Algorithmic Randomness 5

Rod Downey

Victoria University

Wellington

New Zealand

Measure-Theoretical Injury

- I would like to discuss some relatively poorly known (and sometimes re-proven) theorems of especially Stuart Kurtz using interesting arguments, I call measure-theoretical injury.
- We will need from Lecture 3 the 0-1 Laws

Effective 0-1 Laws

- Lemma (Kučera-Kautz) Let $n \geq 1$. Let T be a Π_n^D class of positive measure. Then T contains a member of every $D - n-$random degree.
- We will need this especially for $D = \emptyset'$, so that it reads
- Every Π_2^0 class of positive measure contains members of every 2-random degree.

Martin's Theorem

- Recall that a degree $\boldsymbol{a}$ is called hyperimmune if there is a function computable from $\boldsymbol{a}$ not majorized by any computable function.
- We have seen that there are hyperimmune-free randomns.
- Are typical randoms hyperimmune?

- (Martin's Theorem) Suppose that $\boldsymbol{a}$ is 2-random. Then $\boldsymbol{a}$ is hyperimmune.
- Claimed by Kautz to hold for weakly 2-randoms, but every Kurtz random hyperimmune free degree is also weakly 2-random by Lecture 3. (Recall if $A \in \cap_n U_n$, then enumerate enough till A occurs and use majorization.)

The Paris-Martin Method

- The Proof:
- "risking measure": failing on some small measure.
- By 0-1 Law: $\mu(\{A : A$ bounds a hyperimmune degree $\}) > 0$. (Exact calculation later)
- We define partial computable functional Ξ so that
- $\mu(\{A : \Xi^A$ is total and not dominated by any computable function $\} \geq \frac{1}{2}$.
- We consider here Ξ as a continous partial computable partial functional from strings to strings.

- Requirements $R_e : \varphi_e$ total $\rightarrow$ Ξ^A is not majorized by φ_e.

- Suppose that we *knew* whether φ_e was total.

- Then we do this: Begin with R_0 at λ.

- φ_0 was not total, do nothing. Move on to R_1

- φ_0 total, pick witness n_0, compute $\varphi_0(n_0)$ and then set $\Xi^\lambda(n_0) = \varphi_0(n_0) + 1$.

- Then φ_0 cannot majorize Ξ^A for any A.

- Then use a new witness for R_1 etc.

- *However*, even for a witness n_0, we can't decide if $\varphi_0(n_0) \downarrow [s]$
- The solution: The idea then is to try to implement this process in such a way that should we fail to meet some R_e then the overall cost (in terms of measure) will be small. (Here small will be $2^{-(e+2)}$.)

- For R_0 alone.
- Pick 4 arenas: $[00], [01], [10], [11]$.
- Initially (and perhaps forever) R_0 *only* deals with $[00]$. The other requirements will have the arena $\overline{[00]}$ for their pleasure.

- Pick a witness $n_0 = n_0^{00}$ for [00] and R_0.
- *Don't* define $\Xi^\sigma(n_0)$ for *any* σ extending 00 *unless* $\varphi_0(n_0^{00}) \downarrow [s]$.
- Should such an s occur, define $\Xi^{00}(n_0) = \varphi_0(n_0) + 1$.
- *And* then R_0 releases control of [00], since now it is *met* there. The *point* is that now for any strings σ extending 00, $\Xi^\sigma(n_0) = \varphi_0(n_0) + 1$.
- If we *fail* to define Ξ^{00}, then n_0^{00} is a witness to the fact that φ_0 is not total, and don't define Ξ on a cone of measure at most $\frac{1}{4}$. *But* R_0 is met *globally*.

- Now whilst we are devoting [00] to testing R_0, we cannot stop Ξ^ν being defined for strings *not* extending 00.
- Thus, for instance, by the stage s that we get to define $\Xi^{00}(n_0) \downarrow [s] = \varphi(n_0) + 1$, we might well have defined $\Xi^\sigma(m)$ for various σ extending 01.

- Now, we meet R_0 in the cone [01] (and then [10], then [11], always risking 2^{-2} each time). Thus R_1 will assert control of [01]

- R_0 asserts control of all xtensions of 01 of length s. There will be 2^{s-2} such extensions $\nu_1 \dots \nu_{2^{s-2}}$ and note that

$$\sum_{i \leq i \leq 2^{s-2}} 2^{-|\nu_i|} = 2^{-2}.$$

- Pick a new large n_0^{01} which will serve for *all* of the ν_i's.

