

On Decisive and Non U-Shaped Learning

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An Example

Learner reading data and outputting hypotheses.

Data

2
2,3
2,3,5
2,3,5,1
2,3,5,1,8
2,3,5,1,8,4

Hypotheses

Set of even numbers;
Set of all numbers;
Set of prime numbers;
Set of Fibonacci numbers;
Set of Fibonacci numbers;
Set of all numbers.

Learner returns to abandoned hypothesis.

When can this be avoided – topic of this talk.

Explanatory Learning

Given: Class C of r.e. languages known to the learner, selected $L \in C$ unknown to the learner.

Input: Text for L , that is, an infinite sequence w_0, w_1, \dots consisting of all elements of L and perhaps $\#$ in arbitrary order.

Output: Sequence of hypotheses e_0, e_1, \dots such that all but finitely many hypotheses are the same correct index e of the selected language L .

Formally: $e_n = M(w_0 w_1 \dots w_n)$ for all n where M is a total recursive function.

Some Learnable Classes

Finite Classes like $\{A, B\}$.

Assume that $B \not\subseteq A$, say $0 \in B - A$. Let a an index for A and b an index for B . Then

$$M(w_0 w_1 \dots w_n) = \begin{cases} a & \text{if } 0 \notin \{w_0, w_1, \dots, w_n\}; \\ b & \text{if } 0 \in \{w_0, w_1, \dots, w_n\}. \end{cases}$$

For example, $M(235) = a$ and $M(2350\#) = b$.

Class of Finite Sets

On input $w_0 w_1 \dots w_n$ the learner outputs a canonical index for $\{w_0, w_1, \dots, w_n\} - \{\#\}$.

Class of Self-Describing Sets: L is in C if $L = W_{\min(L)}$.

The learner always outputs minimum of data seen so far.

Unlearnable Classes

Example [Gold 1967]

A class **C** consisting of one infinite set **A** and all of its finite subsets cannot be learned from positive data.

Proof Idea: Let **M** be a learner. There is a finite sequence $w_0 w_1 \dots w_n$ of data from **A** such that **M** conjectures **A** as long as data from **A** follows. Thus **M** does not learn the set

$$\{w_0, w_1, \dots, w_n\} - \{\#\}.$$

Further topologically difficult example

The class of all cofinite sets is unlearnable.

Computationally difficult example

The class of all graphs of recursive functions is unlearnable.

Behaviourally Correct Learning

Definition [Case and Lynes 1982]

A learner **M** behaviourally correct identifies **C** if **M** outputs on any text of any language $L \in C$ a sequence e_0, e_1, \dots of hypotheses such that $W_{e_n} = L$ for almost all n .

Example

The class $\{K \cup \{0\}, K \cup \{1\}, \dots\}$ has a behaviourally correct but no explanatory learner.

Behaviourally correct learner implicitly given by

$$W_{M(w_0w_1\dots w_n)} = K \cup \{w_0, w_1, \dots, w_n\} - \{\#\}$$

while an explanatory learner **N** is nonrecursive: Starting with a locking sequence, $x \notin K$ iff **N** outputs a new hypothesis after seeing the locking sequence, x and perhaps some further elements of **K**.

Vacillatory Learning

Definition [Case 1999]

M is a vacillatory learner for **C** iff **M** is a behaviourally correct learner which outputs on any text for any language in **C** only finitely many hypotheses.

Theorem [Case 1999]

There are behaviourally correctly learnable classes which are not vacillatorily learnable.

There are vacillatorily learnable classes which are not explanatorily learnable.

That is, vacillatory learning is properly between explanatory learning and behaviourally correct learning.

Vacillatory Learning is Restrictive

Theorem The class $\{\mathbf{K} \cup \{0\}, \mathbf{K} \cup \{1\}, \dots\}$ is behaviourally correct learnable but not vacillatorily learnable.

Proof Assume that \mathbf{M} vacillatorily learns the class.

There is a locking sequence $w_0 w_1 \dots w_n$ and a finite set \mathbf{E} of indices of \mathbf{K} such that

$$\mathbf{M}(w_0 w_1 \dots w_n x v_0 v_1 \dots v_m) \in \mathbf{E}$$

for all $x, v_0, v_1, \dots, v_m \in \mathbf{K}$. But for every $x \notin \mathbf{K}$ there are m and $v_0, v_1, \dots, v_m \in \mathbf{K}$ with

$$\mathbf{M}(w_0 w_1 \dots w_n x v_0 v_1 \dots v_m) \notin \mathbf{E}.$$

Thus $\overline{\mathbf{K}}$ is r.e. relative to the Turing degree of \mathbf{M} and there is **no recursive** vacillatory learner for this class.

Vacillatorily Learnable Classes

Theorem The class

$$\{\mathbf{L} : \mathbf{L} \neq \emptyset \wedge \exists e < \min(\mathbf{L}) (\mathbf{L} = \mathbf{W}_e)\}$$

is vacillatorily learnable but not explanatorily learnable.

