PA sets, 1-random sets, Π^0_1 classes

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Splendors and miseries of Π^0_1 classes

- A Π^0_1 class ($\subseteq 2^\omega$) is a family of sets (reals) which can be expressed as the family of infinite paths through a recursive tree $T \subseteq 2^{<\omega}$.
- Π^0_1 classes: the solution sets to many problems in logic, combinatorics, algebra and other areas.
- Typical examples
  - the set of complete extensions of a given axiomatizable theory in first-order logic
  - the class of sets which separate a pair of disjoint r.e. sets.
- There are nonempty Π^0_1 classes with no recursive element

- (Kreisel, Shoenfield) Every nonempty Π^0_1 class has an element of degree $\leq \mathbf{0}'$
- (Jockusch, Soare) Every nonempty Π^0_1 class has
  - i) an element of low degree (Low Basis Theorem)
  - ii) an element of r.e. degree
- Interesting structure of the (inclusion) lattice of Π^0_1 classes (in some way analogous to the lattice of Π^0_1 subsets of ω). There are various kinds of "slim" Π^0_1 classes (thin, small, minimal, ...).
  Here: "thick" Π^0_1 classes.

The relation $<<$

Definition (Simpson)

$\mathbf{b} << \mathbf{a}$ means that every infinite tree $T \subseteq 2^{<\omega}$ of degree $\leq \mathbf{b}$ has an infinite path of degree $\leq \mathbf{a}$.

Definition

1. A function $f \in \omega^\omega$ is called DNR if $f(x) \neq \varphi_x(x)$ for all x.
2. Especially: $\mathbf{DNR}_2$ = the class of 0-1 valued DNR functions (i.e. $f \in 2^\omega$)

Theorem (D. Scott and others)

The following conditions are equivalent:

1. **a** is a degree of a DNR_2 function
2. $\mathbf{a} >> \mathbf{0}$
3. **a** is a degree of a complete extension of PA
4. **a** is a degree of a set separating some effectively inseparable pair of r.e. sets.

Remark

$\mathbf{DNR}_2$ is a kind of a "universal" Π^0_1 class

DNR_2 functions are also called PA sets and degrees $>> \mathbf{0}$ are called PA degrees.

(Simpson) Partial ordering $<<$ is dense.

Question

Is $<<$ first-order definable in $\mathcal{D}$ (partial ordering of the degrees)?

PA degrees (i.e. $>> \mathbf{0}$) can be used for a natural definition of $\mathbf{0}'$.

$\mathbf{0}' = \inf \{ \ \mathbf{a} \cup \mathbf{b} : \ (\mathbf{a}, \mathbf{b})$ is a minimal pair of degrees $>> \mathbf{0} \ \}$.

Proof. Idea:

1. Split a maximal set A s.t. $\overline{A}$ dominates all p. r. functions into two nonrecursive r.e. sets A_1, A_2
   Sep (A_1, A_2) is a Π^0_1 class without recursive members sup of any two members $\neq^*$ is above $\mathbf{0}'$.

2. in fact, $\mathbf{a} \cup \mathbf{b} > \mathbf{0}'$ (for $\mathbf{a}, \mathbf{b}$ as above) since no Δ^0_2 DNR$_2$ (or DNR) can form a minimal pair

Or:

$$\mathbf{0}' = \inf \{\, \mathbf{a} : \mathbf{0} << \mathbf{a}\ \&$$
$$\forall \mathbf{c}(\mathbf{0} < \mathbf{c} \leq \mathbf{a} \rightarrow \exists \mathbf{b}(\mathbf{0} << \mathbf{b} \leq \mathbf{a}\ \&$$
$$(\mathbf{c} \not\leq \mathbf{b})))\}$$

i.e. if $\mathbf{a} >> \mathbf{0}$ & $\mathbf{a} \not\geq \mathbf{0}'$
then for some $\mathbf{c}$, $\mathbf{0} < \mathbf{c} < \mathbf{a}$
all PA $\mathbf{b}$ below $\mathbf{a}$ are above $\mathbf{c}$.

