Computability-Theoretic and Proof-Theoretic Aspects of Vaughtian Model Theory

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• In this talk, all languages are computable and all structures have domain ω.

• A structure $\mathcal{A}$ is *computable* if its atomic diagram is computable.

• $\mathcal{A}$ is *decidable* if its elementary diagram is computable.

• These definitions can be relativized.

• In this talk, only decidability will be mentioned, but all results hold for computability as well.

Theorem (Knight). If $\mathcal{A}$ is X-decidable and nontrivial then there is a $\mathcal{B} \cong \mathcal{A}$ whose elementary diagram has the same degree as X.

- Every complete decidable theory has a decidable model.

- But what if we want a special model, such as an atomic, saturated, or homogeneous model?

- These are particularly interesting due to the way their constructions are tied to type realization and type omitting.

- In particular, atomic and saturated models are unique when they exist.

- More generally, two homogeneous models realizing the same types are isomorphic.

- So we can build copies of a given special model without explicitly building an isomorphism.

• Let T be a theory.

• A *partial n-type* of T is a set of formulas in the free variables $x_1, \ldots, x_n$ consistent with T.

• A *(complete) n-type* of T is a maximal partial n-type of T.

• A type Γ is *principal* if $\exists\varphi\, \forall\psi \in \Gamma\ [T \vdash \varphi \to \psi]$.
The formula φ is called a *generator* of Γ.

• $\mathcal{A} \models T$ *realizes* a type Γ if $\exists \overline{a} \in \mathcal{A}\, \forall\psi \in \Gamma\ [\mathcal{A} \models \psi(\overline{a})]$.
Otherwise $\mathcal{A}$ *omits* Γ.

• The *type spectrum* of $\mathcal{A}$ is the set of all types realized in $\mathcal{A}$.

• The n-types of T can be thought of as paths on a binary tree with no dead ends, with principal types being isolated paths.

• We can code a binary tree with no dead ends $\mathcal{P}$ into a theory T in the language with unary predicates $R_0, R_1, \ldots$ via the axioms:

• For $\sigma \in \mathcal{P}$ and $m \in \omega$:

$$\exists^{\geqslant m} x \, \Big(\bigwedge_{\sigma(k)=1} R_k x \;\wedge\; \bigwedge_{\sigma(k)=0} \neg R_k x \Big)$$

• For $\sigma \notin \mathcal{P}$:

$$\neg \exists x \, \Big(\bigwedge_{\sigma(k)=1} R_k x \;\wedge\; \bigwedge_{\sigma(k)=0} \neg R_k x \Big)$$

• If $\mathcal{P}$ is computable then T is decidable and the principal 1-types of T correspond to the isolated paths of $\mathcal{P}$.

- A complete theory T is *atomic* if every formula consistent with T is contained in a principal type of T.

- $\mathcal{A} \models T$ is *atomic* if every type realized in $\mathcal{A}$ is principal.

- $\mathcal{A} \models T$ is *prime* if $\forall \mathcal{B} \models T \ [\mathcal{A} \preceq \mathcal{B}]$.

- T has an atomic model iff T is atomic.

- $\mathcal{A}$ is atomic iff it is prime.

- If $\mathcal{A}, \mathcal{B} \models T$ are both atomic then $\mathcal{A} \cong \mathcal{B}$.

Theorem (Goncharov and Nurtazin; Millar). There is a complete decidable atomic theory T with no decidable atomic model.
T can be chosen to have all types computable.

Theorem (Goncharov and Nurtazin; Harrington). Let T be a complete decidable atomic theory. TFAE

1. T has a decidable atomic model.
2. Given a formula φ consistent with T, we can effectively produce a principal type of T containing φ.
3. There is a computable listing of the principal types of T.

Corollary (Millar; Denisov; Drobotun). Every complete decidable atomic theory has a $\emptyset'$-decidable atomic model.

Theorem (Goncharov and Nurtazin; Millar). There is a complete decidable atomic theory T with no decidable atomic model.
T can be chosen to have all types computable.

By coding a tree into a theory, it is enough to build a computable tree $\mathcal{T}$ s.t.

1. $\mathcal{T}$ has no dead ends.
2. The isolated paths are dense in $\mathcal{T}$.
3. Every path of $\mathcal{T}$ is computable.
4. There is no effective procedure that, given $x \in \mathcal{T}$, produces an isolated path in $\mathcal{T}$ extending x.

