Computable Measure Theory

Decheng DING

joint work with

Yongcheng WU

Nanjing University, China

1 Computable Analysis

1.1. Motivations

One of the important contributions of recursion theory to mathematics is the concept of computability. In the classical mathematics the computability is not considered. Therefore there is a big gap between the mathematics and computer science.

- **Theoretical motivation**: Computable analysis studies the computability of the reals and real valued functions, etc.

  Computable analysis (recursive analysis) wants to find which computations in analysis are possible and which is not.

- **Practical motivation**: To provide a sound algorithmic foundation for numerical computations.
  It is a well-known problem that there are some problems in the numerical computations applying floating-point arithmetic.

- **Example:** Consider the following system of linear equations [Sch03]:

$$\begin{cases} 40157959.0 \cdot x + 67108865.0 \cdot y &= 1 \\ 67108864.5 \cdot x + 112147127.0 \cdot y &= 0. \end{cases}$$

Applying the well-known formula

$$x = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} = \frac{b_1 a_{22} - b_2 a_{12}}{a_{11}a_{22} - a_{21}a_{12}}, y = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} = \frac{b_1 a_{22} - b_2 a_{12}}{a_{11}a_{22} - a_{21}a_{12}},$$

the solution computed by the floating point arithmetic with *double precision* (IEEE standard 754,53bit mantissa) is

$$\widetilde{x} = 112147127, \widetilde{y} = -67108864.5$$

However, the correct solution is namely

$$x = 224294254, y = -134217729.$$

- The reason for this error is that the computed value for $a_{11}a_{22} - a_{21}a_{12}$ is 1.0 whereas the correct value is 0.5.
- Increasing the size of the mantissa does not help substantially, because other systems of linear equations remain unsolvable.
- Exact Computation: To solve such problem, a new research field ”exact computation” has been established. a key idea is to represent the real numbers exactly [Yap96, Yan04].
- Computable analysis is a theoretical foundation of Exact Computation.

1.2. History and different kinds of approaches:

Computable analysis is a new emerging subject of research. There are several different approaches in the area.

Unlike the classical computability theory, there has not been a generally accepted definition of computability of the reals and the real functions.

- A. Turing: In [Tur36], he gave a concise definition of computable real numbers. In [Tur37], he also noticed that the binary and ternary representations of the reals are not applicable to define a reasonable definition of computable real functions.
- Banach and S. Mazur: In [Maz63], they defined the so-called sequential computability of real functions.
- A. Grzegorczyk [Grz55] and D. Lacombe [Lac55a] advised to "name" a real number by a quickly converging sequence of rational numbers, and defined that: a real function f is computable iff there is some machine(digital computer, Turing machine) which computes the name of $f(x)$ with each name of x.

- Based on the work of A. Grzegorczyk and D. Lacombe， M. Pour-El and J. Richards studied a variety of problems in classical analysis and mathematical phyiscs. E.g., they have studied computability of Hilbert space, L^p-spaces and more generally, arbitrary Banach spaces [PER89, PE99].
- Based on the work of A. Grzegorczyk and D. Lacombe， K. Weirauch and C. Kreitz innovated and developed the Type-2 Theory of Effectivity, TTE for short, which is characteristic of its representation theory and Type-2 Turing machines [KW85, Wei00].
- Ko, Ker-I applied NP-completeness theory to study the computational complexity of some basic numerical computations such as maximum and integration [Ko91, Ko98].

- BCSS and their real-RAM [BCSS96, BCSS98].
- Domain theory: A. Edlat and other peoples [Eda95a, Eda96, Eda97].
- The Markov constructive mathematics and Markov algorithms [Kus84, Kus99].
- Subjects related closely: Brouwer's intuitionist analysis [Bro75, Bro75a]; Bishop-Bridge's constructive analysis [BB85].
- A lot work to do: (1) many basic problems unsolved; (2) many algorithms in numerical analysis should be reconsidered in the more sound sense of computable analysis.

2 Type-2 theory of effectivity

2.1. Characteristics

- It uses naming systems of computational objects to define computability about the computational objects. Different naming systems induce accordingly different types of computability.
- It applies generalized Turing machines, called Type-2 Turing machines, as its computation models.
- It is a natural extension of the classical recursion theory.
- Therefore, it is sound and realistic.

2.2. Naming systems

- **Definition** TTE uses finite as well as infinite strings to name the real numbers, real valued functions, and so on. Therefore there are two kinds of naming systems. Let Σ be a set of symbols including $\{0, 1, /, -\}$. Let Σ^* resp. Σ^ω be the set of all finite resp. infinite strings on Σ.

  1. A *notation* of a non-empty set X is a surjective mapping $\nu :\subseteq \Sigma^* \to X$.
  2. A *representation* of a non-empty set X is a surjective mapping $\delta :\subseteq \Sigma^\omega \to X$.

  Let δ be a naming system of X. If $\delta(p) = x$, p is called a δ-*name* of x.

- **Examples of Notations**

  1. $\nu_{\mathbb{N}} :\subseteq \Sigma^* \to \mathbb{N}$ is a notation of $\mathbb{N}$ using the binary expansions to name natural numbers, where $\mathbb{N}$ is the set of natural numbers.
  2. $\nu_{\mathbb{Z}} :\subseteq \Sigma^* \to \mathbb{Z}$ is a notation of $\mathbb{Z}$ defined by $\nu_{\mathbb{Z}}(w) = \nu_{\mathbb{N}}(w)$ and $\nu_{\mathbb{Z}}(-w) = -\nu_{\mathbb{N}}(w)$, where $\mathbb{Z}$ is the set of integers.
  3. $\nu_{\mathbb{Q}} :\subseteq \Sigma^* \to \mathbb{Q}$ is a notation of $\mathbb{Q}$ defined by $\nu_{\mathbb{Q}}(u/v) = \nu_{\mathbb{Z}}(u)/\nu_{\mathbb{Z}}(v)$, where $\mathbb{Q}$ is the set of rational numbers.

