Algorithmic Randomness 4

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Calibrating Randomness

- How should we attempt to calibrate levels of randomness?
- Among randoms?
- Among non-randoms.
- How does this relate to Turing and other reducibilities, etc?

Measures of Relative Randomness

- A pre-ordering $\leq$ on reals is a *measure of relative randomness* if it satisfies the *Solovay property*:

$$\text{If } \beta \leq \alpha \text{ then}$$
$$\exists c\ (\forall n\ (K(\beta \upharpoonright n) \leq K(\alpha \upharpoonright n) + c)).$$

- Notice that if α is random and $\alpha \leq \beta$ then by Schnorr's Theorem, β is random too.

- Can also use C, and others.

- The idea is that *if* we can characterize *randomness* by initial segment complexity, then we oght to be able to calibrate *randomness* by comparing initial segment complexities.

- Of course this is open to question, and we could also suggest other programs such as using tests and maybe effective Hölder transformations (for instance) to attempt such a calibration. These are unexplored.

Solovay Reducibility

- We talk about *the* halting problem, whereas of course we really mean HALT_U for a universal U. But... they are all the same (Myhill)
- Solovay introduced a reduction to address this for randomness.
- $(\alpha \leq_S \beta)$ α is Solovay or domination reducible to β iff there is a constant d, and a partial computable φ, such that for all rationals $q < \beta$

$$\varphi(q) \downarrow \wedge\ d(\beta - q) > |\alpha - \varphi(q)|.$$

- Intuitively, however well I can approximate β, I can approximate α just as well. Clearly $\leq_S$ implies $\leq_T$.
- A formal way to say this is
- Lemma (Calude, Hertling, Khoussainov, Wang) For c.e. reals, $\alpha \leq_S \beta$ iff for all c.e. $q_i \to \beta$ there exists a total computable g, and a constant c, such that, for all m,

$$c(\beta - q_m) > \alpha - r_{g(m)}.$$

- S-reducibility is a measure of relative randomness (Solovay)

- This follows by : Let d be given. Then there is a constant $c = c(d)$ such that for all n: if σ and τ have length n and $|\sigma - \tau| < 2^{-n+d}$, $K(\sigma) + c > K(\tau)$.

- If $U(\gamma) = \sigma$, then if $\sigma_1 \ldots, \sigma_{2^{2d+1}}$ denote the possible τ in lex order, have $M(1^i\gamma) = \sigma_i$.

- Then suppose $x \leq_S y$ with constant d and partial computable f. To show, for instance, $x \upharpoonright n$ K-below $y \upharpoonright n$, consider the machine M which does the following. For each ν with $U(\nu)\downarrow$, M applied f to $U(\nu)$. If this halts, for each of the $2^{c(d)}$ many strings τ_i, within 2^{-n+d} of $f(U(\nu))$, we define $M(1^{i+1}\nu) = \tau_i$.

  This procedure applied to $y \upharpoonright n$ will result in a program for $x \upharpoonright n$ from amongst these programs. Note that this really is a reduction, in that we get to generate x from y in the limit.

$\leq_S$ and $+$

- Actually for c.e. reals, $\leq_S$ is a simple arithmetical relation.
- (Downey, Hirschfeldt, Nies) $x \leq_S y$ iff there exists a $c \in \mathbb{N}$ and a c.e. real z such that $cy = x + z$.

- Recall: $\alpha \leq_S \beta$ iff there are a computable f and a constant d such that $\alpha - \alpha_{f(n)} < d(\beta - \beta_n)$ for all n. Want $cy = x + z$.
- Roughly, the proof works by synchronizing the enumerations so that the approximation to x is "covered" by one for y, (i.e. $x_{s+1} - x_s$ generates a change in cy of the same order.) Then we use the amount needed for x for x and the excess goes into z.

Only one random c.e. real

- A c.e. real is Ω-*like* if it dominates all c.e. reals.
- (Solovay) Any Ω-like real is random.
- Proof : By Schnorr since then $K(\alpha \upharpoonright n) \geq n - d$.

Solovay proved that Ω-like reals posessed many of the properties that Ω posessed. He remarks:

"It seems strange that we will be able to prove so much about the behavior of $K(\Omega \upharpoonright n)$ when, a priori, the definition of Ω is thoroughly model dependent. What our discussion has shown is that our results hold for a class of reals (that include the value of the universal measures of ...) and that the function $K(\Omega \upharpoonright n)$ is model independent to within $O(1)$."

