A Computational Prospect of Infinity:

ω_1-*Recursion Theory*

Noam Greenberg

August 10, 2005

Definition

A set $A \subset \omega_1$ is ω_1-*recursively enumerable* if it is $\Sigma_1(L_{\omega_1})$-definable.

Definition

A set $A \subset \omega_1$ is ω_1-*recursive* if it is ω_1-r.e. and ω_1-co-r.e.

Definition

A (perhaps partial) function $f : \omega_1 \longrightarrow \omega_1$ is (partial) *recursive* if its graph is ω_1-r.e.

Enumeration Theorem

There is a complete ω_1-r.e. set.

Induction Theorem

If $I : L_{\omega_1} \to \omega_1$ is computable, then there is a (unique) computable $f : \omega_1 \to \omega_1$ such that for all $\beta < \omega_1$, $f(\beta) = I(f \restriction \beta)$.

Turing Reducibility

Let $\mathcal{S} = 2^{<\omega_1}$.
If $\Phi \subset \mathcal{S}^2$ and $A \in 2^{\leqslant\omega_1}$ then

$$\Phi(A) = \cup\{\sigma \ : \ \exists\tau \in \mathcal{S} \ (\tau \subset A \ \& \ (\tau, \sigma) \in \Phi)\}$$

Definition

An ω_1-*Turing functional* is an ω_1-r.e. $\Phi \subset \mathcal{S}^2$ which is *consistent*: for all $A \in 2^{\leqslant\omega_1}$, $\Phi(A) \in 2^{\leqslant\omega_1}$.

TURING REDUCIBILITY

Let $\mathcal{S} = 2^{<\omega_1}$.
If $\Phi \subset \mathcal{S}^2$ and $A \in 2^{\leqslant\omega_1}$ then

$$\Phi(A) = \cup\{\sigma : \exists\tau \in \mathcal{S}\ (\tau \subset A\ \&\ (\tau,\sigma) \in \Phi)\}$$

DEFINITION

An ω_1-*Turing functional* is an ω_1-r.e. $\Phi \subset \mathcal{S}^2$ which is *consistent*: for all $A \in 2^{\leqslant\omega_1}$, $\Phi(A) \in 2^{\leqslant\omega_1}$.

FACT

The following are equivalent for $A, B \in 2^{\omega_1}$:

- *B is $\Delta_1(L_{\omega_1}, A)$-definable.*
- *There is some ω_1-Turing functional Φ such that $\Phi(A) = B$.*

Theorem

Let $X \subset \omega_1$. Then there is a linear ordering $\mathcal{L}_X$ such that for all $Y \subset \omega_1$, there is a Y-computable copy of $\mathcal{L}_X$ iff Y computes X.

THEOREM

There is no embedding of the 1-3-1 lattice into the ω_1*-r.e. degrees.*

Theorem

Let α be an admissible ordinal. Suppose that there is an embedding of the 1-3-1 lattice into the α-r.e. degrees. Then $\mathrm{Th}(\mathcal{R}_\alpha)$ *is not hyperarithmetical.*

THEOREM

Let α be an admissible ordinal. Suppose that there is an embedding of the 1-3-1 lattice into the α-r.e. degrees. Then α is effectively countable: *$\mathbf{0}'_\alpha$ can compute both a partial counting of α and a cofinal ω-sequence in α.*

THEOREM

Suppose that α is an effectively countable admissible ordinal. Then models of arithmetic, in the style of Slaman-Woodin, together with specified non-hyperarithmetical sets, can be coded and decoded in $\mathcal{R}_\alpha$.

Remark

For any admissible ordinal α, the 1-3-1 lattice embeds in $\mathcal{R}_\alpha$ iff α is effectively countable.

COROLLARY

For any admissible ordinal α, $\mathcal{R}_\alpha \not\equiv \mathcal{R}_\omega$.