- **Examples of Representations**

1. $\rho :\subseteq \Sigma^\omega \to \mathbb{R}$ is a representation of $\mathbb{R}$ (the set of real numbers) using a converging sequence of open intervals with rational ends to represent a real.

2. $\overline{\rho}_< :\subseteq \Sigma^\omega \to \overline{\mathbb{R}}$ is a representation of $\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$ using a converging from below sequence of rational numbers or "$-\infty$" to represent an extended real.

3. $\overline{\rho}_> :\subseteq \Sigma^\omega \to \overline{\mathbb{R}}$ is a representation of $\overline{\mathbb{R}}$ using a converging from above sequence of rational numbers or "$+\infty$" to represent an extended real.

2.3. Type-2 Turing machines

- **TT-machine**: (1) several one-way read-only input tape; (2) several two-way working tapes; (3) a one-way write-only output tape, which means that the output can not be revised. TT-machine can transform finite strings as well as infinite strings.
- **Finiteness property**: Each prefix of the output string is determined by a prefix of the input string.

  This implies that a string function is computable only if it is continuous with respect to the discrete topology of Σ^* and the Cantor topology of Σ^ω.

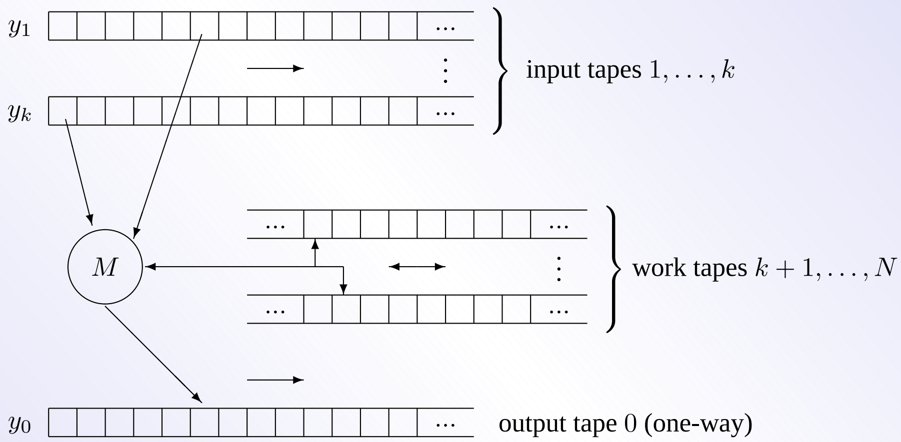


Figure 1: A Type-2 Turing machine computing $y_0 = f_M(y_1, \dots, y_k)$

2.4. Computability on Σ^* and Σ^ω

- Computable string: (1)Every word $\omega \in \Sigma^*$ is computable (2)A sequece $p \in \Sigma^\omega$ is computable if the constant function $f : () \to \Sigma^\omega$, $f() = p$ is computable.
- A subset $A \subseteq \Sigma^*$ is called recursive or decidable if its characteristic function is computable.
- A subset $A \subseteq \Sigma^*$ is called r.e. if it is the domain of a computable function $f : \Sigma^* \to \Sigma^*$.
- Computable string function: $f :\subseteq X \to Y$, where X, Y is Σ^* or Σ^ω is computable if is is computed by a type-2 machine M.

2.5. Computability via naming systems

Let X, Y be non-empty sets with naming systems δ, γ respectively.

- An element $x \in X$ is said to be δ-*computable* if there is a computable δ-name of x.
- A set $A \subseteq X$ is called δ-*r.e.* if $\delta^{-1}(A)$ is r.e.
- A function $f :\subseteq X \to Y$ is said to be (δ, γ)-*computable* if there is a computable string function g s.t. $f \circ \delta = \gamma \circ g|_{\mathrm{dom}(f \circ \delta)}$.

3 Computable measure theory

3.1. Motivations

- Computable measure theory studies computability of functions related to measures. E.g.: (1) Is the measure of a measure space computable? (2) Are the union, intersection, etc., of measurable sets computable? (3) Does a classical theorem still holds in computable sense?
- It is intended to provide a sound theoretical foundation for the computations related to measures in computer science.

3.2. Contributions

- Šanin [San68] introduces computable measurable sets of real numbers as limits of fast converging sequences of simple sets w.r.t. the pseudo-metric $d(A, B) := \mu(A \Delta B)$.
- Ko [Ko91] applies this idea to define polynomial time approximable sets and functions and studies their behavior under some operations.
- Edalat [Eda95a] applies Domain Theory to investigate dynamical systems, measures and fractals.
- Computability on probability measures and on random variables has been studied in the framework of TTE by Müller and Weihrauch [Mue99, Wei99a].

3.3. Our work

We studied several basic problems related to an infinite measure space $(\Omega, \mathcal{A}, \mu)$:

- How to represent the measurable sets?
- How about computability of the measure and the set operations on the measurable sets?
- Show a computable version of the classical Daniell-Stone theorem.
- Besides, we have studied how to represent the measurable functions and computability of the arithmetic operations on the measurable functions.

4 The definition of computable measure space

4.1. Basic concepts in the classical measure theory

Let Ω be a non-empty set.

- A *ring* in Ω is a set $\mathcal{R}$ of subsets of Ω such that
  1. $\emptyset \in \mathcal{R}$,
  2. $A \cup B \in \mathcal{R}$ and $A - B \in \mathcal{R}$ if $A, B \in \mathcal{R}$.
- A σ-*algebra* in Ω is a set $\mathcal{A}$ of subsets of Ω such that
  1. $\Omega \in \mathcal{A}$,
  2. $A^c = \Omega - A \in \mathcal{A}$ if $A \in \mathcal{A}$,
  3. $\bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$ if $A_1, A_2, \ldots \in \mathcal{A}$.

