Positive-measure domination

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Thanks

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Recall from Peter Cholak's talk:

Definition

$f : \omega \to \omega$ is *uniformly a.e. dominating* if for measure-one many X, and all $g \leq_T X$, f dominates g.

- There is a relationship between such functions and effective randomness, the precise nature of which is not yet known.
- I will consider a variation, positive-measure domination, which has so far been easier to deal with.
- As a byproduct we will get a new characterization of K-trivial reals, which does not mention randomness or Kolmogorov complexity directly.

Turing functionals

Definition

Let Φ be a Turing functional.

$$\text{Tot}(\Phi) = \{X : \Phi^X \text{ is a total function in } \omega^\omega\}.$$

We will only be interested in the case where $\text{Tot}(\Phi)$ has positive measure.

Definition

Let Φ be a Turing functional. We write

$$\Phi < A$$

if either $\mu\,\text{Tot}(\Phi) = 0$, or for some $f \leq_T A$,

$$\mu\{X \in \text{Tot}(\Phi) : \Phi^X \text{is dominated by } f\} > 0.$$

Positive-measure domination

Definition

A is positive-measure dominating (PMD) if for each Turing functional Φ, $\Phi < A$.

- Suppose Φ_i, $i \in \omega$ are all the Turing functionals.
- Recall from Peter Cholak's talk: The functional Ψ given by $\Psi^{0^i 1 X} = \Phi_i^X$ is universal for uniform a.e. domination.
- However, $\Psi < 0$.
- Find an example of a functional Ξ with $\Xi \not< 0$?

A functional $\Xi \not< 0$

- Fix a constant c. Let $\Xi^X(s)$ be the least stage $t > s$ at which X looks like it is 2-random, with constant c. That is,

$$\Xi^X(s) = (\mu t > s)(\forall n \leq t\ K_t^{0'_t}(X \upharpoonright n) \geq n - c).$$

- Ξ is total for positive-measure many X, all of which are 2-randoms.
- If $\Xi < 0$ then we get a Π^0_1 class of 2-randoms, contradiction.
- This Ξ is actually universal for positive-measure domination.

A Σ^0_3 class

- Fix Ξ that is universal for positive-measure domination (PMD).
- A is positive-measure dominating iff $\Xi < A$.
- $\exists f \leq_T A, \exists \epsilon > 0, \forall n, \exists s,$

$$\mu\{X : \Xi^X \upharpoonright n \text{ is total and majorized by } f \text{, by stage } s\} > \epsilon.$$

- $\{A : A \text{ is positive-measure dominating}\}$ is a Σ^0_3 class.
- By Jockusch-Soare (1972), each member computes a path in a recursive tree without recursive paths.
- Cholak-Greenberg-Miller show there is a PMD degree which is not DNR.
- Mathias and Cohen generics are not PMD.

Definition (PA reducibility)

$A \leq_{PA} B$ if for each computable tree T, if A computes a path in T then so does B.

Definition (Low-for-random reducibility)

$A \leq_{LR} B$ if each B-random is A-random.

Definition (Running time function)

$\varphi^X(n) = (\mu s)(\forall m < n)(\Phi_s^X(m)\downarrow)$.

Definition (Positive-measure domination reducibility)

$A \leq_{PMD} B \Leftrightarrow \forall \Phi(\varphi < A \rightarrow \varphi < B)$.

Theorem (Nies, Hirschfeldt, Stephan, Terwijn)

The following are equivalent for $A \in 2^\omega$:

- *A is low for random: each Martin-Löf random real is Martin-Löf random relative to A.*
- *A is K-trivial: $\exists c \forall n\, K(A \upharpoonright n) \leq K(\emptyset \upharpoonright n) + c$.*
- *A is low for K: $\exists c \forall n\, K(n) \leq K^A(n) + c$.*
- *$\exists Z \geq_T A$, Z is ML-random relative to A.*
- *$A \leq_T 0'$ and Ω is ML-random relative to A*

All are defined in terms of randomness or Kolmogorov complexity, K.

- We have $A \leq_{PA} 0 \Rightarrow A \leq_{PMD} 0$.
- The only Δ^0_2 degree with $A \leq_{PA} 0$ is 0.
- What about Δ^0_2 degrees with $A \leq_{PMD} 0$?

Theorem (Main Theorem)

The Δ^0_2 degrees with $A \leq_{PMD} 0$ are exactly the K-trivials. That is:

$$A \leq_T 0' \;\;\&\;\; A \leq_{PMD} 0 \Leftrightarrow A \leq_{LR} 0.$$

Moral: The K-trivials are those Δ^0_2 reals that are no good at dominating Turing functionals for positive-measure many oracles.

Theorem

A is positive-measure dominating iff $0' \leq_{LR} A$.

An earlier result with Hirschfeldt had this equivalence for $A \leq_T 0'$ only.

Proof information

Proof deals only with the notion low for random, not the other equivalent forms.
Except, the proof relies on the fact that low for random $\Rightarrow \Delta_2^0$.

Definition

Π_1^μ denotes the collection of all Π_1^0 classes of positive measure.

Lemma (Kučera)

For each $A \in 2^\omega$, each A-random is a tail of each $\Pi_1^\mu(A)$ class.

Lemma (implicit in Dobrinen and Simpson)

Let Φ be a Turing functional and $B \in 2^\omega$. The following are equivalent:

- *Tot(Φ) has a $\Pi_1^\mu(B)$ subclass.*
- $\varphi < B$.

Theorem

Let A, B be reals. The following are equivalent:

1. $A \leq_{LR} B$
2. *Each* $\Pi^{\mu}_1(A)$ *class has a* $\Pi^{\mu}_1(B)$ *subclass.*
3. *Some* $\Pi^{\mu}_1(A)$ *class consisting entirely of A-random reals has a* $\Pi^{\mu}_1(B)$ *subclass.*

More generally

Theorem

Let $A, B \in 2^\omega$. The following are equivalent:

- $A' \leq_{LR} A \oplus B$.
- *Each* $\Pi^\mu_1(A')$ *class has a* $\Pi^\mu_1(A \oplus B)$ *subclass.*
- *Each* $\Pi^\mu_2(A)$ *class has a* $\Pi^\mu_1(A \oplus B)$ *subclass.*
- $\forall \Phi$, *if* $Tot(\Phi^A)$ *has positive measure then it has a* $\Pi^\mu_1(A \oplus B)$ *subclass.*
- $\forall \Phi(\varphi^A < A \oplus B)$ *($A \oplus B$ is PMD relative to A).*

In particular with $A = 0$ we get the characterization of PMD degrees in terms of randomness.

Why are $A \leq_{LR} 0$ and $A \leq_{PMD} 0$ equivalent for $A \leq_T 0'$?

Lemma

Let $A \in 2^\omega$. The following are equivalent:

1. *Each $\Pi_1^\mu(A)$ class has a Π_1^μ subclass.*
2. *$A \leq_T 0'$ and for each Φ, if Tot(Φ) has a $\Pi_1^\mu(A)$ subclass then $\varphi < 0$.*

(1)⇒(2) uses external facts (low for random reals are Δ_2^0).
(2)⇒(1) uses the fact that if $A \leq_T 0'$ then each $\Pi_1^0(A)$ class is a Π_2^0 class and hence is Tot(Φ) for some Φ.