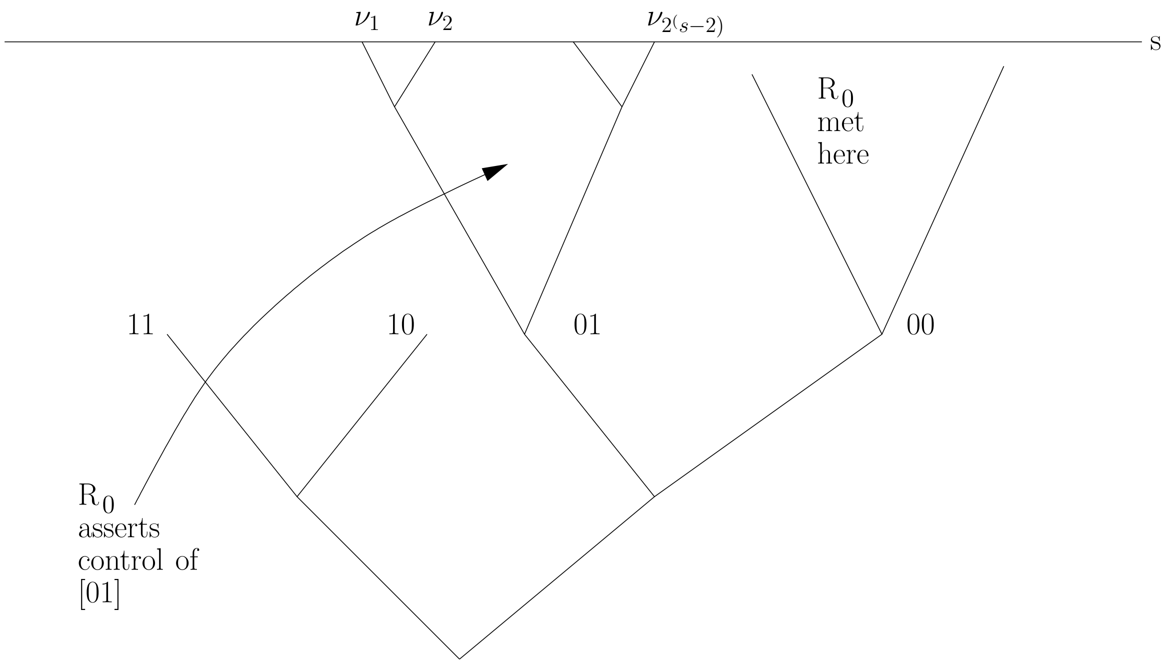

Figure 1: R_0 changes to the next cone

- Now if we ever see a stage $t > s$ such that $\varphi_0(n_0^{01}) \downarrow [t]$, we will be free to define $\Xi^{\nu_i}(n_0^{01}) \downarrow [t] = \varphi_0(n_0^{01}) + 1$, for *all* ν_i simultaneously.

- In general, R_e deals with the cones $[\sigma_i]$ at length $e + 2$.

- In this way, we see that at any stage each R_e has exactly $2^{-(e+2)}$ of measure risked because of it, of $R - e$ is completely met.

- Thus the overall measure where Ξ is not defined is bounded by $\sum_{e \in \mathbb{N}} 2^{-(e+2)} = \frac{1}{2}$.

The NST Proof

- There is a very nice proof of Martin's Theorem by Nies, Stephan, and Terwijn.
- Recall from Lecture 3, A is 2-random iff for all computable time bounds g with $g(n) \geq n^2$,

$$\exists d \exists^\infty n (C^g(A \upharpoonright n) \geq n - d).$$

- Let A-computable : $f(k) = \mu n[\exists p_1, \ldots, p_k \leq n](C^g(A \upharpoonright p_i) \geq p_i - d)$.
- Claim f is not dominated by any computable h.

- If not, Define the Π^0_1 class
  $P = \{Z : \forall k[\exists p_1, \ldots, p_k \leq h(k)]$
  $(C^g(Z \upharpoonright p_i) \geq p_i - d)\}$.

- $P \neq \emptyset$ and contains only 2-randoms!

Every 2-random is CEA

- Theorem (Kurtz) plus Kautz observation.
  Suppose that A is 2-random. Then A is CEA.