- For any system $\mathcal{E}$ of subsets of Ω let $\sigma(\mathcal{E})$ be the smallest σ-algebra in Ω containing $\mathcal{E}$.
- Let $\mathcal{A}$ be a σ-algebra in Ω. A *measure* on $\mathcal{A}$ is a function $\mu : \mathcal{A} \to \overline{\mathbb{R}}$ such that

  1. $\mu(\emptyset) = 0$, $\mu(A) \geq 0$ for $A \in \mathcal{A}$, and
  2. $\mu(\bigcup_{i=1}^{\infty} A_n) = \sum_{i=1}^{\infty} \mu(A_n)$ if $A_1, A_2, \ldots \in \mathcal{A}$ are pairwise disjoint.

  A measure μ is called *finite* resp. *infinite* if $\mu(\Omega) < \infty$ resp. $\mu(\Omega) = \infty$.
- $(\Omega, \mathcal{A}, \mu)$ is called a *measure space* if $\mathcal{A}$ is a σ-algebra in Ω and μ is a measure on $\mathcal{A}$. $(\Omega, \mathcal{A}, \mu)$ is said to be *finite* resp. *infinite* if μ is *finite* resp. *infinite*.

4.2. Computable measure space

In order to introduce computability to measure theory, we use the concept of computable measure space to replace the more abstract classical one.

Definition 4.1 *A* computable measure space *(abbr.* CMS*) is a quintuple* $(\Omega, \mathcal{A}, \mu, \mathcal{R}, \alpha)$ *such that*

1. $(\Omega, \mathcal{A}, \mu)$ *is a measure space with* $\mathcal{A} = \sigma(\mathcal{R})$*,*
2. $\mathcal{R}$ *is a countable ring of finitely measurable sets with* $\Omega = \bigcup \mathcal{R}$*,*
3. $\alpha :\subseteq \Sigma^* \to \mathcal{R}$ *is a notation of* $\mathcal{R}$ *with recursive domain,*
4. μ *is* (α, ρ)*-computable and the union and the set difference are* (α, α, α)*-computable.*

A computable measure space is essentially the classical concept $(\Omega, \mathcal{A}, \mu)$ extended by the effective part $(\mathcal{R}, \alpha)$, which provides fundamental and necessary conditions of computability.

4.3. Properties

These lemmas apply to show that in the computable measure space $(\Omega, \mathcal{A}, \mu, \mathcal{R}, \alpha)$ the class $\mathcal{A}$ of the measurable sets can be approximate effectively by the countable subcalss $\mathcal{R}$.

Lemma 4.1 *$\mathcal{A} = m(\mathcal{R})$, where $m(\mathcal{R})$ is the minimal class closed under the limits of the monotone sequence in $\mathcal{R}$.*

Lemma 4.2 *There is a $(\nu_{\mathbb{N}}, \alpha)$-computable partition (D_n) of Ω.*

5 Undecidability of the measurable sets

5.1. Undecidability of the a.e. equality and a.e. inclusion

Let $(\Omega, \mathcal{A}, \mu, \mathcal{R}, \alpha)$ be a CMS with $\mu(\Omega) = \infty$. Let δ be a representation of $\mathcal{A}$.

Theorem 5.1 $\{(A, B) \in \mathcal{A} \times \mathcal{A} : A = B\ a.e.\}$ *is not r.e. with respect to any representation of* $\mathcal{A}$.

Theorem 5.2 $\{(A, B) \in \mathcal{A} \times \mathcal{A} : A \subset B\ a.e.\}$ *is not r.e. with respect to any representation of* $\mathcal{A}$.

5.2. Contradiction on computability of the measure and the set operations

Theorem 5.3 *1. If μ is $(\delta, \rho_>)$-computable on $\mathcal{A}_0$, then the intersection $\cap$ is not computable with respect to δ on $\{(A, B) : A, B \in \mathcal{A}_{\infty\infty}, A \cap B \in \mathcal{A}_0\}$.*

2. If μ is $(\delta, \overline{\rho}_>)$-computable on $\mathcal{A}_\infty$, then the intersection $\cap$ is not computable with respect to δ on $\{(A, B) : A, B \in \mathcal{A}_{\infty\infty}, A \cap B \in \mathcal{A}_\infty\}$.

Theorem 5.4 *1. If μ is $(\delta, \rho_>)$-computable on $\mathcal{A}_0$, then the difference "$-$" is not computable with respect to δ on $\{(A, B) : A, B \in \mathcal{A}_{\infty\infty}, A - B \in \mathcal{A}_0\}$.*

2. If μ is $(\delta, \overline{\rho}_>)$-computable on $\mathcal{A}_\infty$, then the difference "$-$" is not computable with respect to δ on $\{(A, B) : A, B \in \mathcal{A}_{\infty\infty}, A - B \in \mathcal{A}_\infty\}$.

6 Representations and computability of the measurable sets

6.1. Problems

1. How to represent the measurable sets from a CMS with an infinite measure? The difficulty is how to use a sequence of finitely measurable sets to approximate a general measurable set.

2. What is the computability of the measure and the set operations induced by the representations obtained?

We will introduce three different representations of $\mathcal{A}$: $\delta_{\mathcal{T}_1}$, $\delta_{\mathcal{T}_2}$ and $\delta_{\mathcal{M}}$.

- $\delta_{\mathcal{T}_1}$ and $\delta_{\mathcal{T}_2}$ are defined via computable topological spaces derived from $(\Omega, \mathcal{A}, \mu, \mathcal{R}, \alpha)$.
- $\delta_{\mathcal{M}}$ is defined via a computable $\mathcal{L}^*$-space derived from $(\Omega, \mathcal{A}, \mu, \mathcal{R}, \alpha)$.
- To define a representation instead of a *multi-representation*, we identify every pair of measurable sets A, B such that $\mu(A \triangle B) = 0$.

