FIVE LECTURES ON ALGORITHMIC RANDOMNESS

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This article is devoted to some of the main topics in algorithmic randomness, at least from my idiosyncratic view.

Contents

1. Introduction

In this article, I plan to give a course around certain highlights in the theory of algorithmic randomness. At least, these will be some highlights as I see them.

I will try not to cover too much of the material already covered in my notes [15]. However, since I plan to make these notes relatively self-contained I will by necessity need to include some introductory material concerning the basics of Martin-Löf randomness and Kolmogorov complexity.

There will also be a certain intersection with the articles Downey, Hirschfeldt, Nies and Terwijn [24] and Downey [14].

I will not be able to cover the background computability theory needed, and here refer the reader to Soare [79] and Odifreddi [69, 70]. While I plan to write sketches of proofs within these notes, full details can be found in the forthcoming monograph Downey and Hirschfeldt [19]. Other work on algorithmic randomness can be found in well-known sources such as Calude [5] and Li-Vitanyi [50].

The subject of algorithmic randomness is a vast one, and has been the under intense development in the last few years. With only five lectures I could not hope to cover all that has happened, nor even report on the history. Here I certainly recommend the reader look at the surveys Downey, Hirschfeldt, Nies and Terwijn [24] and Downey [14].

Probably the most brutal omission is the work on triviality and lowness, which has its roots in Solovay's [82], and its first modern incarnation in Kučera-Terwijn [42] and Downey, Hirschfeldt, Nies, and Stephan [23]. This work is of central importance as has been shown especially through the powerful results of (Hirschfeldt and) Nies [65–67], and subsequently Hirschfeldt, Stephan (e.g. [32]), Slaman and others. The reason for this lamentable omission is that I don't think it is possible to give a fair treatment to Martin-Löf lowness and triviality, as well as other important triviality lowness notions for Schnorr and computable randomness, in even one or two lectures. They could have five to themselves! A short account can be found in Downey, Hirschfeldt, Nies, Terwijn [24], and a full account is, or will be soon, found in Downey and Hirschfeldt [19].

Finally these notes will certainly contain more material than I could possibly cover in any set of lectures in the hope that the extra material will help the participants of the meeting *Computational Prospects of Infinity.* This is especially true of the many results I won't have time to prove in the lectures, but whose proofs I have included.

In these notes in Section 2, I will first develop the basic material on Kolmogorov Complexity. Whilst this approach does not follow the historical development of the subject, it does make logical sense. I will include proofs of the fundamental results including Kraft-Chaitin, the Coding Theorem, and Symmetry of Information.

In Section 3, I will discuss the background material on the three basic approaches to randomness, via measure theory, prediction, and compression. Here I will also include the exciting recent results of Miller and Yu classifying 1-randomness in terms of plain complexity.

In Section 4, I will look at some classic theorems concerning randomness for general classes of reals. This material will include the Kučera-Gács Theorem and other results of Kučera. Other central results treated will be van Lambalgen's Theorem, effective 0-1 laws, and results on PA and FPF degrees. We also introduce n-randomness and variations and look at the exciting recent work showing that 2-randomness is the same as Kolmogorov randomness.

In Section 5, I will look at various methods of calibrating randomness using initial segment methods. This will include the Slaman-Kučera Theorem and other work on computably enumerable reals, and the work of Solovay, and Miller-Yu on van Lambalgen reducibility and its relationship with $\leq_K$ and $\leq_C$.

Finally, in Section 6, I plan to give sketches of proofs of the poorly known results of Stuart Kurtz from his thesis [45], and some refinements later by Steven Kautz [34]. None of this work has ever been published aside from the presentations in these theses. The techniques, whose origins go back to work of Paris and of Martin, are powerful and are extremely interesting.

We remark that throughout these notes we will be working over the alphabet $\{0, 1\}$, and hence will be looking at $2^{<\omega}$, and 2^ω for "reals" meaning members of Cantor Space. This sapce will be equipped with the basis of clopen sets $[\sigma] = \{\alpha : \sigma \prec \alpha\}$, and comes with the usual Lebesgue measure $\mu([\sigma]) = 2^{-|\sigma|}$. The whole development can also be done over other alphabets with little change. Most notation is drawn from Soare [79]. We will use λ to denote the empty string.