- That is, there is a X with $X <_T A$ and A being Σ_1^X.

- We build Ξ so that $\Xi^A = X$ on a set of postive measure.

- $\mu(\{A : \Xi(A)$ total and $\Xi(A) \not\leq_T A\}) \geq \frac{1}{4}$, and$A$ is c.e. in $\Xi(A)$ whenever $\Xi(A)$ is total.

- To make A c.e. in $\Xi(A)$, $n \in A$ iff $\exists m(\langle n, m\rangle \in \Xi(A))$.
- Requirements: $R_e : A \neq \Phi_e^{\Xi(A)}$.
- Need to notion of *acceptable strings*

Acceptable strings

- ξ is *acceptable* for a string σ iff

$$\xi(\langle m, n \rangle) = 1 \rightarrow \sigma(n) = 1.$$

- We make A c.e. in Ξ^A. Thus we will always require that $\Xi^\sigma[s]$ is acceptable.
- (Notation) We will also try, whenever possible, to use ξ to represent string in the range of $\Xi[s]$.

The Problem

- It is very difficult for *us* to force $A \neq \Phi_e^{\Xi(A)}$.
- It is *our opponent* who controls Φ_e.
- In some sense we can *monitor* what the opponent tries to do. Kurtz' strategy is based around this idea.

Kurtz's Idea

- We will inductively be working above some string β, to which Kurtz assigns a *state* via a colour blue_e.
- blue_e means we are currently "happy"(do be described!)
- We consider R_e in the cone $[\beta]$ for now. The argument is actually infinitary.
- We don't define (perhaps forever, again) Ξ on some cone of small measure in the cone $[\beta]$.

- Ξ^σ will be defined for σ extending $\beta 1^{e+2}$, should it be possible for us to actually perform some kind of diagonalization.
- This testing is signalled by state red_e.
- While we wait: work in $[\nu_1], \ldots, [\nu_{2^{(e+2)}-1}]$ (length $e+2$ extensions of β)
- Give these colour yellow_e. Only will define Ξ^τ for $\nu_i \preceq \tau$, for now.

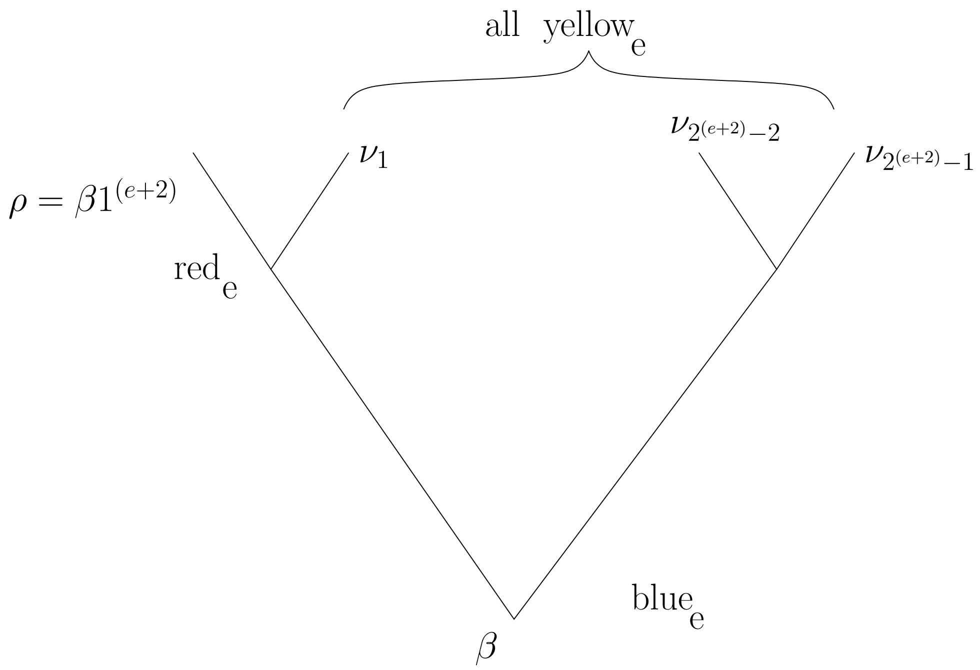

Figure 2: R_e's basic module

$\mathbf{Purple}_e$

- The construction will be defining Ξ^σ for various θ extending ν_i in the cone $[\beta]$.
- Under what circumstances would a string θ look bad from R_e's point of view?
- It would look bad if it looked like the initial segment of a real α with $\Phi_e^{\Xi(\alpha)} = \alpha$.