- **Theorem** (Calude, Hertling, Khoussainov, and Wang) If a c.e. real is Ω-like then it is an Ω-number. That is, a halting probability.

- Proof: We have $\Omega \leq_S \alpha$ with enumerations $\Omega_s \to \Omega$, $\alpha_s \to \alpha$

- We know that if we use a stage by stage approximation, then essentially $c(\alpha - \alpha_s) \geq \Omega - \Omega_s$.

- Use Kraft-Chaitin. If $U(\tau) = \sigma$ then $2^{-|\tau|}$ enters the domain of $U - U_s$. We re-cycle this by defining $M(\tau) = \sigma$ with $2^{-|\tau|}$ entering $\alpha - \alpha_s$, this keeping $K_M(\sigma) \leq K_U(\sigma) + c$. etc.

Kučera-Slaman Theorem

- **Theorem** (Kučera-Slaman) If a c.e. real is random then it is Ω-like.
- ie all random c.e. reals are the "same" and are halting probabilities. (even though it might be possible for it to be as high as $n + 2\log n$ all oscillations occur at the "same" n's.)

- Proof: Suppose that α is random and β is a c.e. real. We need to show that $\beta \leq_S \alpha$. We enumerate a Martin-Löf test $F_n : n \in \omega$ in stages.

  Let $\alpha_s \to \alpha$ and $\beta_s \to \beta$ computably and monotonically. We assume that $\beta_s < \beta_{s+1}$.

- At stage s if $\alpha_s \in F_n^s$, do nothing, else put $(\alpha_s, \alpha_s + 2^{-n}(\beta_{s+1} - \beta_{t_s}))$ into F_n^{s+1}, where t_s denotes the last stage we put something into F_n.

- One verifies that $\mu(F_n) < 2^{-n}$. Thus the F_n define a Martin-Löf test. As α is random, there is a n such that for all $m \geq n$, $\alpha \notin F_m$. This shows that $\beta \leq_S \alpha$ with constant 2^n.

Variations

- It follows that if α is c.e. real and random it is Turing complete. This is true in a very strong way.
- (Downey and Hirschfeldt) Suppose that A is a c.e. set and α is a 1-random c.e. real. Then $A \leq_{wtt} \alpha$, and this is true with identity use. ("$\alpha \leq_{ssw} \beta$")

- Proof: We construct $\Gamma^\alpha = A$, where $\gamma(x) = x$.
- We use KC and we know our coding constant e. We know $K_U(\alpha \upharpoonright n) \geq n - c$ and have $\alpha_s \to \alpha$.
- Initially $\Gamma^{\alpha_s}(n) = 0$. We want to change this at some $t > s$ should n enter A_{s+1}. We need a change in $\alpha_s \upharpoonright n$.
- Enumerate the KC axiom $\langle 2^{n-c-e-1}, \alpha_s \upharpoonright n \rangle$.
- This causes $K_{s+1}(\alpha_s \upharpoonright n)$ to drop below $n - c - 1$. Thus $\alpha_t \upharpoonright n \neq \alpha_s \upharpoonright n$ for some $t > s$.

- Similar methods show
- (Kučera) Suppose that A is a random set of c.e. degree. Then A is Turing complete
- (Downey and Hirschfeldt) Suppose that A is a random set of c.e. wtt-degree. Then A is wtt-complete.

Structure

- The c.e. reals using $\leq_S$ forms a upper semilattice, called the Solovay degrees.
- (Downey, Hirschfeldt, Nies)
  (i) + induces a join
  (ii) It is distributive
  (iii) dense
  (iv) $[\Omega]$ is the only join inaccessible element.

- Proof of (i). $x, y \leq_S z$ implies there is a c, p, q such that $cz = x + p = y + q$. So $2cz = (x + y) + (p + q)$. So $x + y \leq_S z$. Clearly $x, y \leq x + y$.
- (ii) (distributive) $z \leq_S x_1 + y_1$. Run the enumerations of and cover the $z_{s+1} - z_s$ using bits of $x_{1,s+1} - x_s, y_{s+1} - y_s$.
- density and $[\Omega]$ being join inaccessible more intricate.