Theorem (Millar; Denisov; Drobotun). Every complete decidable atomic theory has a $\emptyset'$-decidable atomic model.

Theorem (Csima). Every complete decidable atomic theory has a low-decidable atomic model.

Theorem (Csima, Hirschfeldt, Knight, Soare). Let $A \leqslant_{\mathrm{T}} \emptyset'$. TFAE

1. A is nonlow$_2$.
2. Any complete decidable atomic theory has an A-decidable atomic model.

• This result shows there is a difference between the existence of low atomic models and the Low Basis Theorem.

Theorem (Csima, Hirschfeldt, Knight, Soare). Let $A \leqslant_{\mathrm{T}} \emptyset'$. TFAE

1. A is nonlow$_2$.
2. Any complete decidable atomic theory has an A-decidable atomic model.
3. Let $\mathcal{T}$ be a computable binary tree with no dead ends in which the isolated paths are dense. Given $x \in \mathcal{T}$, we can A-effectively produce an isolated path of $\mathcal{T}$ extending x.
4. Let $\mathcal{T}$ be a computable binary tree with no dead ends and let $S_0, S_1, \ldots \subseteq \mathcal{T}$ be uniformly Δ^0_2 dense sets. Given $x \in \mathcal{T}$, we can A-effectively produce a path of $\mathcal{T}$ extending x and meeting each S_i.
5. Every Δ^0_2 set contains the range of some A-limitwise monotonic function.

• Let $A \leqslant_{\mathrm{T}} \emptyset'$ be nonlow$_2$.

• Let $\mathcal{T}$ be a computable binary tree with no dead ends in which the isolated paths are dense. Given $x \in \mathcal{T}$, we can A-effectively produce an isolated path of $\mathcal{T}$ extending x.

• This implies that any complete decidable atomic theory has an A-decidable atomic model.

Theorem (Martin). $A \leqslant_{\mathrm{T}} \emptyset'$ is low$_2$ iff there is an $f \leqslant_{\mathrm{T}} \emptyset'$ that dominates every $g \leqslant_{\mathrm{T}} A$.

• So if $A \leqslant_{\mathrm{T}} \emptyset'$ is nonlow$_2$ then for each $f \leqslant_{\mathrm{T}} \emptyset'$ there is a $g \leqslant_{\mathrm{T}} A$ s.t. $\exists^\infty n\, (g(n) > f(n))$.

• Let $A \leqslant_{\mathrm{T}} \emptyset'$ be low$_2$.

• There is a computable binary tree with no dead ends $\mathcal{T}$ in which the isolated paths are dense s.t. there is no A-effective procedure that, given $x \in \mathcal{T}$, produces an isolated path of $\mathcal{T}$ extending x.

• This implies that there is a complete decidable atomic theory with no A-decidable atomic model.

Theorem (Jockusch). $A \leqslant_{\mathrm{T}} \emptyset'$ is low$_2$ iff there is a uniformly Δ^0_2 collection of sets containing every set $\leqslant_{\mathrm{T}} A$.

Theorem (Goncharov and Nurtazin; Millar). There is a complete decidable atomic theory T with no decidable atomic model.
T can be chosen to have all types computable.

• We have seen that, without the assumption that all types are computable, this result can be extended.

• But what if we do add in this assumption?

Theorem (Hirschfeldt). Let $A >_{\mathrm{T}} \emptyset$ and let T be a complete decidable theory all of whose types are computable. Then T has an A-decidable atomic model.

• This extends Csima's result for the $A \leqslant_{\mathrm{T}} \emptyset'$ case.

• Restatement: Let $A >_{\mathrm{T}} \emptyset$ and let $\mathcal{T}$ be a computable binary tree with no dead ends, all of whose paths are computable. Then there is an A-computable listing of the isolated paths of T.

Corollary (Slaman; Wehner). There is a structure with copies of every nonzero degree but no computable copy.

Omitting Types Theorem. Let T be a complete theory and S a countable set of (partial) types of T. There is a model of T omitting all the nonprincipal types in S.

Theorem (Millar). Let T be a complete decidable theory.
Let S_0 be a computable set of complete types of T.
Let S_1 be a computable set of nonprincipal partial types of T.
There is a decidable model of T omitting all nonprincipal types in S_0 and all types in S_1.