Comment : More on that later.

The role of DNR_2 functions (= PA sets).

Important: Coding

Finitary coding

For any nonempty Π^0_1 class $\mathcal{A} \subseteq \mathbf{DNR}_2$

$\exists x_0 \exists f_0, f_1 \in \mathcal{A}\ (f_i(x_0) = i\ ;\ i = 0, 1)$

i.e. both values 0, 1 are consistent
with $\mathcal{A}$ at x_0

("undecidable formula")

proof: Gödel incompleteness phenomenon

Infinitary coding

For any nonempty Π^0_1 class $\mathcal{A} \subseteq \mathbf{DNR}_2$

there is a recursive sequence (increasing) $(x_i : i \in \omega)$ s.t.

$\forall C \exists f \in \mathcal{A}\ (f(x_i) = C(i) \text{ for all } i)$

(i.e. any infinite binary information is consistent with $\mathcal{A}$ at places given by x_i) ("flexible column")

Variants

static or dynamic coding (i.e. combined with an r.e. set)

Remark

By coding a recursive information

(by this way) we keep Π^0_1 classes $\subseteq$ $\mathbf{DNR}_2$.

(Typically: Σ^0_2 or Π^0_2 event).

This phenomenon is implicit in the connection of DNR_2 functions (i.e. PA sets) to r.e. sets.

Examples

Theorem (K, 86-87)

For every DNR_2 (even DNR) $f \leq_T \emptyset'$ there is a nonrecursive r.e. set A recursive in f (A is even PS).

Remark

This gives a priority-free solution to Post's problem.

Similar techniques give: Friedberg-Muchnik and standard finite-injury argument.

For $\emptyset''$-arguments one needs "infinitary" coding into DNR_2 functions (i.e. PA sets).
Typical example: Minimal pair construction

Fact

There are two PA sets A_1, A_2 below $\emptyset''$, s.t. (A_1, A_2) form a minimal pair and each A_i bounds a nonrecursive r.e. set B_i.
Thus, (B_1, B_2) form a minimal pair of r.e. sets.

Remark

1) Minimality conditions are satisfied explicitely for PA sets rather than for r.e. sets.
2) Easy to combine with jump classes, avoiding an upper cone, etc.
3) There is no minimal pair of PA sets below $\emptyset'$.

Proposition

For any $\emptyset''$-recursive sequence of $\emptyset'$ -indices of DNR_2 functions $f_i \leq_T \emptyset'$ there is a nonrecursive r.e. set A recursive in all f_i.

Question

Does for every r.e. set $A <_T \emptyset'$ exist incomplete PA set $<_T \emptyset'$ which is above A ? (a witness of an incompleteness of A).

Answer: NO.

Theorem (Slaman, K, unpublished)
There is low$_2$ r.e set A s.t. A joins to $\emptyset'$ every DNR function $\leq_T \emptyset'$,
i.e. $A \oplus f \equiv_T \emptyset'$ for all such f.

Remark
Such A cannot be low. In fact, for every low A there is always a low PA above.

Hint: code A into a Π^0_1 class of PA sets "in one step" and apply relativized LBT.

Coding into random (= chaotic) objects is more "complicated" and limited.

Algorithmic randomness

Main approaches:

- stochasticity (frequency stability)
- chaoticness (Kolmogorov complexity)
- typicalness (measure - theoretic approach)

Kolmogorov complexity:
plain: $C(y) \; = \; min\{|x| : U(x) = y\}$
where U is a universal TM

prefix free: $K(y) \; = \; min\{|x| : U(x) = y\}$
where U is now a universal **prefix-free** TM

Definition

A set A is 1-random (Chaitin-random) if
$K(A \upharpoonright n) \geq n + c$ for all n.

Measure-theoretic approach.

Definition

A class $\mathcal{A} \subseteq 2^\omega$ is of Σ^0_1 measure zero if there is a computable sequence of (indices of) Σ^0_1 classes $\{ \mathcal{B}_n : n \in \omega \}$ s.t.