The Density Theorem

- Splits into two cases.
- Theorem (Downey, Hirschfeldt, Nies)

  (i) If $\boldsymbol{a}$ is incomplete and $\boldsymbol{b} <_S \boldsymbol{a}$, then there exist $\boldsymbol{a_1}|_{\boldsymbol{S}}\boldsymbol{a_2}$ such that $\boldsymbol{b} < \boldsymbol{a_1}, \boldsymbol{a_2}$, and $\boldsymbol{a} = \boldsymbol{a_1} \vee \boldsymbol{a_2}$. That is every incomplete degree splits over all lesser ones.

  (ii) If $[\Omega] = \boldsymbol{a} \vee \boldsymbol{b}$ then either $[\Omega] = \boldsymbol{a}$ or $[\Omega] = \boldsymbol{b}$.

- (ii) is a straightforward finite injury argument.

- We give the idea for (i), that $\alpha <_S \alpha <_S \Omega$. There are β^0 and β^1 s.t. $\alpha <_S \beta^0, \beta^1 <_S \alpha$ and $\beta^0 + \beta^1 = \alpha$.
- Recall: $\alpha \leq_S \beta$ iff there are a computable f and a constant d such that $\alpha - \alpha_{f(n)} < d(\beta - \beta_n)$ for all n.

We want to build β^0 and β^1 such that

- $\beta^0, \beta^1 \leq_S \alpha$,
- $\beta^0 + \beta^1 = \alpha$, and
- the following requirement is satisfied for each $e, k \in \omega$ and $i < 2$:

$$R_{i,e,k} : \Phi_e \text{ total}$$

$$\Rightarrow \exists n(\alpha - \alpha_{\Phi_e(n)} \geq k(\beta^i - \beta^i_n)).$$

- Two requirements show the problems.
  - $R_0 : \Phi$ total $\Rightarrow \exists n(\alpha - \alpha_{\Phi(n)} \geq k(\beta^0 - \beta^0_n))$
  - $R_1 : \Psi$ total $\Rightarrow \exists n(\alpha - \alpha_{\Psi(n)} \geq l(\beta^1 - \beta^1_n))$

- We assume Φ, Ψ total for the following.
  - two containers, labeled β^0 and β^1, and
  - a large funnel, through which bits of α are being poured.
- R_0 and R_1 fight for control of the funnel.
- Bits of α must go into the containers at the same rate as they enter α to make $\beta^0 + \beta^1 = \alpha$.
- R_0 says put the bits into β^1 till satisfied. R_1 the opposite.
- R_0 is *satisfied through n at stage s* if $\Phi(n)[s]\downarrow$ and $\alpha_s - \alpha_{\Phi(n)} > k(\beta^0_s - \beta^0_n)$.

- The idea is R_0 sets a quota for R_1 into β^0.
- If the quota is 2^{-m} and R_0 finds that either
  - it is unsatisfied or
  - the least number through which it is satisfied changes,

  then it sets a new quota of $2^{-(m+1)}$ for how much may be funneled.
- Lemma: There is an n through which R_0 is eventually permanently satisfied, that is,

  $\exists n, s\, \forall t > s\, (\alpha_t - \alpha_{\Phi(n)} > k(\beta^0_t - \beta^0_n))$.
- The proof is that R_1's quota $\to 0$ and

its noise is computable, then a' la Sacks.

The Dilemma

- So now, R_0 is permanently satisfied, and R_1 has a final quota 2^{-m} that it is allowed to put into β^0.
- If we *knew* when s occurred with $\alpha - \alpha_s < 2^{-m}$, then we could use the same strategy.
- If we are too quick R_1 can't be satisfied.

- Idea : R_1 *uses* Ω *as an investment advisor.*

- After the final stage u where R_1's final quota is set, R_1 puts as much of $\alpha_{t+1} - \alpha_t$ into β^0 as possible so that the total amount put into β^0 since stage s *does not exceed* $2^{-m}\Omega_t$*.*

- Since Ω settles last, we can show

- There is a stage t after which R_1 is allowed to funnel all of $\alpha - \alpha_t$ into β^0.

- (Downey and Hirschfeldt) This works for any Σ^0_3 measure of relative randomness where $+$ is a join, the 0 degree includes the computable reals, and the top degree is Ω.