6.2. Computable topological spaces

For introducing computability to a set of continuous-typed objects, some concrete information about the topology should be known.

Definition 6.1 *(Weihrauch)*

1. An *effective topological space* is a triple $\mathbf{S} = (M, \sigma, \nu)$ where M is a non-empty set, σ is a countable collection of subsets of M such that

$$x = y\, if\ \{A \in \sigma :\ x \in A\} = \{A \in \sigma :\ y \in A\}$$

   and $\nu :\subseteq \Sigma^* \to \sigma$ is a notation of σ.

2. A *computable topological space* is an effective topological space for which the equivalence problem

$$\{(u, v) :\ u, v \in \mathrm{dom}(\nu) \text{ and } \nu(u) = \nu(v)\} \text{ is r.e..}$$

Notations:

1. let $w = a_1 a_2 ... a_n$, then $\langle w \rangle := 110a_1 0a_2 0 \cdots 0a_n 011$.

2. $w \lhd p$ means that w is a substring of the sequence p.

Definition 6.2 Let $\mathbf{S} := (M, \sigma, \nu)$ be an effective topological space. Define the *standard representation* $\delta_{\mathbf{S}} : \subseteq \Sigma^\omega \to M$ of $\mathbf{S}$ by $\delta_{\mathbf{S}}(p) := x$ for all $x \in M$ and $p \in \Sigma^\omega$ such that $\{A \in \sigma : x \in A\} = \{\nu(w) : \langle w \rangle \lhd p\}$ and $\{w : \langle w \rangle \lhd p\} \subseteq \mathrm{dom}(\nu)$.

6.3. The first representation $\delta_{\mathcal{T}_1}$

Let $\sigma_1 := \{\uparrow(E,r), \downarrow(E,r) : E \in \mathcal{R}, r \in \mathbb{Q}^+\}$ where

$$\begin{cases} \uparrow(E,r) := \{A \in \mathcal{A} : \mu(E-A) < r\} \\ \downarrow(E,r) := \{A \in \mathcal{A} : \mu(A-E) < r\}. \end{cases}$$

$\nu_{\sigma_1} : \subseteq \Sigma^* \to \sigma_1$ is defined by

$$\begin{cases} \nu_{\sigma_1}(0\langle u\rangle\langle v\rangle) := \uparrow(\alpha(u), \nu_{\mathbb{Q}}(v)) \\ \nu_{\sigma_1}(1\langle u\rangle\langle v\rangle) := \downarrow(\alpha(u), \nu_{\mathbb{Q}}(v)). \end{cases}$$

We have

Proposition 6.1 $(\mathcal{A}, \sigma_1, \nu_{\sigma_1})$ *is a computable topological space.*

Let $\mathbb{T}_1 := (\mathcal{A}, \sigma_1, \nu_{\sigma_1})$, $\delta_{\mathcal{T}_1}$ the standard representation of $\mathbb{T}_1$.

Theorem 6.1 *1.* μ *is* $(\delta_{\mathcal{T}_1}, \rho)$*-computable on* $\mathcal{A}_0$ *and* $(\delta_{\mathcal{T}_1}, \overline{\rho}_<)$*-computable on* $\mathcal{A}_\infty$.

2. $\cup$ *is computable w.r.t.* $\delta_{\mathcal{T}_1}$.

3. $\cap$ *is computable w.r.t.* $\delta_{\mathcal{T}_1}$ *on* $\{(A, B) : A, B \in \mathcal{A}_0, \text{ or } A \cap B \in \mathcal{A}_\infty\}$ *but not on the complement.*

4. $(\cdot)^c$ *is computable w.r.t.* $\delta_{\mathcal{T}_1}$ *on* $\mathcal{A}_0$ *but not on* $\mathcal{A}_\infty$.

5. "$-$*" is computable w.r.t.* $\delta_{\mathcal{T}_1}$ *on* $\mathcal{A} \times \mathcal{A}_0$ *but not on the complement.*

6.4. The second representation $\delta_{\mathcal{T}_2}$

Use the computable partition sequence (D_n) of Ω.
Let $C_n := \cup_{i\leq n} D_i$ for all $n \in \mathbb{N}$.
Let $\sigma_2 := \{E(i) : E \in \mathcal{R}, i \in \mathbb{N}\}$ where

$$E(i) := \{A \in \mathcal{A} : \mu((A \triangle E) \cap C_i)) < 2^{-i}\}.$$

Define $\nu_{\sigma_2} :\subseteq \Sigma^* \to \sigma_2$ by

$$\nu_{\sigma_2}(\langle u\rangle\langle v\rangle) := E(i),$$

where $\alpha(u) = E$ and $\nu_{\mathbb{N}}(v) = i$.
Proposition 6.2 $(\mathcal{A}, \sigma_2, \nu_{\sigma_2})$ *is a computable topological space.*

Let $\mathbb{T}_2 := (\mathcal{A}, \sigma_2, \nu_{\sigma_2})$ and denote by $\delta_{\mathcal{T}_3}$ the standard representation of $\mathbb{T}_2$.

Theorem 6.2 *1.* $\cup$, $\cap$, $(\cdot)^c$ *and "$-$" are computable w.r.t.* $\delta_{\mathcal{T}_2}$.
2. μ *is* $(\delta_{\mathcal{T}_2}, \overline{\rho}_<)$*-computable on* $\mathcal{A}$ *but not* $(\delta_{\mathcal{T}_2}, \rho_>)$*-computable on* $\mathcal{A}_0$.

6.5. Computable $\mathcal{L}^*$-space

Definition 6.3 A limit space ($\mathcal{L}^*$-space) is a pair $(X, \rightarrow)$ where X is a set and "$\rightarrow$" is a limit relation.

Definition 6.4 A *computable $\mathcal{L}^*$-space* is a quadruplet $(X, \rightarrow, D, \nu)$ such that $(X, \rightarrow)$ is a $\mathcal{L}^*$-space, D is a countable dense subset of X and ν is a notation of D.

