Some properties of c.e. reals in the *sw-degrees*

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July 25, 2005

Introduction and definitions

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Definition

A set A is nearly computably enumerable if there is a computable approximation $\{A_s\}_{s \in w}$ such that $A(x) = \lim_s A_s(x)$ for all x and $A_s(x) > A_{s+1}(x) \Rightarrow \exists y < x(A_s(y) < A_{s+1}(y))$.

Definition

A real α is computably enumerable (c.e) if $\alpha = 0.\chi_A$ where A is a nearly c.e. set. A real α is strongly computably enumerable (strongly c.e.) if $\alpha = 0.\chi_A$ where A is a c.e. set.

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Definition

Let $A, B \subseteq N$. We say that B is strongly weak truth table reducible (*sw-reducible*) to A, and write $B \leq_{sw} A$, if there is a Turning reduction Γ such that $B = \Gamma^A$ and the use $\gamma(x) \leq x + c$ for some constant c. For reals $\alpha = 0.\chi_A$ and $\beta = 0.\chi_B$, we say that β is sw-reducible to α, and write $\beta \leq_{sw} \alpha$ if $B \leq_{sw} A$.

The *sw* degrees have a number of nice aspects

For instance, Downey, Hirschfeldt, and Nies proved *sw*-reducibility satisfies Solovay property and

Theorem (Downey, Hirschfeldt, Laforte)

Let α *and* β *be c.e. reals such that* $\liminf_n H(\alpha \upharpoonright n) - H(\beta \upharpoonright n) = \infty$*. Then* $\beta \leq_{sw} \alpha$*.*

Furthermore if α is a c.e. real which is noncomputable, then there is a noncomputable strongly c.e. real $\beta \leq_{sw} \alpha$, and this is not true in general, for $\leq_S$.

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Theorem

If β is strongly c.e. and α is c.e. then $\alpha \leq_{sw} \beta$ implies $\alpha \leq_{S} \beta$.

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the structure of c.e. reals in the *sw-degrees*

However, we still are interested in *sw*-reducibility since it has some nice properties and it is helpful for studying Turing-degrees by exploring the *sw*-degrees. Further, we may study the structure of c.e. reals in the *sw-degrees*.

Definition

Let A be a nearly c.e. set. The sw-canonical c.e. set A^* associated with A is defined as follows. Begin with $A_0^* = \emptyset$. For all x and s, if either $x \notin A_s$ and $x \in A_{s+1}$, or $x \in A_s$ and $x \notin A_{s+1}$, then for the least j with $< x, j > \notin A_s^*$, put $< x, j >$ into A_{s+1}^*.

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Theorem (Downey, Hirschfeldt, Laforte)

If A is nearly c.e. and noncomputable then there is a noncomputable c.e. set $A^* \leq_{sw} A$. Hence there are no minimal sw-degrees of c.e. reals.

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Theorem (Downey, Hirschfeldt, Laforte)

There exist nearly c.e. sets A and B such that for all nearly c.e. $W \geq_{sw} A, B$ there is a neraly c.e. Q with $A, B \leq_{sw} Q$ but $W \not\leq_{sw} Q$. Thus the sw-degrees of c.e. reals do not form an uppersemilattatice.

Yu Liang and Ding Decheng pointed out that we can not characterize randomness by *sw*-reducibility by proving that there is no a largest c.e. *sw*-degree.

Theorem (Yu and Ding)

There is no sw-complete c.e. real. Even more, there is a pair of c.e. reals for which there is no c.e. real above both of them respect to sw-reducibility.

Theorem (Fan and Lu)

Let $\{\alpha_e\}_{e\in\omega}$ be an effective enumeration of strongly c.e. reals. Then there are strongly c.e. reals β_0, β_1 such that $\beta_0 \not\leq_{sw} \alpha_e$ or $\beta_1 \not\leq_{sw} \alpha_e$ for every α_e.

Proof

$$R_{e,i} : \Phi_i^{\alpha_e} \neq \beta_0 \vee \Psi_i^{\alpha_e} \neq \beta_1,$$

where $\phi_i(x) \leq x + i$ and $\psi_i(x) \leq x + i$.