Other structure

- The Solovay degrees of c.e. real is not a lattice (Downey and Hirschfeldt)
- Minimal pairs exist etc.
- (Downey, Hirschfeldt, LaForte) The structure of the S-degrees of c.e. reals has an undecidable theory.
- This is proven using Nies' method of effective dense boolean algebras.
- Little else known.

Other Measures

- S-reducibility is a measure of relative randomness, but not the only one, and it has some problems.

  (i) Restricted to c.e. reals.

  (ii) Too fine.

  (iii) Too uniform.

sw

- Another measure of relative randomness is *sw-reducibility*:

  $\beta \leq_{sw} \alpha$ if there is a functional Γ s.t. $\Gamma^\alpha = \beta$ and the use of Γ is bounded by $x + c$ for some c. If $c = 0$ called ssw-reducibility, used by Soare and Csima in differential geometry.

- sw-reducibility is incomparable with S-reducibility.

- sw-reducibility says that there is an *efficient way* to convert the *bits* of α into those of β.

The Yu-Ding Theorem

- Even though Kučera-Slaman says that any two versions of Ω are "the same", there is no efficient way to convert the bits of one into another
- Theorem (Yu and Ding)

  (i) There is no sw-complete c.e. real.

  (ii) There are two c.e. reals β_0 and β_1 so that there is no c.e. real α with $\beta_0 \leq_{sw} \alpha$ and $\beta_1 \leq_{sw} \alpha$.

- The proof, roughly works by picking two long intervals $\beta_0 \upharpoonright [n, n+t], \beta_1 \upharpoonright [n, n+t]$, to diagonalize against some α and sw reduction Γ_e with use $n + e$.
- Initially the reals are 0 on this interval.
- Then alternating between β_0 and β_1 adding $2^{-(n+t)}$ each time, where time here means "expansionary stages."
- Yu and Ding observed that this process will cause α to be too large.
- This is proven by induction, and the reason I think, is that when α has lots of 1's, it can only change large.

- Here is an example.

  stage 1: $\beta_{0,1} = 0.001$, $\beta_{1,1} = 0$ and $\alpha_1 = 0.001$

  stage 2: $\beta_{0,2} = 0.001$, $\beta_{1,2} = 0.001$ and $\alpha_2 = 0.010$

  stage 3: $\beta_{0,3} = 0.010$, $\beta_{1,3} = 0.001$ and $\alpha_3 = 0.100$

  stage 4: $\beta_{0,4} = 0.010$, $\beta_{1,4} = 0.010$ and $\alpha_4 = 0.110$

  stage 5: $\beta_{0,5} = 0.011$, $\beta_{1,5} = 0.010$ and $\alpha_5 = 0.111$

  stage 6: $\beta_{0,6} = 0.011$, $\beta_{1,6} = 0.011$ and $\alpha_6 = 1.000$

  stage 7: $\beta_{0,7} = 0.100$, $\beta_{1,7} = 0.011$ and $\alpha_7 = 1.100$

stage 8: $\beta_{0,8} = 0.100$, $\beta_{1,8} = 0.100$ and $\alpha_8 = 10.000$

- You must prove that α's best strategy is the *least effort* one. (Definition of Barmpalias and Lewis).

- Similar methods can be used to prove :

- Theorem (Barmpalias and Lewis) There is a c.e. real α such that for any random c.e. real β, $\alpha \not\leq_{sw} \beta$.

- Using a different argument, Hirschfeldt constructed a real α such that for all random reals β, $\alpha \not\leq_{sw} \beta$.

$\leq_{rK}$

- Would like a measure of relative randomness combining the best of S-reducibility and sw-reducibility.
- one such is $A \leq_{rK} \beta$ iff there for all n, $K(A \upharpoonright n | B \upharpoonright n + c) = O(1)$.
- Again $+$ is a join, etc so it is dense.
- Little else known. Known that $\leq_C$ does not imply $\leq_{rK}$ on the c.e. reals. (Downey, Greenberg, Hirschfeldt, Miller)

Results on $\leq_K$ and $\leq_C$

- Recall $A \leq_K B$ to mean $K(A \upharpoonright n) \leq K(B \upharpoonright n) + O(1)$, all n.
- Thanks to the work of Miller and Yu (mainly) we know a lot about the structure of K and C degrees on *randoms.*
- The first thing we find is that $\leq_C$ and $\leq_K$ are not really *reducibilities*
- Yu, Ding, Downey If X is random then $\{Y : Y \leq_Q X\}$ is uncountable. Moreover it contains members of each Turing degree.
- The proof is to observe: If Y is very

sparse then its complexity is low, but we can code any degree into a sparse set.

