gamma_{\tau}(y)$ into A.

Permitting Marker of a γ-Marker

Definition

1. For $x = \gamma_\tau(y)$, define the *permitting marker of x* by
$$m(x) = \max\{\phi^*(d_\tau^\beta(y)) \mid d_\tau^\beta(y)\downarrow\}$$
2. $x = \gamma_\tau(y) \in A_{s+1} - A_s$ iff
$B_{s+1} \upharpoonright (m(x)+1) \neq B_s \upharpoonright (m(x)+1)$.

Agitation

Let $\Delta(B;k)\downarrow\neq K(k)$, and v be the stage at which $\Delta(B;k)$ was defined.

Case 1. There is $y > n$, $\gamma(y) \leq \delta(k)[v]$.

- let m be the greatest y, defined by the choice algorithm,
- let $p(\alpha) = \max\{\phi^*(d_\tau^{\leq\alpha}(y))[v], b^\alpha[v] \mid \gamma(y) \leq \delta(k)[v]\}$,
- define a conditional restraint by $\vec{r}(\alpha) = (p(\alpha), \delta(k)[v])$,
- if $p(\alpha) \leq \phi(d_\tau^\alpha(k))[v]$, then define $g(\alpha) = g^\alpha(m)[v]$,
- define the *valid use* $\delta^*(k) = p(\alpha)$, and
- enumerate $d_\tau^\alpha(m)$ into D.

Special Attention I

If there is a *b* such that

1. $g(\alpha) \downarrow < b \leq p(\alpha)$
2. *b* enters *B*, then

- Update the conditional restraint to be $(g(\alpha), \delta(k)[v])$,
- Say that α *receives special attention.*

Waiting for Response

1. B changes below $p(\alpha)$.
   - Special attention I
   - Otherwise, and B has changed below the most recent $p(\alpha)$, and then it is cancelled, and all delayed permission in this cycle of rectification are repaid.
2. While waiting for an α-expansionary stage,
   – all agitators of α honest, and $\Gamma(m)$ has been lifted.

Building f

Case 2. O.W. Then:

- Note that $\Theta(A)[v] \upharpoonright (\phi(d_\tau^\alpha(k))[v] + 1)$ have been preserved since stage v,
- we define for each $x \leq \phi(d_\tau^\alpha(k))[v]$, $f(x) = B(x)$,
- reset Δ.

Special Attention II

- Request that for any future stage, if there is a b with $bm(\alpha)[v] < b \leq b^{\alpha}[v]$, which enters B, then
- α receives special attention by defining a conditional restraint as

$$\vec{r}(\alpha) = (bm(\alpha)[v], \delta(k)[v])$$

$p(\alpha)[s]$ Is Decreasing

$p(\alpha)$ is decreasing, until either

1. B changes below the most recent $p(\alpha)$, so that all lost permission of agitators of nodes $\beta \supseteq \alpha$ are repaid, or
2. There is no more agitator $d_\tau^\alpha(y)$ available, which means that $\Theta(A)[v] \upharpoonright (\phi(d_\tau^\alpha(k))[v] + 1)$ have been cleared of all γ-markers that α has influence, so that α starts to build $f = B$, and in which case, Δ is set to be totally undefined, called *reset*, or
3. The base marker $bm(\alpha)$ has changed, in which case, both f and Δ are reset.
   [This is the reason why $b^\alpha[v]$ is included in the definition of $p(\alpha)$.]
   In any case, every inequality $\Delta(B;k) \downarrow \neq K(k)$ will be eventually rectified.

Well Ordering of $d_\tau^\alpha(m)$

Lemma

For a fixed m, $\exists\, s_0$ *say, after which if we rectify* $\Delta(B; k) \downarrow \neq K(k)$ *and enumerate* $d_\tau^\alpha(m)$ *into D, then* $m \leq k$.

This allows us to prove:

Lemma

- *If* $g^\alpha(m)[s]$ *are bounded, then so are* $d_\tau^\alpha(m)[s]$.

Proof. By the choice lemma, and the special action I.

Correctness of α

1. If the base markers $bm(\alpha)[s]$ of α unbounded, then $\Phi(B, X)$ is partial.
2. O.W. and f is built infinitely many times, then $f =^* B$.
3. O.W., and $\Delta(B)$ is total, then $\Delta(B) =^* K$.