Pick a large number $k_{e,i}$ for $R_{e,i}$ such that $k_{e,i} > e, i$ and $k_{e,i} > 3k_{e',i'}$ for all $e' < e$ or $e = e', i' < i$.
We only put numbers between $k_{e,i}$ and $3k_{e,i}$ into B or C for $R_{e,i}$.

our results of c.e. reals in the sw-degrees

Theorem (Fan and Lu)

Let $\{\alpha_e\}_{e\in\omega}$ be an effective enumeration of strongly c.e. reals. Then there is a c.e. real β such that $\alpha_e \leq_{sw} \beta$ for every α_e.

Proof

$$R_e : \Gamma_e^{\beta} = \alpha_e,$$

where Γ_e is defined by us such that $\gamma_e(x) \leq x + e + 3$.

1) Check whether there exist some R_e such that $\alpha_e(x)$ changes.
2) Choose the least $e \leq s$ such that
$\exists(x \leq s)[\alpha_{e,s+1}(x) \neq \alpha_{e,s}(x)]\}$.
3) Set $\beta_{s+1} = \beta_s \upharpoonright (x + e + 3) + 2^{-(x+e+3)}$.

Definition (Yu)

A c.e.real α is sw-cuppable if there is a c.e. real β such that there is no c.e. real above both of them respect to sw-reducibility.

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Theorem (Yu)

There exists a sw-cuppable c.e. real.

Theorem (Fan and Lu)

For any c.e. real α, there exists a c.e. real β such that β is sw-cuppable and $\alpha \leq_{sw} \beta$.

Theorem (Fan and Lu)

Let $\{\alpha_e\}_{e\in\omega}$ be an effective enumeration of c.e. reals. Then there is a strongly c.e. real β_0 and a c.e. real β_1 such that $\beta_0 \not\leq_{sw} \alpha_e$ or $\beta_1 \not\leq_{sw} \alpha_e$ for every e.

Proof

$$R_e : \Phi_e^{\alpha_e} \neq \beta_0 \vee \Psi_e^{\alpha_e} \neq \beta_1,$$

For simplicity, we assume that $\phi^{\alpha}(x) = x$ and $\phi^{\alpha}(x) = x$.

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Proof

$$R_e : \Phi_e^{\alpha_e} \neq \beta_0 \vee \Psi_e^{\alpha_e} \neq \beta_1,$$

Lemma

Given (n, k), there is a strongly c.e. real β_0, a c.e. real β_1 and l such that there exists a function $\Gamma : [0, 1) \times [0, 1) \rightarrow R$ satisfies $\Gamma(\beta_0 \upharpoonright l, \beta_1 \upharpoonright l) \geq n$ and $\beta_0 \upharpoonright k = 0$. Moreover, β_0, β_1 and l can be computed uniformly from (n, k).

For simplicity, we assume that $\phi^{\alpha}(x) = x$ and $\phi^{\alpha}(x) = x$.

The proof of the lemma is divided into two cases: (1) the induction on n; (2) the induction on k.
Now consider the case for the induction on n. Fixed n, assume that $\Gamma(\beta_{0,i,k} \upharpoonright l_{i,k}, \beta_{1,i,k} \upharpoonright l_{i,k}) \geq i$ and $\beta_{0,i,k} \upharpoonright k = 0$ for every $i \leq n, k \in N$. Let $l_{n+1,0}$ be equal to $(l_{n,l_{n,0}} + 1)$.
Step 1. Imitate our programme for putting numbers into $A_{n,l_{n,0}} \upharpoonright l_{n,l_{n,0}}, B_{n,l_{n,0}} \upharpoonright l_{n,l_{n,0}}$, and do the similar action on the natural number between 2 and $l_{n+1,0}$.
Note that $\Gamma(\beta_{0,n,l_{n,0}} \upharpoonright l_{n,l_{n,0}}, \beta_{1,n,l_{n,0}} \upharpoonright l_{n,l_{n,0}}) \geq n$, $\beta_{0,n,l_{n,0}} \upharpoonright l_{n,0} = 0$.
It must be $\Gamma(\beta_{0,n,l_{n,0}} \upharpoonright l_{n,l_{n,0}}, \beta_{1,n,l_{n,0}} \upharpoonright l_{n,l_{n,0}}) = n$ and $\beta_{0,n,l_{n,0}} \upharpoonright l_{n,0} = 0$ at some stage t. Hence, at stage t, $\Gamma_t(\beta_{0,n+1,0,t} \upharpoonright l_{n,l_{n,0}}, \beta_{1,n+1,0,t} \upharpoonright l_{n,l_{n,0}}) = n/2$, and $A_{n+1,0,t} \upharpoonright l_{n,0} + 1 = 0$, $B_{n+1,0,t} \upharpoonright 1 = 0$.