- Replace with:
- Yu, Ding, Downey $\mu(\{B : B \leq_K A\}) = 0$. Hence uncountably many K degrees.
- Yu, Ding In fact $2^{\aleph_0}$. (Actually this follows from the above by a Theorem of Silver and the fact that $\leq_K$ is Borel.)
- (Miller and Yu) For almost all pairs $A|_K B$.
- (Miller and Yu) for all $n \neq m$, $\Omega^{(n)}|_K \Omega^{(m)}$. (This extends earlier work of Yu, Ding, Downey; Solovay)
- (Miller and Yu) However, there are random A, B with $B <_K A$. (This result is the most difficult!)

- (Miller and Yu) Each K-degree of a random countable.
- (Miller) There is an uncountable K-degree.
- (Csima and Montalbán) There are minimal pairs of K-degrees.

A unified approach

- (von Lambalgen reducibility) For $x, y \in 2^\omega$, write $x \leq_{vL} y$ if $(\forall z \in 2^\omega)\ x \oplus z$ is 1-random $\implies$ $y \oplus z$ is 1-random.
- is the same as Define $y \leq_{\mathcal{L}R} x$ if $(\forall z \in 2^\omega)\ z$ is 1-x-random $\implies$ z is 1-y-random, on the randoms.
- inspired by van Lambalgen: $A \oplus B$ is random iff A is B-random and B is A-random.
- (Miller and Yu) If $\alpha \leq_{vL} \beta$ and α is n-random, then β is n-random.

- The proofs of most of these are relatively easy once you figure out what to do.
- Suppose that α n-random and $\alpha \leq_{vL} \beta$. Use Kučera's Theorem that there is a random z with $z \equiv_T \emptyset^{(n-1)}$. Then $\alpha \oplus z$ is random, and hence $\beta \oplus z$ is random and hence β is 1-z-random, that is β is n-random.

- (Miller and Yu) If $y \leq_T x$ and y is 1-random, then $x \leq_{vL} y$.
- (Miller and Yu) If $m \neq n$, then $\Omega^{\emptyset^{(m)}}$ and $\Omega^{\emptyset^{(n)}}$ have no upper bound in the vL-degrees.
- (Miller and Yu) If x is n-random and $y \leq_T x$ is 1-random, then y is n-random.
- (Miller and Yu) $x \leq_K y \implies x \leq_{vL} y$.
- (Hence)(Yu, Ding, Downey) for randoms $\mu(\{\beta : \beta \leq_K \alpha\}) = 0$.
- Proof: If β is 1-α-random, then $\beta \not\leq_{vL} \alpha$ and hence, since $\mu(\{\beta : \beta$ is 1-α-random $\}) = 1$, we get

$\mu(\{\beta : \beta \leq_K \alpha\}) = 0$, since $\leq_K$ implies $\leq_{vL}$.

Miller's Theorems

- In unpublished work, Miller has used these techniques to establish other fascinating results on $\leq_K$.
- Theorem (Miller)

  (i) If α, β are random, and $\alpha \equiv_K \beta$, then $\alpha' \equiv_{tt} \beta'$. As a consequence, every K-degree of a random real is countable.

  (iii) If $\alpha \leq_K \beta$, and α is 3-random, the $\beta \leq_T \alpha \oplus \emptyset'$.

- Note that (ii) implies that the cone of K-degrees above a 3-random is countable.

- (Miller and Yu) There are upper K-cones that are uncountable above a 1-random.

- (Miller-Yu) This is proven using a variation on the Miller-Yu proof that there are K-comparable randoms.
- That proof uses the following difficult result.
- (Miller and Yu) Suppose that $\sum_n 2^{f(n)} < \infty$, then there is a 1-random Y with

$$K(Y \upharpoonright n) < n + f(n),$$

for almost all n.

- Then to get K-comparible reals, use the result taking $g(n) = K(B \upharpoonright n) - n$ for random B, which is convergent by the Ample Excess Lemma, then use the above on some convergent function f with $g - f \to \infty$.

