Cupping Computably Enumerable Degrees in the Difference Hierarchy

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  (cf. In $\mathcal{R}$, there exist noncuppable degrees.)

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$\mathcal{P}_e$: $E \neq \Phi_e^D$.

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- If n enters K, and we want to rectify $\Gamma^{A,D}(n)$ (defined as 0) at stage s, we put $\gamma(n)[s])$ (or smaller) into D, to undefine $\Gamma^{A,D}(n)$.
- If A changes blow $\gamma(n)[s])$ after stage s, then we can take $\gamma(n)[s])$ out of D since this A change can undefine $\Gamma^{A,D}(n)$.

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- Step 1: Choose x and k.

  If K changes below k, then we start from the beginning, except that we keep k the same.

  - ◊ Such a refresh (or reset) procedure can happen at most k many times.
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- Step 4: Take $\gamma(k)$ out of D and put x into E.

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- Reach step 4 eventually. Again, $\mathcal{P}$ is satisfied.
- Stop at step 3 infinitely many times. Then A is computable, which can be called a pseudo-outcome of $\mathcal{P}$.

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- Between almost universal cupping degree and $\mathbf{0}'$, there are no c.e. degrees.
- There is no d.c.e. universal cupping degree.
- Maximal incomplete degrees are almost universal cupping.

Theorem 1

Almost universal cupping degrees exist.

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- Compare with Arslanov's requirements.

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- Consider the interactions of two $\mathcal{P}$ strategies.
- Two ways to get around the obstacle.

  ◊ Make A ω-c.e. and universal cupping (Li, Song and Wu)

  ◊ Make A d.c.e. but Δ^A is now necessary.

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- How many cupping partners are needed in this definition?

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- How many cupping partners are needed in this definition?
- Answer: infinite.

Three alternative approaches

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  There are c.e. degrees $\mathbf{a} > \mathbf{0}$, $\mathbf{b}$, $\mathbf{c}$, with $\mathbf{b} \not\geq \mathbf{c}$ such that $\mathbf{b}$ cups any nonzero c.e. degree below $\mathbf{a}$ above $\mathbf{c}$.

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- (Slaman)

  There are c.e. degrees $\mathbf{a}, \mathbf{b} > \mathbf{0}$ such that for any $\mathbf{c} \leq \mathbf{a}$, if $\mathbf{c} \not\leq \mathbf{b}$, then $\mathbf{c} \cup \mathbf{b} = \mathbf{0}'$.

Theorem 2 (Plus-cupping for d.c.e.)

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There are a c.e. degree $\mathbf{a} > \mathbf{0}$ and an incomplete d.c.e. degree $\mathbf{d}$ such that $\mathbf{d}$ cups each nonzero c.e. degree below $\mathbf{a}$ to $\mathbf{0}'$.

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There are d.c.e. degrees $\mathbf{a}, \mathbf{d}$ such that if $\mathbf{c}$ is a nonzero c.e. degree below $\mathbf{a}$ then $\mathbf{c} \cup \mathbf{d} = \mathbf{0}'$, and if $\mathbf{c}$ is a c.e. degree not below $\mathbf{a}$ then $\mathbf{c} \cup \mathbf{a} = \mathbf{0}'$.

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- Consider the interactions of these two $0'''$ arguments.

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Li and Yi's cupping implies Theorem 2.

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(3) These c.e. degrees can be low.

Questions

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- How to define computably enumerable degrees in the Δ_2^0 degrees?

Thank you!