Definition 6.5 Let $\mathcal{X} := (X, \rightarrow, D, \nu)$ be a computable $\mathcal{L}^*$-space. A function δ is said to be a *standard representation* of $\mathcal{X}$, if δ is an admissible representation of the limit space $(X, \rightarrow)$, and for each $p \in \mathrm{dom}(\delta)$, $p = \langle u_1, u_2, \cdots \rangle$ such that $u_i \in \mathrm{dom}(\nu)$ and $\nu(u_i) \rightarrow \delta(p)$.

6.6. The third representation $\delta_{\mathcal{M}}$

To construct a computable $\mathcal{L}^*$-space of $\mathcal{A}$, we need to define an appropriate limit relation on $\mathcal{A}$.

An admissible limit relation should embody the topology induced by the pseudo-metric d on $\mathcal{A}_0$.

Definition 6.6 The limit relation $\to_\mu :\subseteq \mathcal{A}^\infty \to \mathcal{A}$ is defined by that $A_n \to_\mu A$ if $A \in \mathcal{A}_0$ and $\mu(A_n \Delta A) \leq 2^{-n}$ $(n \in \mathbb{N})$, or if $A \in \mathcal{A}_\infty, \mu(A_n - A) \leq 2^{-n}$ and $\mu(A \cap C_n - A_n) \leq 2^{-n}$ $(n \in \mathbb{N})$.

Proposition 6.3 *Let* (A_n) *be a sequence such that* $A_n \to_\mu A$*.*

1. *If* $A \in \mathcal{A}_0$ *then* $\lim_n \mu(A_n \Delta A) = 0$*, where* $\mathcal{A}_0 := \{A \in \mathcal{A} : \mu(A) < \infty\}$*.*
2. *If* $A \in \mathcal{A}_\infty$ *then* $\lim_n \mu((A_n \Delta A) \cap C_k) = 0$ *for all* $k \in \mathbb{N}$*, where* $\mathcal{A}_\infty := \{A \in \mathcal{A} : \mu(A) = \infty\}$*.*

This proposition shows that our choice of the convergence is reasonable.

Proposition 6.4 *$\mathcal{R}$ is dense in $(\mathcal{A}, \to_\mu)$.*

Theorem 6.3 *$(\mathcal{A}, \to_\mu, \mathcal{R}, \alpha)$ is a computable $\mathcal{L}^*$-space.*

We denote by $\delta_\mathcal{M}$ the standard representation of $(\mathcal{A}, \to_\mu, \mathcal{R}, \alpha)$ (Definition 6.5).

Theorem 6.4 *1. μ is $(\delta_\mathcal{M}, \overline{\rho})$-computable.*

2. $\cup$ is $(\delta_\mathcal{M}, \delta_\mathcal{M}, \delta_\mathcal{M})$-computable w.r.t. $\delta_\mathcal{M}$.

3. $\cap$ is $(\delta_\mathcal{M}, \delta_\mathcal{M}, \delta_\mathcal{M})$-computable on $\{(A, B) : A$ or $B \in \mathcal{A}_0,$ or $A \cap B \in \mathcal{A}_\infty\}$.

4. $-$ is $(\delta_\mathcal{M}, \delta_\mathcal{M}, \delta_\mathcal{M})$-computable on $\{(A, B) : A$ or $B \in \mathcal{A}_0,$ or $A - B \in \mathcal{A}_\infty\}$.

5. $(\cdot)^c$ is $(\delta_\mathcal{M}, \delta_\mathcal{M})$-computable on $\mathcal{A}_0 \cup \mathcal{A}_{\infty\infty}$.

6.7. Remarks on the three representations

- By Theorem 6.1 and Theorem 6.4, the computability of the measure and the set operations induced by $\delta_{\mathcal{M}}$ is more strong than that induced by $\delta_{\mathcal{T}_1}$.
- It can be shown directly that $\delta_{\mathcal{M}} < \delta_{\mathcal{T}_2}$, namely $\delta_{\mathcal{M}}$ is more strong than $\delta_{\mathcal{T}_2}$. However by Theorem 6.2 and Theorem 6.4, the computability of the measure and the set operations induced by these two different representations have different superiorities.

7 Computable Daniell-Stone theorem

7.1. Motivation

- There are two ways to introduce measure and integration: first measure and then integration or vice versa.
- As a fundamental result, these two ways are essentially equivalent (Daniell-Stone theorem [Bau01], also of this type is Riesz representation theorem).
- For a constructive version, see [BB85].
- We show it in the computable sense that we show that computable premises lead to computable consequences.

7.2. The classical theorem

Let $\sigma(\mathcal{F}) := \{\{f > a\} : f \in \mathcal{F}, f \geq 0, a \in \mathbb{R}\}$ be the smallest σ-algebra in Ω such that every function $f \in \mathcal{F}$ is measurable.

Theorem 7.1 (Daniell-Stone Theorem) *Let $\mathcal{F}$ be a Stone vector lattice with abstract integral I. Then there is a measure μ on $\sigma(\mathcal{F})$ such that f is μ-integrable and $I(f) = \int f\, d\mu$ for all $f \in \mathcal{F}$. Furthermore, if there is a sequence $(f_i)_i$ in $\mathcal{F}$ such that $(\forall x \in \Omega)(\exists i)[f_i(x) > 0]$, then the measure μ is uniquely defined.*

The theorem is shown in the following two steps:

1. Define the measure:

$$\mu(A) := \sup_n I(f_n) \quad \text{if } A \in \sigma(\mathcal{F}) \text{ and } f_n \nearrow 1_A,$$

   where the 1_A is the characteristic function of A and $f_n \nearrow 1_A$ means that the sequence (f_n) converges to 1_A.

2. Show that, for each $f \in \mathcal{F}$, $f \in \mathcal{L}(\mu)$ i.e. f is μ-integrable, and $I(f) = \int f \, d\mu$.