Theorem (Millar). There is a complete decidable theory T and a computable set S of partial types of T s.t. no decidable model of T omits all nonprincipal types in S.

Theorem (Millar). There is a complete decidable theory T and a computable set S of partial types of T s.t. no decidable model of T omits all nonprincipal types in S.

Proof. Let T be a complete decidable atomic theory T with all types computable but no decidable atomic model.

Let L be a uniform enumeration of all c.e. partial types of T.

By padding, replace each partial type Γ in L by an equivalent computable partial type $\widehat{\Gamma}$ to get S.

L includes all the computable types of T, which are all the complete types of T.

Omitting $\widehat{\Gamma}$ is the same as omitting Γ, so a model of T omitting all nonprincipal types in S is atomic, and hence not decidable.

- How far can this result be extended?

Theorem (Csima). Let $\emptyset <_{\mathrm{T}} A \leqslant_{\mathrm{T}} \emptyset'$. Let T be a complete decidable theory and let S be a computable set of partial types of T. There is an A-decidable model of T omitting all nonprincipal types in S.

Corollary (Csima). Let $\emptyset <_{\mathrm{T}} A \leqslant_{\mathrm{T}} \emptyset'$ and let T be a complete decidable theory all of whose types are computable. Then T has an A-decidable atomic model.

• We have seen that the corollary holds without the hypothesis that $A \leqslant_{\mathrm{T}} \emptyset'$.

• What about the theorem?

Theorem (Csima, Hirschfeldt, Shore). TFAE

1. A has hyperimmune degree (i.e., there is an $f \leqslant_{\mathrm{T}} A$ not dominated by any computable function).

2. Let T be a complete decidable theory and let S be a computable set of partial types of T. There is an A-decidable model of T omitting all nonprincipal types in S.

• So building atomic models by realizing types is "easier" than building them by omitting types.

• We can also take the reverse mathematical approach to the results studied above.

• Here are some examples involving atomic and prime models, from joint work with Csima and Shore.

Atomic Model Theorem. A complete theory has an atomic model iff it is atomic.

Prime Model Theorem. A complete theory has a prime model iff it is atomic.

Atomic Model Uniqueness. Any two atomic models of a given complete theory are isomorphic.

Prime Model Uniqueness. Any two prime models of a given complete theory are isomorphic.

Atomic-Prime Equivalence. A structure is atomic iff it is prime.

- In RCA_0, we can show the following.
  - If a complete theory has an atomic model then it is atomic.
  - If a structure is prime then it is atomic.

Theorem (Csima, Hirschfeldt, Shore). TFAE

1. ACA_0.
2. Atomic-Prime Equivalence.
3. Atomic Model Uniqueness.
4. Prime Model Theorem.

- Prime Model Uniqueness follows from ACA_0, but otherwise its status is open.

- The Atomic Model Theorem is an interesting case.

• Want to show that Atomic-Prime Equivalence, Atomic Model Uniqueness, and the Prime Model Theorem all imply ACA_0.

• Build a complete decidable theory T with decidable atomic models $\mathcal{A}$ and $\mathcal{B}$ s.t. if $\mathcal{C} \preccurlyeq \mathcal{A}, \mathcal{B}$ then we can get $\emptyset'$ from $\mathcal{C}$ and the embeddings.

• Work in the language with unary predicates R_n and $Q_{n,s}$ for $n, s \in \omega$.

• T is specified by the following axioms (with quantifier elimination and completeness being straightforward to show).

Axioms for T:

- The R_n define infinite, pairwise disjoint sets.
- Each $Q_{n,s}$ defines a subset of R_n.
- $Q_{n,s}(x) \to Q_{n,s+1}(x)$.
- If $n \notin \emptyset'_s$ then $\neg Q_{n,s}(x)$.
- If n enters $\emptyset'$ at s then

$$\exists^\infty x\, Q_{n,s}(x) \quad \wedge \quad \exists^\infty x\, \neg Q_{n,s}(x).$$

- If $n \in \emptyset'_s$ then $\neg Q_{n,s}(x) \to \neg Q_{n,s+1}(x)$.