- We say that a string θ is *threatening* a requirement R_e is there is a yellow$_e$ strings ν extending its blue$_e$ predecessor β with $\nu \preceq \theta$, and a strings ξ acceptable to θ such that

  (i) $|\theta| \leq s$

  (ii) $|\xi| = s$

  (iii) $\Xi^{\nu}[s] \preceq \xi$

  (iv) $\Phi_e^{\xi}(k) = \nu(k) = 0[s]$ for some k such that $|\beta| < k \leq |\nu|$, and

  (v) no initial segment of θ has colour purple$_e$ (a minimality requirement).

- θ which is threatening R_e colour purple$_e$. While this situation holds, we will not define Ξ on any extension of θ.

Too much purple$_e$

- Maybe: β, the measure of the strings coloured red$_e$ (namely $2^{-(|\beta|+e+2)}$) *plus* the measure of purple$_e$ strings remains *small enough*.

- So R_e's injury to defining Ξ is small in $[\beta]$.

- The R_e is met and $\beta 1^{(e+2)}$ tells us so.

- ν_i are chosen so that they each differ from 1^{e+2} in at least one place where they are a 0. Let $\rho = \beta 1^{e+2}$.

- By (iv) of "threatening" for such ξ, for some k with $\rho(k) = 1$ we must have $\Phi_e^{\xi}(k) = \nu(k) = 0 \neq \rho(k) = 1$, by fact that $\rho = \beta 1^{e+2}$.

- Hence, if we define Ξ^{α} on extensions α of ρ to emulate Ξ^{ν}, then it *cannot be* that $\Phi_e^{\Xi^{\alpha}} = \alpha$ as it must be wrong on ρ.

- If θ is purple$_e$, let θ' denote the unique string of length $|\theta|$ extending the red$_e$ string ρ with $\theta(m) = \theta'(m)$ for all $n \geq |\rho|$.
- When the density of purple$_e$ strings above the blue$_e$ string β exceeds $2^{-(e+3)}$, there must be some yellow$_e$ string ν such that the density of purple$_e$ strings above ν must also exceed $2^{-(e+3)}$, by the Lebesgue Density Theorem.

- Let $\{\theta_1, \ldots, \theta_n\}$ list the purple$_e$ strings above ν
- For θ_i, let ξ_i be the least string which witnesses the threat to R_e

- Then we know:

  (i) ξ_i is acceptable for θ_i, (and for ρ),

  (ii) $\Xi^\nu \preceq \xi_i$, and

  (iii) $\Phi_e^{\xi_i}(k) = 0 = \theta_i(k)$ for some k with $\rho(k) = 1$.

- We can win R_e in the cone above ρ, and on a set of large measure.

$\mathbf{Green}_e$

- Define Ξ to mimic the action of Ξ on the θ_i above ν, so that $\Xi^\nu[s] = \Xi^\rho[s]$. The action is that

  (a) Any string extending β loses its colour.

  (b) β loses colour blue$_e$.

  (c) Each string θ'_i is given colour green$_e$.

  (d) We define $\Xi^{\theta'_i} = \xi_i$, which will then force $\Phi_e^{\xi_i}(k) \neq \theta'(k)$ for some k with $|\beta| < k \leq |\nu|$.

- Green$_e$ is best of all (and saves whales)