7.3. How to construct a “computable measure” from the Stone vector lattice $\mathcal{F}$?

- How to represent $\mathcal{F}$?
- How to represent the measure constructively?
- How to guarantee the computability of the measure?

- A computable pseudometric space is a quadruplet $\mathcal{M} = (M, d, A, \alpha)$ such that (M, d) is a pseudometric space, $A \subseteq M$ is dense and $\alpha :\subseteq \Sigma^* \to A$ is a notation of A such that $\mathrm{dom}(\alpha)$ is recursive and the restriction of the pseudometric d to A is (α, α, ρ)-computable.
- The factorization $(\overline{M}, \overline{d})$ of the pseudometric space (M, d) is a metric space defined canonically as follows: $\overline{x} := \{y \in M \mid d(x, y) = 0\}$, $\overline{M} := \{\overline{x} \mid x \in M\}$, $\overline{d}(\overline{x}, \overline{y}) := d(x, y)$

7.4. Solution:

- Represent $\mathcal{F}$ with abstract integral I by a *computable Stone vector lattice*:

$$(\Omega, \mathcal{F}, I, \mathcal{D}, \gamma)$$

  $\mathcal{D}$ is a countable subset of $\mathcal{F}$ which is dense under the metric d_I defined by $d_I(f, g) := I(|f - g|)$; γ is a notation of $\mathcal{D}$.

  Practically, the computable stone vector lattice is represented by $\mathcal{D}$ and its notation γ.

  $(\mathcal{F}, d_I, \mathcal{D}, \gamma)$ is a computable pseudo-metric space. Let $\overline{\mathcal{F}}$ be the factorized space of $\mathcal{F}$ under the pseudo-metric. Denote its Cauchy representation by $\delta_{\overline{\mathcal{F}}}$.

- Represent the measure to be constructed by a computable measure space:

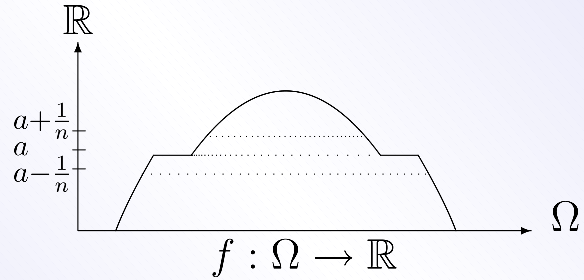
$$(\Omega, \mathcal{A}, \mu, \mathcal{R}, \alpha).$$

- Construct $\mathcal{R}$ and define its notation α.

  A natural choice:

$$\mathcal{R} := \left\{\{f > a\} : f \in \mathcal{D}^+, a \in \mathbb{Q}^+\right\}.$$

However, the measure μ will not be computable on such $\mathcal{R}$. E.g.,



since $\{f > a\} = \sup_n \left\{f > a + \frac{1}{n}\right\}$, $\mu\{f > a\}$ is approximated from below by the sequence $(nI(f \wedge (a + \frac{1}{n}) - f \wedge a))_n$.
However, $\{f > a\} \neq \inf_n \left\{f > a - \frac{1}{n}\right\}$. We can not obtain a sequence approximate $\mu\{f > a\}$ from above.

- To solve the problem, we show that there is a (γ, ρ)-computable correspondence $\Phi : \mathcal{D}_+ \rightrightarrows \mathbb{R}_c$ such that for each γ-name of $f \in \mathcal{D}_+$, a ρ-name p of a computable real $a \in \Phi(f)$ is computed such that $\mu(\{f > a\})$ is computable. Then define $\mathcal{R} := \{\{f > a\} : f \in \mathcal{D}_+, a \in \Phi(f)\}$.
- Consider the relation between $\mathcal{F}$ and $\mathcal{L}(\mu)$.

  Generally speaking, a function $f \in \mathcal{F}$ is not μ-integrable. However, we find a computable isometric between the two pseudo metric spaces $(\mathcal{F}, d_I)$ and $(\mathcal{L}(\mu), d_\mu)$, where

$$d_I(f, g) := I(|f - g|) \text{ and } d_\mu(u, v) := \int |u - v| d\mu.$$

  Therefore, in a sense, $\mathcal{F}$ can be represented by $\mathcal{L}(\mu)$.

- Denote by RSF the class of the *rational step functions* on $\mathcal{R}$, where a *rational step function* on $\mathcal{R}$ is is of the form:

$$f = \sum_{i=1}^{n} a_i 1_{A_i}$$

where $a_i \in \mathbb{Q}$ and $A_i \in \mathcal{R}$ for $1 \leq i \leq n$.

It is well-known that RSF is dense in $\mathcal{L}(\mu)$ under the pseudo-metric d_μ.

- Let ν_{RSF} be a canonical notation of ν_{RSF} defined with $\nu_{\mathbb{Q}}$ and α. Then $(\mathcal{L}(\mu), d_\mu, \mathrm{RSF}, \nu_{\mathrm{RSF}})$ is a computable pseudo-metric space.

- Let $\delta_{\overline{\mathcal{L}(\mu)}}$ denote the Cauchy representation of its factorized space $\overline{\mathcal{L}(\mu)}$.

