Decidability and Definability in the Σ^0_2-Enumeration Degrees

My First Beamer Talk

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Enumeration Degrees

- Introduced by Friedberg and Rogers in 1959.
- A is enumeration reducible to B ($A \leq_e B$) if we can enumerate A given any enumeration of B.

Definition

$A \leq_e B$ iff there is c.e. set Φ such that
$A = \Phi^B = \{x : \exists \langle x, P \rangle \in \Phi \ (P \text{ finite and } P \subseteq B)\}$

- Similar to Turing reducibility:
  $A \leq_T B$ iff for $\forall x \ A(x) = \Phi^B(x) = y$
  And $\Phi^B(x) = y$ iff $\exists \langle x, y, P, N \rangle \in \Phi \ (P \subseteq B \text{ and } N \subseteq \overline{B})$.

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- Analysis of Types in Effective Model Theory.
- Existentially Closed Groups in Computable Group Theory.
- Computable Analysis.
- And much, much more!!!

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Enumeration Degrees: Global Structure

Notation

$\mathfrak{D}_e$ is the set of all e-degrees.
$\mathfrak{D}_T$ is the set of all Turing-degrees.

Fact

Minimal element is $\mathbf{0}_e$ = the set of all c.e. sets.

Definition

$\deg_e(A) \vee \deg_e(B) =_{\text{def}} \deg_e(A \oplus B)$

Theorem (Case, 1971)

Every countable ideal in $\mathfrak{D}_e$ has an exact pair.

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$\mathfrak{D}_T$ as a Substructure of $\mathfrak{D}_e$.

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range(ι) = TOT = the set of total enumeration degrees.

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Facts about $\mathfrak{D}_e$

Theorem (Gutteridge, 1971)

$\mathfrak{D}_e$ has no minimal elements.

Definition

A degree $\mathbf{a}$ is quasi-minimal if $\mathbf{a} > \mathbf{0}_e$ and for all $\mathbf{b} \in$TOT, $\mathbf{b} \leq \mathbf{a}$ implies $\mathbf{b} = \mathbf{0}_e$.

Theorem (Cooper, 1989)

$\mathfrak{D}_e$ is not dense.

Theorem (Calhoun, Slaman, 1996)

There exist Π_2^0-degrees $\mathbf{a}, \mathbf{b}$ such that $\mathbf{a} < \mathbf{b}$ and $(\mathbf{a}, \mathbf{b}) = \emptyset$.

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Σ_2^0-Enumeration Degrees

Defining the Jump

Definition

We define the enumeration jump as follows:

- $K_A =_{\text{def}} \{x : x \in \Phi_x^A\}$
- $J(A) = K_A \oplus \overline{K_A}$
- $\mathbf{0}'_e = \deg_e(J(\emptyset)) = \deg_e(\overline{K})$

Theorem (Cooper, 1984)

A is Σ_2^0 iff $A \leq_e J(\emptyset)$.

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The Σ_2^0-enumeration degrees are dense.

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Undecidability of Complete Theories

Given a theory, we can ask the following questions:

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Is the theory undecidable?

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What is the n, if any, s.t. the Π_n-theory is decidable and the Π_{n+1}-theory is undecidable?

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Theorem

The Π_1-theory of Σ^0_2-enumeration degrees decidable (in the language of $\{\leq\}$).

Proof.

- Σ_1 sentence ψ is of the form $\exists \overline{x} \varphi(\overline{x})$.
- $\varphi(\overline{x})$ describes an ordering on $\overline{x}$.
- Can embed any finite p.o. in the Σ^0_2-enumeration degrees.
- If $\varphi(\overline{x})$ describes a p.o., ψ is true, otherwise $\neg\psi$ is true.

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Sets A and B are *computably inseparable* if there is no computable set C with $A \subseteq C$ and $B \cap C = \emptyset$.

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A set S of first order sentences is *Hereditarily Undecidable* if $\overline{S}$ and $S \cap V$ are computably inseparable (V all valid sentences in language of S).