- Now build decidable atomic models $\mathcal{A}$ and $\mathcal{B}$ s.t. if $\mathcal{C} \preccurlyeq \mathcal{A}, \mathcal{B}$ then we can get $\emptyset'$ from $\mathcal{C}$ and the embeddings.

Atomic Model Theorem. A complete theory has an atomic model iff it is atomic.

- The "only if" direction follows from RCA_0.

- $\mathsf{ACA}_0 \vdash \mathsf{AMT}$.

Theorem (Csima, Hirschfeldt, Knight, Soare). Let $A \leqslant_{\mathrm{T}} \emptyset'$. TFAE

1. A is nonlow$_2$.
2. Any complete decidable atomic theory has an A-decidable atomic model.

Corollary. $\mathsf{WKL}_0 \nvdash \mathsf{AMT}$.

Proof. There is a low degree bounding a model of WKL_0.

• Let T be a complete decidable atomic theory.

• By combining the proofs of the previous theorem and the existence of a low atomic model of T, we can get a low atomic model of T below any given nonlow$_2$ degree $\mathbf{a} \leqslant_{\mathrm{T}} \mathbf{0}'$.

• By iterating, we can build an ω-model of AMT below any given nonlow$_2$ degree $\mathbf{a} \leqslant_{\mathrm{T}} \mathbf{0}'$.

• Take **a** to be c.e.

• Then **a** does not bound a model of WKL_0.

• Thus RCA_0+ AMT $\nvdash$ WKL_0.

• Also, RCA_0+ AMT $\nvdash$ RT^2_2 (or any of several principles provable in RT^2_2, such as SRT^2_2, COH, CAC, etc.).

• Most of these nonimplication results also follow from a general conservativity result.

Theorem (Csima, Hirschfeldt, Shore). AMT is conservative over RCA_0 for sentences of the form

$$\forall X\, (P(X) \to \exists Y\, \forall n\, R(X, Y, n)),$$

where P is arithmetic and R is computable.

• By a result of Hirschfeldt and Shore, the same is true of AMT + COH.

• Note that statements such as WKL and RT^2_2 can be written in the above form.

Question. Does $\mathrm{RCA}_0 + \mathrm{RT}^2_2 \vdash \mathrm{AMT}$? What if we replace RT^2_2 by weaker principles?

• A structure $\mathcal{A}$ is *homogeneous* if

$$\forall \overline{a}, \overline{b} \in \mathcal{A}\ [(\mathcal{A}, \overline{a}) \equiv (\mathcal{A}, \overline{b}) \rightarrow (\mathcal{A}, \overline{a}) \cong (\mathcal{A}, \overline{b})].$$

• Atomic and saturated structures are homogeneous.

• If $\mathcal{A}$ and $\mathcal{B}$ are homogeneous and realize the same types then $\mathcal{A} \cong \mathcal{B}$.

• Every theory has a homogeneous model.

Theorem (Goncharov). There is a complete decidable theory with no decidable homogeneous model.

• A collection of types is the type spectrum of a homogeneous model iff it is closed under certain basic operations, type-type amalgamation, and type-formula amalgamation.

• A listing of types $\mathcal{L}$ has the *effective extension property* if type-formula amalgamation can be performed effectively within $\mathcal{L}$.

Theorem (Goncharov; Peretyat'kin). Let $\mathcal{A}$ be homogeneous. Then $\mathcal{A}$ has a decidable copy iff there is a listing of its type spectrum with the effective extension property.

• Let X have PA-degree.

• That is, there is an X-computable nonstandard model of Peano Arithmetic.

• Equivalently, there is an X-computable Scott set, i.e., a Turing ideal $\mathcal{S}$ s.t. every infinite binary tree in $\mathcal{S}$ has a path in $\mathcal{S}$.

• $\mathcal{S}$ can be chosen so that there are partial X-computable functions witnessing the fact that it is a Scott set.

• Let T be a complete decidable theory.

• The collection of types of T coded by sets in $\mathcal{S}$ can be shown to satisfy the requirements for being the type spectrum of a homogeneous model, and to have the X-effective extension property.

• Thus every complete decidable theory has an X-decidable homogeneous model.

Theorem (Csima, Harizanov, Hirschfeldt, Soare). There is a complete decidable theory T s.t. every homogeneous model of T has PA-degree.

Corollary. TFAE

1. X has PA-degree.
2. Every complete decidable theory has an X-decidable homogeneous model.