Step 2. At stage $t+1$, let $A_{n+1,0}$ active and $B_{n+1,0}$ waiting, set $A_{n+1,0,t+1}(1)=1$, which forces $\Gamma_{t+1}(\beta_{0,n+1,0,t+1} \upharpoonright l_{n,l_{n,0}}, \beta_{1,n+1,0,t+1} \upharpoonright l_{n,l_{n,0}})$ equal to $n/2+1/2$. At stage $t+2$, let $A_{n+1,0}$ be waiting and $B_{n+1,0}$ active, set $B_{n+1,0,t+2}(1)=1$, $B_{n+1,0,t+2}(q)=0$ $(q>1)$, which forces $\Gamma_{t+2}(\beta_{0,n+1,0,t+2} \upharpoonright l_{n+1,0}, \beta_{1,n+1,0,t+2} \upharpoonright l_{n+1,0}) = n/2+1$.
Step 3. Imitate the programme of the changes of $A_{n,0} \upharpoonright l_{n,0}, B_{n,0} \upharpoonright l_{n,0}$. Note that $A_{n+1,0,t+2}(x) = B_{n+1,0,t+2}(x) = 0$ $(2 \leq x \leq l_{n,0}+1)$, do the following similar actions. Imitate the programme
Note that $\Gamma(\beta_{0,n,0} \upharpoonright l_{n,0}, \beta_{1,n,0} \upharpoonright l_{n,0}) \geq n$. The effect of the changes on $[2, l_{n,0}]$ of $A_{n+1,0}$ and $B_{n+1,0}$ induces $\Gamma(\beta_{0,n+1,0} \upharpoonright l_{n+1,0}, \beta_{1,n+1,0} \upharpoonright l_{n+1,0}) \geq n+1$.

Next consider the case for the induction on k. Fix (n, k), assume that $\Gamma(\beta_{0,i,j} \upharpoonright l_{i,j}, \beta_{1,i,j} \upharpoonright l_{i,j}) \geq i$ and $\beta_{0,i,j} \upharpoonright j = 0$ for every $i \leq n$ or $j \leq k$. We can win by controlling $A_{n,k+1} \upharpoonright l_{n,k+1}, B_{n,k+1} \upharpoonright l_{n,k+1}$ as follows. Let $l_{n,k+1}$ be equal to $l_{n-1,l_{n,k}} + 1$.

Step 1. Imitate the programme of the changes of $A_{n-1,l_{n,k}} \upharpoonright l_{n-1,l_{n,k}}, B_{n-1,l_{n,k}} \upharpoonright l_{n-1,l_{n,k}}$.
Note that $\Gamma(\beta_{0,n-1,l_{n,k}} \upharpoonright l_{n-1,l_{n,k}}, \beta_{1,n-1,l_{n,k}} \upharpoonright l_{n-1,l_{n,k}}) \geq n-1$, $\beta_{0,n-1,l_{n,k}} \upharpoonright l_{n,k} = 0$. It must be $\Gamma(\beta_{0,n-1,l_{n,k}} \upharpoonright l_{n-1,l_{n,k}}, \beta_{1,n-1,l_{n,k}} \upharpoonright l_{n-1,l_{n,k}}) = n+1$, $\beta_{0,n-1,l_{n,k}} \upharpoonright l_{n,k} = 0$ at some stage t. Hence, at stage t, $\Gamma_t(\beta_{0,n,k+1,t}, \beta_{1,n,k+1,t}) = (n-1)/2$, and $A_{n,k+1,t} \upharpoonright l_{n,k,t} + 1 = 0$.