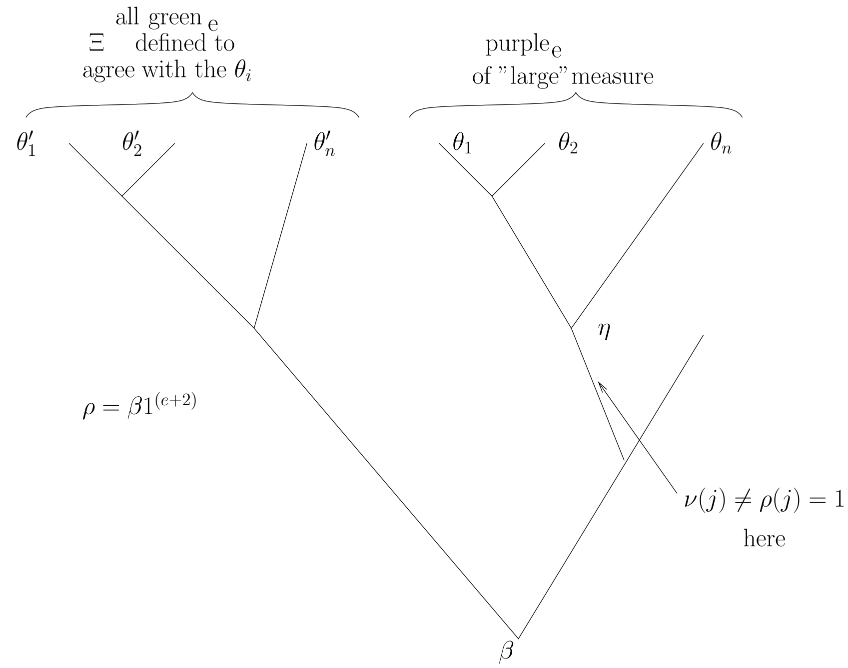

Figure 3: R_e's acts under purple$_e$ pressure

- Each time we are forced to use the red_e string ρ, we are guaranteed to succeed on set of measure at least $2^{-(e+5)}$ above β
- With a "catch up stage" for the acceptability of the ξ, we then replicate this e module on the boundary.
- Notice $\{Xi^A$ is total $\}$ is a Π^0_2 class of positive measure, and hence 2-randomness is enough.

1-Generic degrees

- Recall that A is n-generic iff for all Σ^0_n sets of strings S, either there is a $\sigma \prec A$ with $\sigma \in S$, or there is a $\sigma \prec A$ such that, for any $\tau \in S$, $\sigma \not\preceq \tau$.
- Corollary (Kurtz) Every 2-random real bounds a 1-generic one.
- This follows from
- Folklore Suppose that A is CEA(B) with $B <_T A$. Then A bounds a 1-generic degree.

- This is Shore's proof:
- $A = \cup_s A_s$ B-computable enumeration. and let $c(n) = \mu s(A_s \upharpoonright n = A \upharpoonright n)$.
- Let V_e denote the e-th c.e. set of strings
- Construction: $G_{s+1} = G_s$ unless (for some least e) we see some extension $\gamma \in V_{e,c(s+1)}$ with $G_s \prec \gamma$. In this latter case, let $G_{s+1} = \gamma$.

- Suppose G is not 1-generic. Claim $A \leq_T B$.
- Suppose G is dense in V_e and not meeting it. Suppose e now has priority.
- To compute c, assume that G_s is known. Compute a minimal stage t such that some extension γ_s of G_s occurs in $V_{e,t}$. The it must be that $t > c(s+1)$, lest we would act for e. This allows us to compute $c(s+1)$, and hence G_{s+1}.

More 1-Generic degrees

- Kurtz used the risking measure method to prove the following remarkable result.
- Recall a class $\mathcal{C}$ of degrees is *downward dense* below a degree $\boldsymbol{a}$ iff for all nonzero $\boldsymbol{b} < \boldsymbol{a}$ there is a degree $\boldsymbol{c} \leq \boldsymbol{b}$ with $\boldsymbol{c} \in \mathcal{C}$.
- Theorem The 1-generic degrees are downward dense below any 2-random degree.

- The proof uses an idea akin to that of the CEA proof for 1-genericity.
- One key ingredient is the fact (Lecture 4) that if Φ^A is noncomputable, then the cone of reals Turing above Φ^A has measure 0.
- This allows for "low measure noise" in the construction of Ξ^{Φ^A}.
- *But*, sometimes Φ^A is computable, and this must be dealt with another way.
- The proof is either "elegant" or "horrible" depending on your outlook.

Analogs

- Actually there are a number of analogs that can be pursued here.
- For example, Yu Liang has proven a analog of van Lambalgen's Theorem: $x \oplus z$ is n-generic iff x is n-z-generic and z is n-generic.
- We can ask if there are reals low for 1-genericity.
- Theorem (Greenberg, Miller, Yu) No.

- One last analog: (Csima, Downey, Greenberg, Hirschfeldt, Miller)

  (i) If A is 2-generic and B is n-generic, and for $n \geq 3$, $A \leq_T B$. Then A is n-generic.

  (ii) (MUST BE CHECKED) There is a 1-generic $X \leq Y$ with Y 2-generic and X not 2-generic.