- We will call α *weakly low for K* if $(\exists^\infty n)[K(n) \leq K^\alpha(n) + O(1)]$.
- The information in α is so useless that it cannot help to compress n.
- (i) If α is 3-random it is weakly low for K.

  (ii) If α is weakly low for K, and also random, then α is strongly Chaitin random in that

$$(\exists^\infty n)\big[K(\alpha \restriction n) \geq n + K(n) - O(1)\big].$$

Outside of the Randoms

- Little is known about $\leq_K$ and $\leq_C$ outside of the random reals.
- (Downey and Hirschfeldt) The C- and K- degress of c.e. reals form a dense uppersemilattices.
- This is because the Σ_3^0 density theorem holds, again.
- This uses Downey, Hirschfeldt, Nies, Stephan that $+$ is a join.
- To see this, given $x, y < z$, run the enumerations, and have one z-program if $x_s \upharpoonright n$ stops first, and one if $y_s \upharpoonright n$ first.

Loveland-Chaitin-Stephan

- $\leq_C$ implies $\leq_T$ on c.e. reals, generalizing Loveland's and Chaitin's Theorems(Stephan)
- Loveland $C(\alpha \upharpoonright n|n) = O(1)$ iff α computable.
- Chaitin $A \leq_C 1^\omega$ iff A is computable.

- The proofs use the "Π_1^0 class method" each time.
- A Π_1^0 class with a finite number of paths only has computable ones.
- Loveland Proof: Only finitely many programs to consider for the $C(X \upharpoonright n | n) = O(1)$. Knowing these and the maximum hit infinitely often will allow for the construction of the Π_1^0 class.

- Chaitin's is the same proof PLUS:
- $|\{\sigma : C(\sigma) \leq C(n) + d \wedge |\sigma| = n\}| = O(2^d)$. (Chaitin)
- Since we know that between n and 2^n there are C-random lengths with $C(n) = \log n$, we can then apply the Lemma. (i.e. to construct the Π^0_1 class.)
- Stephan's is a kind of relativization of this. (Together with enumerations)
- Is this true for $\leq_K$? Intuitively yes, but.....

K-trivial reals:

- (Solovay) There exist noncomputable reals α such that for all n

$$K(\alpha \upharpoonright n) \leq K(1^n) + d.$$

- These are called K-trivial reals. specifically, $KT(d)$.
- What goes wrong with the Π^0_1 class method. The answer is "nothing" except that the tree is no longer computable, but a $\emptyset'$-computable tree with a finite number of paths.
- Thus (Chaitin) all K-trivial reals are Δ^0_2 and for each d, $KT(d)$ has $O(2^d)$ members. (Zambella)

- How many are there? Let $G(d)$ denote the number of $KT(d)$ reals. We know $G(d) \leq_T \emptyset'''$. We know $G(d) \not\leq_T \emptyset'$. We know $\sum \frac{G(d)}{2^d}$ is convergent. Is $G(d)$ machine dependent in its complexity? (Downey, Miller, Yu)

- Related to Csima-Montalbán functions. f such that $K(A \upharpoonright n) \leq K(n) + f(n) + O(1)$ implies A is K-trivial. CM if F is nondecreasing, and weakly CM if liminf $\to \infty$.

- Such A can be c.e. *sets*. (DHNS, and others)

- Solovay's 1974 proof is very complicated. Here is a simplified version proving a stronger result.

- (DHNS) There is a c.e. noncomputable set A such that for all n

$$K(A \upharpoonright n) \leq K(n) + \mathcal{O}(1).$$

- Let

$$A_{s+1} = A_s \cup \{x : W_{e,s} \cap A_s = \emptyset \wedge x \in W_{e,s}$$

$$\wedge \sum_{x \leq j \leq s} 2^{-K(1^j)[s]} < 2^{-(e+1)}\}.$$

- (DHNS) K-trivial reals are never of high degree, so this is an injury free solution to Post's problem.

Nies Theorems

Nies (and Hirschfeldt) has some deep material here using the "golden run" construction:

- Every K-trivial is bounded by a K-trivial c.e. set.
- Every K-low is superlow, and " jump tracable".
- K-trivial = low for Martin-Löf randomness (Meaning random^A iff random) =low for K (meaning $K^A = K$)

- K-trivials are closed under T-reducibility and form the only known natural Σ_3^0 ideal in the Turing degrees.
- The are bounded above by a low$_2$ degree.
- This is a special case of unpublished work of Nies showing that every Σ_3^0 ideal in the c.e. Turing degrees is bounded by a low$_2$ c.e. degree. (Proof in Downey-Hirschfeldt)
- Unknown if this Σ_3^0 ideal has an exact pair.