Step 2. At stage $t+1$, let $A_{n,k+1}$ waiting and $B_{n,k+1}$ active, set $B_{n,k+1,t+2}(1)=1$, $B_{n,k+1,t+2}(q)=0$ $(q>1)$, which forces $\Gamma_{t+1}(\beta_{0,n,k+1,t+1}\upharpoonright l_{n,k+1}, \beta_{1,n,k+1,t+1}\upharpoonright l_{n,k+1})=n/2$.
Step 3. Imitate the programme of the changes of $A_{n,k}\upharpoonright l_{n,k}, B_{n,k}\upharpoonright l_{n,k}$. Here $A_{n,k+1,t+1}(x)=B_{n,k+1,t+1}(x)=0$ $(2\leq x\leq l_{n,k}+1)$.
Note that $\Gamma(\beta_{0,n,k}\upharpoonright l_{n,k}, \beta_{1,n,k}\upharpoonright l_{n,k})\geq(n-1)$, $\beta_{0,n,k}\upharpoonright k=0$.
The effect of the changes on $[2, l_{n,k}+1]$ of $A_{n,k+1}$ and $B_{n,k+1}$ induces $\Gamma(\beta_{0,n,k+1}\upharpoonright l_{n,k+1}, \beta_{1,n,k+1}\upharpoonright l_{n,k+1})\geq n$ and $\beta_{0,n,k+1}\upharpoonright(k+1)=0$.
Since the construction is effective, $\beta_{0,n,k}$ is strongly c.e. and $\beta_{1,n,k+1}$ is c.e. for every n,k.

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Theorem

For any noncomputable c.e. real α, there exist a c.e. real β such that $\beta \not\leq_{sw} \alpha$ and $\alpha \leq_T \beta$.

The proof of the theorem

It suffices to build a c.e. real α to meet the following requirements:

$$R_{<e,n>} : \alpha = \Phi_e^{\beta_e} \Rightarrow \exists \Gamma (\Gamma^{\alpha} = \beta_e)$$

where each $\{\Phi_e, \beta_e\}_{e \in \omega}$ is an enumeration of sw-procedures and c.e. reals with use $\phi_e(x) \leq x + n (n \in \omega)$.

Without loss of generality, suppose that β_e is less than 0.1.

The special programm of the theorem

For any Φ_e, our aim is to make $\alpha \neq \Phi_e^{\beta_e}$ or to define a function Γ such that $\Gamma^\alpha = \beta_e$.

Assume that when we put some number ($\leq l(e,n)$) into α at expansionary stage, $\beta \lceil (\phi_e(x)+1)$ changes at the greatest position, i.e. the change is the slowest.

We assume that in digital expansion of β, there are infinite 1.

The strategy for $n = 0$

1. Wait for an expansionary stage when the first 1 appears in digital expansion of β, say at the position $m < l(e, n)$.
Then we let $\alpha(m-1)$ change to 1.
2. Wait for next expansionary stage and once we find it, $\beta(m_1)$ must change to 1. Then we let $\alpha(m-2)$ change to 1 and wait for next expansionary stage.
Repeating the above strategy until β have to be ready to change at position 1, by our assumption, this is impossible. Hence we win.

The strategy for $n = 1$

1. Wait for an expansionary stage when the first 1 appears in digital expansion of β, say at the position $m < l(e, n)$.
Then we let $\alpha(m-1)$ change to 1.
2. Wait for next expansionary stage and once we find it, $\beta(m-1)$ must change. We let $\alpha(m-2)$ change to 1 and wait for next expansionary stage.
Repeating the above strategy until β have to be ready to change at position 1, by our assumption, this is impossible. Hence we win.