The vacillatory learner conjectures on input $\mathbf{w}_0, \mathbf{w}_1, \dots, \mathbf{w}_n$ that index $e < \min(\{\mathbf{w}_0, \mathbf{w}_1, \dots, \mathbf{w}_n\} - \{\#\})$ for which the minimum of $\mathbf{W}_{e,n} \Delta(\{\mathbf{w}_0, \mathbf{w}_1, \dots, \mathbf{w}_n\} - \{\#\})$ is maximal.

Vacillating between two indices

The class

$$\{\mathbf{L} : \mathbf{L} = \mathbf{W}_e \text{ for } e \in \{\min(\mathbf{L}), \min(\mathbf{L} - \min(\mathbf{L}))\}\}$$

has a vacillatory learner which vacillates between two indices but does not have an explanatory learner.

Decisive Learning

Padding permits to enforce that the learner takes a new unused hypothesis at every mind change. Is this also possible on a semantic level?

Definition

A decisive learner never semantically returns to an abandoned hypothesis.

Question [Osherson, Stob and Weinstein 1986]

Is decisive learning restrictive?

Theorem [Fulk, Jain and Osherson 1994]

There is a class C_{BC} which is behaviourally correctly learnable but not with a decisive learner.

Theorem [Baliga, Case, Merkle and Stephan 2000]

There is an explanatorily learnable class C_{EX} which does not have a decisive behaviourally correct learner.

Constructing the Class C_{EX}

Let M_0, M_1, \dots be a list of all recursive learning machines. One constructs a K -recursive sequences e_0, e_1, \dots and $\sigma_0, \sigma_1, \dots$ such that

- for all x , σ_x is a finite sequence and $\{y : y < x\} \subseteq \text{rng}(\sigma_x) \subset W_{M_{e_x}(\sigma_x)} \subseteq \{y : y \neq x\}$;
- for all e , if M_e learns for infinitely many cosingle sets and never conjectures $\{0, 1, 2, 3, \dots\}$ then there is an x with $e_x = e$.

Let $C_{EX} = \{\text{rng}(\sigma_0), W_{M_{e_0}(\sigma_0)}, \text{rng}(\sigma_1), W_{M_{e_1}(\sigma_1)}, \dots\}$.

Learner for C_{EX} : on text for $L \in C_{EX}$, find least nonelement x and determine $\sigma_x, M_{e_x}(\sigma_x)$ in the limit. Conjecture $\text{rng}(\sigma_x)$ if it coincides with data seen so far and $W_{M_{e_x}(\sigma_x)}$ otherwise.

Special Results

Theorem [Baliga, Case, Merkle and Stephan 2000]

If \mathcal{C} is explanatorily learnable then there is also an explanatory learner which does not return to an abandoned hypothesis twice. So “second-time decisive” learning is not restrictive in the context of explanatory learning.

Proof-Idea Start with a learner which has on every text a locking-sequence. Hypotheses belonging to locking-sequences are not changed, those belonging to other input are redirected into a finite set taken for this purpose only once. Done by outputting hypotheses which simulate the behaviour of learner during the enumeration.

Theorem [Baliga, Case, Merkle and Stephan 2000]

If \mathcal{C} is explanatorily learnable and contains the set $\{0, 1, 2, 3, \dots\}$ of natural numbers then \mathcal{C} has a decisive explanatory learner.

Non U-Shaped Learning

Definition

A non U-Shaped learner never abandons a correct hypothesis for an incorrect one.

So the U-Shape “correct-incorrect-correct” does not occur in any learning process.

Theorem [Baliga, Case, Merkle, Stephan and Wiehagen]
Every behaviourally correct learner for the class C_{BC} is U-shaped on some text for some language; thus “non U-shaped” is restrictive for behaviourally correct learning.

Theorem [Baliga, Case, Merkle, Stephan and Wiehagen]
Every explanatorily learnable class has a non U-shaped explanatory learner.

Constructing the Class C_{BC}

Let M_0, M_1, \dots be a list of all recursive learning machines. For every e , Let C_{BC} contain for each e the following sets:

- C_{BC} contains $L_e = \{(e, 0), (e, 1), (e, 2), \dots\}$;
- for each k let $I_{e,k} = \{(e, 0), (e, 1), \dots, (e, k)\}$ and $J_{e,k}$ the set conjectured by $M_e((e, 0) (e, 1) \dots (e, k))$; C_{BC} contains $I_{e,k}, J_{e,k}$ for the first k satisfying $I_{e,k} \subset J_{e,k} \subseteq L_{e,k}$ whenever such a k exists.

Whenever M_e is a behaviourally correct learner for C_{BC} then there is a k with $I_{e,k}, J_{e,k} \in C_{BC}$ and on some text for $J_{e,k}$, M_e conjectures the U-shape $J_{e,k}, I_{e,k}, J_{e,k}$.

Non U-Shaped Explanatory Learning

Let M be an explanatory learner for C which has a locking-sequence on every text for a language in C .