7.5. The computable Daniell-Stone theorem

Theorem 7.2 (Computable Daniell-Stone, WU and Weihrauch) *Let* $(\Omega, \mathcal{F}, I, \mathcal{D}, \gamma)$ *be a computable Stone vector lattice with abstract integral such that* $(\forall x \in \Omega)(\exists i)[f_i(x) > 0]$. *Then there exists a unique computable measure space* $(\Omega, \mathcal{A}, \mu, \mathcal{R}, \alpha)$ *such that*

1. *there exists a* (γ, α)*-computable correspondence* $\phi : \mathcal{D} \rightrightarrows \mathcal{R}$*, and*
2. *there exists a* $(\delta_{\overline{\mathcal{F}}}, \delta_{\overline{\mathcal{L}(\mu)}})$*-computable isometric* $\psi : \overline{\mathcal{F}} \to \overline{\mathcal{L}(\mu)}$ *such that*

$$I(f) = \int g d\mu \quad (\forall f \in \mathcal{F}, g \in \psi([f])),$$

where $\delta_{\overline{\mathcal{F}}}$ *resp.* $\delta_{\overline{\mathcal{L}(\mu)}}$ *is the Cauchy representation of the computable metric space* $(\overline{\mathcal{F}}, d_I, \mathcal{D}, \gamma)$ *resp.* $(\overline{\mathcal{L}(\mu)}, \mathrm{RSF}, d_\mu, \nu_{\mathrm{RSF}})$.

Remarks:

- By item 1, there is a TT-machine such that, for each $f \in \mathcal{D}$ represented by a γ-name, a set $A \in \mathcal{R}$ represented by an α-name can be computed, and each set in $\mathcal{R}$ can be obtained this way.
- By item 2, there is a TT-machine such that, for each $f \in \mathcal{F}$ represented by a $\delta_{\overline{\mathcal{F}}}$-name, the machine computes a μ-integrable function g represented by a $\delta_{\overline{\mathcal{L}(\mu)}}$-name such that
$$I(f) = \int g d\mu.$$

Thank You !

References

[Bau01] Heinz Bauer, *Measure and Integration Theory*, de Gruyter Studies in Mathematics 26 (Walter de Gruyter, Berlin, New York, 2001).

[BB85] E. Bishop and D.S. Bridges, *Constructive Analysis*, volume 279 of *Grundlehren der Mathematischen Wissenschaften*. Springer, Berlin, 1985.

[BC72] Bishop and Cheng, *Constructive measure theory*, Memoirs of the AMS, 1972.

[BCSS96] Blum, Lenore and Cucker, Felipe and Shub, Mike and Smale, Steve, Complexity and Real Computation: A Manifesto, *International Journal of Bifurcation and Chaos*, 6:3–26, 1996.

[BCSS98] Blum, Lenore and Cucker, Felipe and Shub, Michael and Smale, Steve, *Complexity and Real Computation*, Springer, New York, 1998.

[Bis67] E. Bishop, *Foundations of Constructive Analysis*. McGraw-Hill, New York, 1967.

[Bla99] J. Blanck, Effective domain representations of $H(X)$, the space of compact subsets. *Theoretical Computer Science*, 219:19-48, 1999.

[Bro75] L.E.J. Brouwer, The unreliability of logical principles, *Collected Works, Vol. 1, Philosophy and Foundations of Mathematics*, 1975, edit by Heyting, A., North-Holland, Amsterdam, pages 107–111.

[Bro75a] L.E.J. Brouwer, Fixed cores which cannot be found, though they are claimed to exist by classical theorems, *Collected Works, Vol. 1, Philosophy and Foundations of Mathematics*, 1975, edit by Heyting, A., North-Holland, Amsterdam, pages 516–518.

[Cha74] Y.K. Chan, Notes on constructive probability theory, *Annals of Probability*, 2:51-75, 1974.

[Cheng] Henry Cheng, Constructive mathematics and computer science, downloaded at google.com, 2005.

[Coh80] Donald L. Cohn. *Measure Theory*. Birkhäuser, Boston, 1980.

[DG96] P.D. Gianantonio, Real number computation and domain theory. *Information and computation*, 127:11-25, 1996.

[Eda95a] A. Edalat, Dynamical systems, measures, and fractals via domain theory. *Information and Computation*, 120(1):32–48, 1995.

[Eda95b] A. Edalat, Domain and integration. *Theoretical Computer Scienc*, 151:163-193, 1995.

[Eda96] A. Edalat, Power domains theory and iterated function systems. *Information and Computation*, 124(2):182–197, 1996.

[Eda97] A. Edalat. Domains for computation in mathematics, physics and exact real arithmetic. *Bulletin of Symbolic Logic*, 3(4):401–452, 1997.

[ES99b] A. Edalat and P. Sunderhauf, A domain-theoretic approach to computability on the real line. *Theoretical Computer Scienc*, 210:73-98, 1999.

[Esc97] M.H. Escardo, PCF extended with real numbers: a domain-theoretic approach to higher-order exact real number computation. PhD thesis, Imperial College, University of London, Great Britain, 1997.

[EK] A. Edalat and M. Krznaric, Numerical Integration with Exact Real Arithmetic, downloaded at google.com, 2005.

[Eng78] Ryszard Engelking, *Dimension Theory*, volume 19 of *North-Holland Mathematical Library* (North-Holland, Amsterdam, 1978).

[Eng89] R. Engelking, *General topology*, Heldermann, Berlin (1989).

[Fra65] S.P. Franklin, Spaces in which sequences suffice, Fundamenta Mathematicae 57:107-115, 1965.

[Grz55] Grzegorczyk, Andrzej, Computable functionals, *Fundamenta Mathematicae*, 42:168–202, 1955.

[Hal74] Paul R. Halmos. *Measure Theory* (Springer-Verlag, New York, 1974).

[HW03] P. Hertling and K. Weihrauch. Random elements in effective topological spaces with measure. *Information and Computation*, 181:32-56, 2003.

[Ko91] Ker-I Ko. *Complexity Theory of Real Functions*. Progress in Theoretical Computer Science. Birkhäuser, Boston, 1991.

[Ko95] K.-I. Ko. A polynomial time computable curve whose interior has a nonrecursive measure. *Theoretical Computer Science*, 145:241-270, 1995.

[Ko98] K.-I. Ko. On the computability of fractal dimensions and Hausdorff measure. *Annals of Pure and Applied Logic*, 93:195-216, 1998.

[Kus84] B.A. Kuš. *Lectures on Constructive Mathematical Analysis*, volume 60. American Mathematical Society, Providence, 1984.