The strategy for $n = 2$

β	100	100	100
α	100	100	100

$\Rightarrow$

β	100	100	110
α	100	101	000

$\Rightarrow$

β	100	101	0
α	100	101	1

$\Rightarrow$

β	100	110	0
α	100	111	0

$\Rightarrow$

β	101	000	0
α	101	000	0

Similarly we can get

β	11	000	0
α	11	000	0

$\Rightarrow$

β	100	000	0
α	100	000	0

The strategy for $n = 2$

β	000	100	100
α	100	100	100

$\Rightarrow$

β	000	100	110
α	100	101	000

$\Rightarrow$

β	000	101	0
α	100	101	1

$\Rightarrow$

β	000	110	0
α	100	111	0

$\Rightarrow$

β	001	000	0
α	101	000	0

Similarly we can get

β	01	000	0
α	11	000	0

$\Rightarrow$

β	01	000	0
α	10	000	0

Note that

β	10	000	0
α	11	000	0

can move left forever by using 1 with lifting 1 if it only meet 1 with lifting 1. We call such case 11.

Using 11, the above case can change to

β	100	000	0
α	111	000	0

$\Rightarrow$

β	0110	000	0
α	1000	000	0

$\Rightarrow$

β	100	000	0
α	101	000	0

$\Rightarrow$

β	100	100	0
α	101	110	0

$\Rightarrow$

β	101	000	0
α	110	000	0

$\Rightarrow$

β	1100	000	0
α	1111	000	0

$\Rightarrow$

β	10000	000	0
α	10000	000	0

The strategy for n

1. Wait for an expansionary stage, say s_0 when in β, the number of 1 is $\geq 2^{n-1}+1$. Suppose that the position of the last 1 in β is t_0.
2.Creating a situation such that we can apply $(n-1)$-strategy from next expansionary stage.
a) then add 1 to the position t_0-1 to the α to force β change at t_0-1+n.
b) Waiting for next expansionary stage when in β, there appear a new 1, say at position t_1, then add 1 to the position t_1-1 to the α to force β change at t_1-1+n..
c)repeating until we can get more than $2^{n-2}+1$ new good 1 with lifting $n-1$ position after stage s_0.
Now the numbers on α and β are:

β	$010\cdots01$	$0\cdots0$	$010\cdots01$	$\cdots$	$010\cdots01$
α	$100\cdots00$	$0\cdots0$	$100\cdots00$	$\cdots$	$100\cdots00$

to find the first fixed pair

β	0100 $\cdots$ 01	0 $\cdots$ 00
α	1000 $\cdots$ 00	0 $\cdots$ 0

$\Rightarrow$

β	011 $\cdots$ 001	0 $\cdots$ 00
α	110 $\cdots$ 000	0 $\cdots$ 0

$\Rightarrow$

β	011 $\cdots$ 111	0 $\cdots$ 00
α	111 $\cdots$ 100	0 $\cdots$ 00

$\Rightarrow$

β	100 $\cdots$ 000	0 $\cdots$ 00
α	111 $\cdots$ 110	0 $\cdots$ 00

This is called the first fixed pair. Note that it corresponds to

β	0100 $\cdots$ 01	0 $\cdots$ 00
α	1000 $\cdots$ 00	0 $\cdots$ 00

to find the ability of the first fixed pair

when this first fixed point meet a block, we will prove that it can takeover the block, i.e.,