- Hirschfeldt, Nies, Stephan have shown that if A and B are Δ_2^0 random then there is a K-trivial below both.

- This needs Kučera's proirity-free soution to Post's Problem. And if you are a c.e. set below a incomplete random then you are K-trivial.

- Many, many relationships with other classes. (Nies lecture)

- Is there a low $\boldsymbol{a}$ above all the K-trivials? If so can it be random? (It can't be c.e.) As Kučera points out this would need new coding ideas into randoms.

Hausdorff dimension

- 1895 Borel, Jordan
- Lebesgue 1904 measure
- In any n-dimesnional Euclidean space, Carathéodory 1914

  $$\mu^s(A) = \inf\{\sum_i |I_i|^s : A \subset \cup_i I_i\},$$

  where each I_i is an interval in the space.
- 1919 Hausdorff s fractional. and refine measure 0.
- For $0 \leq s \leq 1$, the s-measure of a clopen set $[\sigma]$ is

  $$\mu_s([\sigma]) = 2^{-s|\sigma|}.$$

- Lutz-Mayordomo-Hitchcock has the following characterization of effective Hausdorff dimension: (also Staiger)
- An *s-gale* is a function $F : 2^{<\omega} \mapsto \mathbb{R}$ such that

$$F(\sigma) = 2^s(F(\sigma 0) + F(\sigma 1)).$$

Similarly we can define s-supergale, etc.

- Theorem (Lutz-Mayordomo-Hitchcock) For a class X the following are equivalent:

(i) $\dim(X) = s$.

(ii) $s = \inf\{s \in \mathbb{Q} : X \subseteq S[d]$ for some s-gale $F\}$.

- Lutz says the folllowing:

  "Informally speaking, the above theorem says the the dimennsion of a set is the *most hostile environment* (i.e. most unfavorable payoff schedule, i.e. the infimum s) in which a single betting strategy can *achieve infinite winnings* on every element of the set."

- Thm Lutz, Mayordomo, Hitchcock: The Hausdorff dimension of a real α is

$$\liminf_{n\to\infty} \frac{K(\alpha \restriction n)}{n} = (\liminf_{n\to\infty} \frac{C(\alpha \restriction n)}{n})$$

Dimensions of strings

- Lutz has introduced a method of assigning dimensions to strings.
- liminf $\frac{K(\alpha \restriction n)}{n}$,
- equivalently, the infimum over all s of the values of $d^s(\alpha \restriction n)$.
- To discreteize this characterization, Lutz used three devices:

  (i) He replaced supergales by *termgales*, which resemble supergales, yet have modifications to deal with the terminations of strings. This is done first via s-termgales and then later by termgales, which are uniform families of s-termgales.

(ii) He replaced $\to \infty$ by a finite threshold.

(iii) He replaced optimal s-supergale by and optimal termgale.

- For $s \in [0, \infty)$, an s-*termgale* is a function d from the collection of terminated strings T to $\mathbb{R}^+ \cup \{0\}$, such that $d(\lambda) \leq 1$, and

$$d(\sigma) \geq 2^{-s}[d(\sigma 0) + d(\sigma 1) + d(\sigma \Box)].$$

  Here $\Box$ is a delimiting symbol, and has vanishing probability as $n \to \infty$.

- (i) A *termgale* is a family $d = \{d^s : s \in [0, \infty)\}$ of

s-termgales such that

$$2^{-s|\sigma|}d^s(\sigma) = 2^{-s'|\sigma|}d'(\sigma),$$

for all s, s' and $\sigma \in 2^{<\omega}$.

(ii) We say that a termgale is *constructive* or Σ^0_1, if d^0 is a Σ^0_1 function.

- Now introduce optimal termgales etc.
- filtering through discrete semimeasures and the Coding theorem, you get
- There is a constant $c \in \mathbb{N}$ such that for all $\sigma \in 2^{<\omega}$,

$$|K(\sigma) - |\sigma| \dim(\sigma)| \leq c.$$