Analysis of Δ

Suppose that $\Delta(B)$ is partial. Let k be the least $y > n$ at which Δ diverges. Then $\theta\phi(d^{\alpha}_{\tau}(k))[s]$ unbounded.

Case 1. $d^{\alpha}_{\tau}(k)[s]$ are unbounded.
$g^{\alpha}(k)[s]$ unbounded. $\Phi(B, X)$ is partial.

Case 2. O.W. and $\phi(d^{\alpha}_{\tau}(k))[s]$ unbounded.
$\Phi(B, X)$ is partial.

Case 3. O.W.
Then $\Theta(A)$ is partial.

Possible Outcomes

1. Refine the outcome b to approximate the least x such that $\gamma(x)[s]$ are unbounded, to provide a well defined environment for lower priority strategies.
2. Guess the least $k > n$ such that $\Delta(B;k)$ diverges, and by property IV, refine furthermore the outcome to approximate the least x such that $\gamma(x)[s]$ are unbounded, if α has verified that $\Phi(B,X)$ is partial.

Comments

Suppose that for some $k > n$, $d^{\alpha}_{\tau}(k)[s]$ are unbounded. Let k_1 be the least k.
Can we approximate the k_1 at this level?

No!
The possible outcome is the least $k > n$ such that $\Delta(B;k)$ diverges.
Let k_0 be the least such k. It is possible that $k_0 < k_1$.

Conditional Restraint Rules

Let s be a stage, and s^- be the least stage $s' < s$ such that $p(\alpha)[t]$ is defined for all $t \in [s^-, s]$. Then we have the following rules:

1. $p(\alpha)$ is decreasing in stages $[s^-, s]$.
2. If $p(\alpha)[s+1] \neq p(\alpha)[s]$, then one of the following conditions occurs
   (a) $p(\alpha)[s+1) < p(\alpha)[s]$,
   (b) B changes below $p(\alpha)[s]$ at stage $s+1$.
   (c) Any agitator whose honestification has been delayed by any $p(\alpha)[t]$ for any $t \in [s^-, s]$, is cancelled.

Valid Use of an Agitator Is Decreasing

Considering more $\mathcal{S}$-strategies:

1. Given an agitator $d_\tau^\beta(y)$, the valid use may be delayed by the collection of $p(\alpha)$ for all $\alpha \subseteq \beta$. Since for each α, $p(\alpha)$ is decreasing in stage.
2. By definition, the valid use $\phi^*(d_\tau^\beta(y))[s]$ is decreasing in stages.

How about the permitting marker of a γ-marker, $x = \gamma_\tau(y)$, say? We need:
$m(x)[s]$ are decreasing in stages.

A Maxmin Lemma

Suppose that $Q_1 \supseteq Q_2 \supseteq \cdots \supseteq Q_n \supset \emptyset$ is a sequence of finite sets Q_j of nodes, $j = 1, 2, \cdots, n$, and that for every j, every node $\alpha \in Q_j$, there is a decreasing sequence $a_1^\alpha, a_2^\alpha, \cdots, a_j^\alpha$. For every $j = 1, 2, \cdots, n$, we define $a_j = \max\{a_j^\alpha \mid \alpha \in Q_j\}$. Then

$$a_1 \geq a_2 \geq \cdots \geq a_n.$$

Proof. By definition.

The Permitting Marker of a γ-Marker

Recall that:
Let $x = \gamma_\tau(y)$. Define the permitting marker $m(x)$ defined by

$$m(x) = \max\{\phi^*(d_\tau^\beta(y)) \mid d_\tau^\beta(y) \downarrow\}.$$

Lemma
Let $x = \gamma_\tau(y)$. The permitting marker $m(x)[s]$ will be decreasing in stages.

Proof. By the maxmin lemma.

New Problems for the Full Proof

1. The $\mathcal{R}$-, $\mathcal{S}$-principles.
2. The conditional restraint rules, the principles of decreasing valid uses, and of decreasing permitting markers.
   Both 1 and 2 are essential, and need:
3. New choice algorithm and choice lemma for the $\mathcal{S}$-strategy below more than one $\mathcal{R}$-strategy.