β	0	$100\cdots00$	$0\cdots00$
α	1	$111\cdots11$	$0\cdots00$

injure it, it changes to

β	00	$10\cdots01$	$0\cdots00$
α	10	$00\cdots00$	$0\cdots00$

$\Rightarrow$

β	01	$10\cdots00$	$0\cdots00$
α	11	$11\cdots10$	$0\cdots00$

$\Rightarrow$

β	00	$110\cdots01$	$0\cdots00$
α	10	$000\cdots00$	$0\cdots00$

$\Rightarrow$

β	01	$1100\cdots00$	$0\cdots00$
α	11	$1111\cdots10$	$0\cdots00$

$\Rightarrow$

β	1	$000\cdots00$	$0\cdots00$
α	1	$111\cdots11$	$0\cdots00$

look for the last fixed pair

the last fixed pair corresponds to

β	0	$1\cdots1$	$0\cdots00$
α	1	$0\cdots0$	$0\cdots00$

By induction suppose that this is

β	$100\cdots00$	$0\cdots00$
α	a	$0\cdots00$

when this last fixed pair meet a lifting n-number, i.e.,

β	1	$100\cdots00$	$0\cdots00$
α	1	a	$0\cdots00$

applying this last fixed pair,

β	1	$11\cdots11$	p	$0\cdots00$
α	1	$11\cdots11$	q	$0\cdots00$

injure it

β	1	$00\cdots00$	$0\cdots00$
α	1	$000\cdots0$	$0\cdots00$

Note that there are 2^{n-1} fixed pairs.

leads to a contradiction

6) Applying the above result repeatedly. Then we can force β bigger enough. If we can applying it infinitely, then we can prove that $\beta \geq 0.1$, which is a contradiction. That is,

β	0.0 1	$11\cdots11$	$0\cdots00$
α	0. 01	$11\cdots11$	$0\cdots00$

then we can get

β	0.1 0	$00\cdots00$	$0\cdots00$
α	0.10	$00\cdots00$	$0\cdots00$

The single strategy for requirement $R_{e,n}$

1. Wait for the first expansionary stage, say s_1. Then compute $\Psi(e, H, s_1)$.

(1) If $\Psi(e, H, s_1) = 1$, then we do nothing and go to next σ.

(2) If $\Psi(e, H, s_1) = 0$, then from this stage we wait for a stage such that either $\Psi(e, H) = 1$ or there are $2^{n-1} + 1 + 2^{n-2} + 1 + \cdots + 2^{2-1} + 1$ times new 1 appear in β.

2. If later at the next expansionary stage after s_1 there are $2^{n-1}+1+2^{n-2}+1+\cdots+2^{2-1}+1$ times new 1 appear in β, then define (or redefine) $\Gamma^{\alpha}(x)=\beta_e(x)$ for any $x \leq l(e,n)$ with use $\gamma(x)=x+C$.

From this stage, at every expansionary stage, we should define and redefine $\Gamma^{\alpha}=\beta_e$. That is, if we find that $\beta(x)$ change to be 1 at some position x some expansionary stage and Γ^{α} do not know, then we put some number $\leq \gamma(x)$ into α, and initialise all strategies with lower priority.

Since we have got the prepared data, we can make a disagreement.

The strategies for two requirements R_0, R_1

Suppose that σ_0 and σ_1 work on R_0-strategy and R_1-strategy respectively. And $\sigma_0 \subseteq \sigma_1$.

The R_1-strategy is:

1. Wait for the first expansionary stage, say s_1. Then compute $\Psi(1, H, s_1)$.
(1) If $\Psi(1, H, s_1) = 1$, then we do nothing and go to next σ.

(2) If $\Psi(1, H, s_1) = 0$, then from this stage we wait for a stage such that either $\Psi(e, H) = 1$ or there are $2^{n_1+r-1} + 1 + 2^{n_1+r-2} + 1 + \cdots + 2^{2-1} + 1$ times new 1 appear in β_1.

2. If at the next expansionary stage, say s_2 there are $2^{n+r-1}+1+2^{n+r-2}+1+\cdots+2^{2-1}+1$ times new 1 appear in β, then define (or redefine) $\Gamma^\alpha(x)=\beta_e(x)$. From s_2, at every expansionary stage, we should define and redefine $\Gamma^\alpha=\beta_1$.

Let $x_0=\min\{\gamma_0(y)|\ \gamma_0(y)$ wants to enter into $\alpha\}$.
$x_1=\min\{\gamma_1(y)|\ \gamma_1(y)$ wants to enter into $\alpha\}$.

If $x_0<x_1$, then use the programme given above to make disagreement and initialise R_1.

If $x_0\geq x_1$, then put x_0 into α, apply the programme given above from the position x_0. If it make a disagreement, then we win. If not, then x_1 will eventually be put into α.
Note that we delay putting x_1 into α here.