[Kus99] B.A. Kuš. Markov‘s constructive analysis: a participant‘s view. *Theoretical Computer Science*, 219:267-285, 1999.

[KW85] C. Kreitz and K. Weihrauch, Theory of representations, *Theoretical Computer Science*, 38:35-53, 1985.

[Lac55a] Lacombe, Daniel, Extension de la notion de fonction récursive aux fonctions d'une ou plusieurs variables réelles I, *Comptes Rendus Académie des Sciences Paris*, 240:2478–2480, jun, 1955.

[Maz63] S. Mazur, *Computable Analysis*, volume 33. Razprawy Matematyczne, Warsaw, 1963.

[MS02] M. Menni and A. Simpson, Topological and Limit-space Subcategories of Countably-based Equilogical Spaces, Mathematical Structures in Computer Science, Vol. 12(2002).

[Mue99] Norbert Th. Müller. Computability on random variables. *Theoretical Computer Science*, 219:287–299, 1999.

[Odi89] P. Odifreddi, *Classical Recursion Theory*, volume 129 of *Studies in Logic and the Foundations of Mathematics*, North-Holland, Amsterdam, 1989.

[PE99] M.B. Pour-El, The structure of computability in analysis and physical theory: An extension of Church's thesis. In E.R. Griffor, editor, *Handbook of Computability Theory*, volume 140 of *Studies in Logic and the Foundations of Mathematics*. pages 449-471, Elsevier, Amsterdam, 1999.

[PER89] M.B. Pour-El and J.I. Richards, *Computability in Analysis and Physics*, Perspectives in Mathematical Logic. Springer, New York, 1985.

[Rud74] Walter Rudin, *Real and Complex Analysis*. McGraw-Hill, New York, 2nd edition, 1974.

[San68] N.A. Šanin, *Constructive Real Numbers and Constructive Function Spaces*, volume 21 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, 1968.

[Sch01] M. Schroder, Admissible representations of limit spaces, Computability and Complexity in analysis, pp. 273-295, Vol. 2064 of Lecture Notes in Computer Science, (Springer, Berlin, 2001)

[Sch03] Matthias Schröder, Admissible representations for continuous computations. Informatik Berichte 299, FernUniversität Hagen, Hagen, April 2003. Dissertation.

[SHT95] V. Stoltenberg-Hansen and J. V. Tucker, Effective algebras. In S. Abramsky, D.M. Gabbay, and T.S.E. Maibaum, editors, *Handbook of Logic in Computer Science*, volume 4, pages 357-527, Oxford, 1995, Clarendon Press.

[SHT99] V. Stoltenberg-Hansen and J. V. Tucker, Concrete models of computation for topological algebras. *Theoretical Computer Science*, 219:347–378, 1999.

[Soa87] R.I. Soare, *Recursively Enumerable Sets and Degrees*, Perspectives in Mathematical Logic, Springer-Verlag, Berlin, Heidelberg, New York, 1987.

[Spi02a] Bas Spitters, *Constructive and intuitionistic integration theory and functional analysis*. PhD thesis, University of Nijmegen, 2002.

[Tur36] A.M. Turing, On computable numbers, with an application to the ”Entscheidungsproblem” (a correction), *Proceedings of the London Mathematical Society*, 43(2):544-546, 1937.

[Tur37] A.M. Turing, On computable numbers, with an application to the ”Entscheidungsproblem”, *Proceedings of the London Mathematical Society*, 42(2):230-265, 1936.

[WD80] K. Weihrauch and T. Deil, Berechenbarkeit auf cpo-s. Schriften zur Informatik und angewandten Mathematik 63. Technische Hochschule Aachen. Aachen, 1980. Eine Vorlesung von Klaus Weihrauch ausgearbeitet von Thomas Deil.

[WD04] Y.C. Wu and D.C. Ding, Computability of measurable sets via effective topologies. Submitted to *Archive for Mathematical Logic*, 2004.

[WDW04] Y.C. Wu, D.C. Ding and K. Weihrauch. Computability of measurable sets via effective metrics. Submitted to *Math Logic Quarterly*, 2005.

[Wei85] K. Weihrauch, Type 2 recursion theory. *Theoretical Computer Science*, 38:17–33, 1985.

[Wei93] K. Weihrauch, Computability on computable metric spaces, *Theoretical Computer Science*, 113:191-210, 1993.

[Wei98] K. Weihrauch, A refined model of computation for continuous problems. *Journal of Complexity*, 14:102-121, 1998.

[Wei99a] K. Weihrauch, Computability on the probability measures on the Borel sets of the unit interval. *Theoretical Computer Science*, 219:421–437, 1999.

[Wei00] K. Weihrauch, *Computable Analysis* (Springer, Berlin, 2000).

[WS81] K. Weihrauch and Ulrich Schreiber, Embedding metric spaces into cpo’s. *Theoretical Computer Science*, 16:5–24, 1981.

[WW04] Y.C. Wu and K. Weihrauch. A computable version of the Daniell-Stone theorem on integration and linear functionals, submitted to *Theoretical Computer Science*, 2004.

[Yap96] C. Yap, White paper on exact computation and reliable geometric software, downloaded at google.com, 2005.

[YD94] C. Yap and T. Dube, The exact computation paradigm, the 2nd edition of *Computing in Euclidean Geometry*, edited by D.-Z. Du and F.K. Hwang, World Scientific Press, 1994.

[Yan04] J.A. Yan, *Lecture on Measure Theory* (Chinese), Scientific Press, Beijing, 2004.

[ZW99] X.Z. Zheng and K. Weihrauch, The arithmetical hierarchy of real numbers, in M. Kutylowski, L. Pacholski and T. Wierzbicki, editors, *Mathematical Foundations of Computer Science* 1999, volume 1672 24th International Symposium, MFCS'99, Szklarska Poreba, Plland, September, 1999.