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Elementary Definability

Given classes of structures $\mathcal{C}$, $\mathcal{D}$ in relational languages $\mathcal{L}_\mathcal{C}$ and $\mathcal{L}_\mathcal{D}$, without equality, we make the following definitions:

Definition

1. A Σ_k-scheme $s(\overline{p})$ for $\mathcal{L}_\mathcal{C}$, in $\mathcal{L}_\mathcal{D}$, consists of Σ_k-formulas in $\mathcal{L}_\mathcal{D}$ that code the universe, each relation $R \in \mathcal{L}_\mathcal{C}$ and the negation of R.
2. $\alpha(\overline{p})$ is a Π_{k+1}-correctness condition for s if it codes #1 in $\mathcal{L}_D$.
3. $\mathcal{C}$ is Σ_k-elementary definable w/ parameters in $\mathcal{D}$ if $\exists$ Σ_k-scheme s s.t. for all $C \in \mathcal{C}$ there is $D \in \mathcal{D}$ and $\overline{p} \in D$ s.t.

   (a) $D \models \alpha(\overline{p})$
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2. $\alpha(\overline{p})$ is a Π_{k+1}-correctness condition for s if it codes #1 in $\mathcal{L}_D$.
3. $\mathcal{C}$ is Σ_k-elementary definable w/ parameters in $\mathcal{D}$ if $\exists$ Σ_k-scheme s s.t. for all $C \in \mathcal{C}$ there is $D \in \mathcal{D}$ and $\overline{p} \in D$ s.t.

    (a) $D \models \alpha(\overline{p})$
    (b) $C \cong \tilde{C}$ structure in D defined by s and $\overline{p}$.

How to Show Theory Fragment is Undecidable

Lemma (Nies Transfer Lemma, 1996)

(For $k \geq 1$, $r \geq 2$.)
If $\mathcal{C}$ is Σ_k-elementary definable with parameters in $\mathcal{D}$ and the Π_{r+1}-theory of $\mathcal{C}$ hereditarily undecidable then the Π_{r+k}-theory of $\mathcal{D}$ hereditarily undecidable.

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Theory of the Σ^0_2-enumeration degrees is Undecidable

Theorem (Slaman, Woodin, 1997)

The First order theory of the Σ^0_2-enumeration degrees in language $\{\leq\}$ is undecidable.

Proof.

- Finite Graphs are Σ_2-elementary definable in Σ^0_2-enumeration degrees with parameters.
- The Π_4-theory of Finite Graphs is hereditarily undecidable.
- Nies Transfer Lemma tells us that $\Pi_{3+2} = \Pi_5$-theory is hereditarily undecidable.

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Π_3-Theory of Σ^0_2-enumeration degrees is Undecidable

To show this, need theory that is:

- Π_3-hereditarily undecidable and
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Theorem (Nies, 1996)

The Σ_2- (and hence Π_3-) theory of the finite bipartite graphs with nonempty left and right domains in the language of one binary relation, but without equality, is hereditarily undecidable.

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(Ahmad Pair) There exists Σ^0_2-enumeration degrees $\mathbf{a}, \mathbf{b}$ such that $\mathbf{a} \not\leq \mathbf{b}$ but $\forall \mathbf{x} < \mathbf{a}, \mathbf{x} \leq \mathbf{b}$.

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Coding the Universe II

Fix Domains $\mathcal{L} = \{0, \ldots, n\}$ and $\mathcal{R} = \left\{\tilde{0}, \ldots, \tilde{m}\right\}$.
Build $\mathbf{a}$, $\mathbf{b}$, $\mathbf{a}_0$, $\ldots$, $\mathbf{a}_n$ such that:

1. $\mathbf{a} \not\leq \mathbf{b}$ and $\mathbf{a}_i \not\leq \mathbf{b}$
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$\mathbf{a}_0$, $\ldots$, $\mathbf{a}_n$ specified by $\mathbf{a}$ and $\mathbf{b}$ (Π_1-formula):

$$\varphi(x) = x < a \;\wedge\; x \not\leq b \;\wedge\; \forall y \leq x\,(y \not\leq b \rightarrow y = x)$$

- Not good enough, we need a Σ_1-formula.

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Build degree **c** such that:

1. $\mathbf{c} \not\leq \mathbf{a}_i$
2. $\mathbf{c} \leq \mathbf{a}_i \vee \mathbf{a}_j$, $i \neq j$.