$\mathcal{B}_n \supseteq \mathcal{B}_{n+1}$ for all n

$\mu(\mathcal{B}_n) \leq 2^{-n}$ for all n

$\mathcal{A} \subseteq \bigcap_n \mathcal{B}_n$

Remark: such sequence is called Martin-Löf test.

Definition

A set A is 1-random (ML-random) if it passes all ML-tests, i.e. $\{A\}$ is not of Σ^0_1 measure zero.

Theorem (ML)

There is a universal ML-test $\{ \mathcal{U}_n : n \in \omega \}$.

Fact

- 1-random sets form a Σ^0_2 class $(= \bigcup_n \overline{\mathcal{U}}_n)$
- **1-Rand** $= \{\ \sigma * A : A \in \overline{\mathcal{U}}_0\ \&\ \sigma \in 2^{<\omega}\}$

Remark:

i) work with Π^0_1 classes of a positive measure

ii) another kind of "thick" Π^0_1 classes.

Theorem (Schnorr)

A set A is 1-random (ML-random) iff A is Chaitin-random.

Theorem (K)
For every Π^0_1 class $\mathcal{A}$ with $\mathcal{A} \cap \overline{\mathcal{U}}_n \neq \emptyset$ (given n) there is (effectively) x s.t. $\mu(\mathcal{A}) > 2^{-x}$.
Remark: weak Gödel's incompleteness phenomenon.

Theorem (Kučera, Gacs)

1. Any set is *wtt*-reducible to a 1-random set
2. $\{ \mathbf{a} : \mathbf{a} \geq \mathbf{0}' \} \subseteq$ 1-random degrees.

Remark
The complexity of coding is, r.sp., of degree $\mathbf{0}'$.

A weaker result (K):
1-random degrees are not closed upwards.
A stronger result (Stephan):
PA degrees and 1-random degrees coincide only and precisely above $\mathbf{0}'$.

Theorem (Calude, Nies)
R.e. 1-random reals are *wtt*-complete.

R.sp., when working with 1-randoms above $\emptyset'$ no serious problem with coding (and/or decoding).

Theorem (K, 1987)

1. There is a high incomplete 1-random set,
2. 1-random sets are in all low-high hierarchy classes,
3. Easy to combine with avoiding an upper cone.

Remark
Here $A <_T \emptyset'$ but A' is used to overcome incompleteness of A (to decode).

Natural question:

To what extent it is possible to code an infinitary information into incomplete 1-randoms. In other words, for which incomplete B there is an incomplete 1-random Z s.t. $B <_T Z$.

Fact

For no incomplete PA set A there is an incomplete 1-random set Z above A.
(Easy corollary of Stephan's result.)

Theorem

For every incomplete 1-random A there is an incomplete 1-random Z s.t. $A <_T Z$.

Idea:

1) to code "chaos" into "chaos"

2) $Z = A \oplus B$

3) B is constructed by using the following strong useful result (and relativized LBT).

Theorem (van Lambalgen)

1. $X \oplus Y$ 1-random implies X is 1-random in Y
2. X is 1-random in Y and Y is 1-random implies $X \oplus Y$ is 1-random.

An important special case (of coding) concerns lowness and K-triviality.

Definition

1) Low(MLRand) denotes the class of sets A s.t. 1-Rand^A = 1-Rand

2) $\mathcal{K}$ denotes K-trivials, i.e. the class of sets A s.t. $K(A \restriction n) \leq K(n) + c \quad$ for all n

3) Low for K denotes the class of sets A s.t. $K(y) \leq K^A(y) + c \quad$ for all y,

4) A set A is a basis for 1-Rand if $A \leq_T Z$ for some Z s.t $Z \in$ 1-Rand^A.

By excellent results by Nies, Hirschfeldt, Downey, ... all these classes are the same.

Theorem

Low(MLRand) = $\mathcal{K}$ = low for K =
= Bases for 1-Rand.