Let $e = M(w_0 w_1 \dots w_n)$. Then N outputs an index e' such that $W_{e'} = \{0, 1, 2, 3, \dots\}$ if there are $v_0, v_1, \dots, v_m \in W_e$ with

$$M(w_0 w_1 \dots w_n) \neq M(w_0 w_1 \dots w_n v_0 v_1 \dots v_m)$$

and $W_{e'} = W_e$ otherwise.

Small further adjustments have to be made to N in order to preserve that the learner is explanatory, the outline here gives only that N is behaviourally correct.

Note that in the construction of C_{EX} , learners outputting $\{0, 1, 2, 3, \dots\}$ were not diagonalized. Thus C_{EX} has an explanatory non U-shaped learner.

Consistent Learning

A consistent learner always outputs hypotheses which generate all data seen so far; consistent learners can be undefined on irrelevant data.

Behaviourally correct learners can by patching easily be made consistent: instead of e_n one outputs a hypothesis for the set

$$W_{e_n} \cup \{w_0, w_1, \dots, w_n\} - \{\#\}.$$

But this operation does not preserve the learner to be non U-shaped. The following result is more involved.

Theorem

Every class having a non U-shaped behaviourally correct learner has also such a learner which is in addition consistent.

Consistent Explanatory Learning

Consistent explanatory learning is quite restrictive; for example, if $\{0, 1, 2, \dots\} \in \mathbf{C}$ and \mathbf{C} is consistently learnable then every set in \mathbf{C} is recursive.

Theorem

Every consistently explanatorily learnable class has a consistent and decisive explanatory learner.

Theorem

The class consisting of the halting problem and the set of all natural numbers is decisively explanatorily learnable but does not have a consistent explanatory learner.

Abandoning Wrong Hypotheses

Definition

A learner is wr-decisive if it never returns to a wrong abandoned hypothesis.

Every decisive learner is wr-decisive but not vice versa.

Theorem

A wr-decisive explanatory learner can be made decisive.

Corollary

The class C_{EX} has a non U-shaped explanatory learner but no wr-decisive behaviourally correct learner.

Theorem

The class C_{BC} has a wr-decisive behaviourally correct learner but not a non U-shaped behaviourally correct learner.

Abandoning Non-Overgnlg. Hypotheses

Definition

A learner is nov-decisive if it never returns to a abandoned hypothesis which is not a subset of the language to be learned.

Theorem

Every explanatorily learnable class has an nov-decisive explanatory learner.

Theorem

Every behaviourally correct learnable class has a nov-decisive and consistent behaviourally correct learner.

Vacillatory Learning

A vacillatory learner vacillates between finitely many hypotheses and is almost always correct. This notion is between explanatory and behaviourally correct learning.

Vacillatory learning combined with restrictions on retaking abandoned hypotheses collapses to the corresponding restricted version of explanatory learning.

Theorem

Every class which is vacillatory learnable but not explanatory learnable does also not have a vacillatory learner which is decisive, non U-shaped, wr-decisive or nov-decisive.

Vacillatory vs. Behaviourally Correct

A more interesting question is the following ...

Question

When is vacillatory learning contained in non U-shaped behaviourally correct learning?

Theorem [Carlucci, Case, Jain, Stephan 2005]

A learner which eventually vacillates among at most two indices can be replaced by a non U-shaped behaviourally correct learner.

This is impossible if the given learner is permitted to vacillate between three indices.

Team Learning

Definition

A team of n machines learns a class C iff for every language L in C and every text for L there is a member of the team which converges to an index of L .

A team is furthermore non U-shaped iff no member of the team ever makes a mind change from a correct to an incorrect hypothesis.

Theorem [Carlucci, Case, Jain, Stephan 2005]

A class has a vacillatory learner which on every text vacillates between at most n hypotheses iff it is learnable by a non U-shaped team of n machines such that all machines in the team converge on any text of any language in the class to a hypothesis.

Several Learners per Team

Theorem

A class is vacillatory learnable such that the learner on every text vacillates between at most **2** hypotheses **iff** it is learnable by a team of three machines such that at least two of them converge to a correct hypothesis.

Corollary

If a **2 out of 3** team learns a class then it has a non U-shaped behaviourally correct learner.

Question

If $m/n > 1/2$ and an **m out of n** team learns a class, does this class then have a non U-shaped behaviourally correct learner?

Open Problems

Consistency

- (1) Can a behaviourally correct decisive learner be made consistent?
- (2) Can a behaviourally correct learner never returning to abandoned wrong hypotheses be made consistent?

Team Learning

- (3) Is the inclusion structure of team learning and non U-shaped team learning the same?
- (4) Assume that C is learnable by an m out of n team with $m/n > 1/2$. Is then C learnable by a non U-shaped behaviourally correct learner? Can also such teams be made non U-shaped?

Summary

Main Results for Learnability

- explanatory \Rightarrow non U-shaped explanatory;
- explanatory \Rightarrow nov-decisive explanatory;
- explanatory $\not\Rightarrow$ decisive explanatory;
- consistent explanatory \Rightarrow consistent decisive explanatory;
- behaviourally correct \Rightarrow consistent nov-decisive behaviourally correct;
- behaviourally correct $\not\Rightarrow$ non U-shaped behaviourally correct.