Represent $i \in \mathcal{L}$ by $[\mathbf{a}_i, \mathbf{a}) - [\mathbf{c}, \mathbf{0}'_e]$ (Δ_0-formula):

$$\varphi(x) \; = \; x < a \;\; \wedge \;\; x \not\leq b \;\; \wedge \;\; x \not\geq c$$

Remark

Each element of each domain is now represented by an equivalence class of degrees.

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Coding the Edge Relationship

Construct degrees $\mathbf{e}_0$, $\mathbf{e}_1$ s.t for $i \in \mathcal{L}$, $\tilde{\imath} \in \mathcal{R}$:

1. $E(i, \tilde{\imath})$ iff $\mathbf{e}_0 \leq \mathbf{a}_i \vee \tilde{\mathbf{a}}_{\tilde{\imath}}$ iff $\mathbf{e}_1 \not\leq \mathbf{a}_i \vee \tilde{\mathbf{a}}_{\tilde{\imath}}$.
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Allows Σ_1-definition of edge relationship:

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The Π_3-theory of Σ^0_2-enumeration degrees is undecidable.

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Corollary

The Π_3-theory of Σ_2^0-enumeration degrees is undecidable.

Question

Can we do better?

Conclusion:

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Can we do better?

A new requirement

Lemma (Cooper, McEvoy 1985)

A set A is low if and only if $B \leq_e A$ implies that B is Δ^0_2.

Modify the construction so that

1. $\mathbf{a} \vee \tilde{\mathbf{a}}$ is low.

We now have that

- Parameters $\mathbf{a}, \mathbf{a}_0, \ldots, \mathbf{a}_n, \mathbf{c}, \mathbf{e}_0, \mathbf{e}_1$ are now low (Δ^0_2).
- Parameter $\mathbf{b}$ is Δ^0_2 (but cannot be low) (Ahmad, 1998).

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Open Questions

1. Is the Π_2-theory of the Σ_2^0-enumeration degrees decidable?
2. Is the Π_2-theory of the Δ_2^0-enumeration degrees decidable?
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4. What about decidability in other languages?

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The Π_2-theory of the Σ^0_2-enumeration degrees

We can translate a Π_2-sentence to the question:

- Given partial orders $\mathcal{P}$, $\mathcal{Q}_0, \ldots, \mathcal{Q}_n$, with $\mathcal{P} \subseteq \mathcal{Q}_i$, does every embedding of $\mathcal{P}$ into the degree structure extend to an embedding of one of the $\mathcal{Q}_i$?

Theorem (Lempp, Slaman, Sorbi, to appear)

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Continuity of Cupping

- Ahmad pointed out that if we can decide if there is an Ahmad pair whose join is complete then can decide all sentences of the form

$$\forall x\, \forall y\, \exists z\, \exists w\, \varphi(x, y, z, w).$$

Theorem (Ambos-Spies, Lachlan, Soare, 1993)

If $\mathbf{a}$ and $\mathbf{b}$ are c.e. Turing degrees such that $\mathbf{a} \vee \mathbf{b} = \mathbf{0}'$, then there exists a c.e. Turing degree $\mathbf{c} < \mathbf{a}$ such that $\mathbf{c} \vee \mathbf{b} = \mathbf{0}'$.

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There exist Σ^0_2-enumeration degrees **a** and **b** such that $\mathbf{a} \vee \mathbf{b} = \mathbf{0}'_e$ and for all $\mathbf{c} < \mathbf{a}$, $\mathbf{c} \vee \mathbf{b} < \mathbf{0}'_e$. (We call **a** a minimal cupping companion for **b**.)

Remark

The set of minimal cupping companions for **b** is an antichain. Very Useful.

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Non-splitting Degrees I

Definition

a is non-splitting if for all **b**, **c** $<$ **a**, **b** $\vee$ **c** $<$ **a**.

- Used in Ahmad pairs. We need better understanding of Ahmad pairs if we wish to settle the decidability of the Π_2-theory.

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Non-splitting Degrees II

Theorem (Kent, Sorbi)

There exists a properly Σ^0_2 non-splitting degree.

Theorem (Kent, Sorbi)

Every Δ^0_2-degree bounds a non-splitting degree.

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Does every Σ^0_2-degree bound a non-splitting degree?

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