History: such sets exist.

K-trivial: Δ^0_2 Solovay (a complicated proof)

Low(MLRand): r.e. Terwijn, Kučera (short and easy construction)

K-trivial: r.e. Downey, Kummer (similar as above)

Low for K: r.e. Muchnik

Basis for 1-Rand: r.e. Kučera

Facts:

All are Δ^0_2 (Nies)

All are low (Kučera: GL_1, thus low by the above)

All are superlow, i.e. $A' \leq_{tt} \emptyset'$ (Nies)

Question:

A nice characterization of this class?

Theorem (Stephan)
If B is r.e., $B \leq_T Z$ for some 1-random Z s.t $\emptyset' \not\leq_T Z$, then Z is 1-random in B. Thus, B is a basis for 1-Rand.

[It doesn't hold, in general, without the assumption "B is r.e." (take 1-randoms).]

Corollary (Nies, Hirschfeldt)
Such B is K-trivial.

Thus, incomplete r.e. sets which have an incomplete 1-random set above are necessarily K-trivial.

Theorem (K)
Z incomplete Δ^0_2 and 1-random (even DNR) then there is r.e. and nonrecursive $B \leq_T Z$.

Corollary Such B is K-trivial .

Nies and Stephan asked: Is this, in fact, a characterization of K-trivials?

Question (Nies, Stephan)
Given K-trivial set B is there a 1-random incomplete Z above B ?

Note. Important case: B r.e. (Nies).

Problem: how to combine coding with LBT when working with 1-randoms ?
A big problem connected with coding into chaos.

Theorem (Nies)
K-trivial sets form a Σ^0_3 ideal in the Δ^0_2 T-degrees, generated by its r.e. members.
(Like any such Σ^0_3 ideal)
$\mathcal{K} \subseteq [\mathbf{0}, \mathbf{b}]$ for some r.e. low$_2$ $\mathbf{b}$.

Remark (The role of PA again)
There is a PA set A s.t.

- $A'' \equiv_T \emptyset''$
- A is above all sets in $\mathcal{K}$
- there is a low$_2$ r.e. set below A which is also above all $\mathcal{K}$.

A stronger version of the preceding question (of Nies and Stephan):
Is there a low 1-random set above a given K-trivial set ?

Another question:
Is there a low (low PA) A above all sets in $\mathcal{K}$?

Probably difficult.

Easy fact

For every nonrecursive Z there is a PA set A s.t. $Z \oplus A \equiv_T A'$ (and $A \leq_T Z \oplus \emptyset'$.)

Proof: 1) it follows from Posner-Robinson and relativized LBT, (but we wish to have more)

2) a direct construction.

Take a flexible column in a Π^0_1 class $\mathcal{B} \subseteq \mathrm{DNR}_2$, code A bit by bit into it and $\emptyset' \oplus A$ will find a finite piece σ of A which

- either is a bound for a Σ^0_1 witness
- or $\sigma * 0$ and $\sigma * 1$ can be used for

(Π^0_1) forcing of a Π^0_1 event.

Decoding: first difference from A.

Key: A cannot be an isolated path through a recursive tree.
Crucial: Since DNR_2 is a 'universal' Π^0_1 class, Π^0_1 restriction over $\mathcal{B}$ is still a Π^0_1 subclass of DNR_2.

Theorem If Z is not Δ^0_n $(n \geq 1)$ then there is a PA set A s.t. $Z \oplus A \equiv_T A^{(n)}$, where $A \leq_T Z \oplus \emptyset^{(n)}$.

Remark

Slaman and Shore proved such a result (without PA) and used that (for $n = 2$) for defining the jump. They used Kumabe-Slaman forcing.

Proof: 1) It follows from Slaman-Shore result and relativized LBT.
But a direct construction which would be adaptable for other "fat" Π^0_1 classes is desirable.
2) Working conjecture: such a construction is possible.

PA versus 1-Rand:

Case $n = 2$ (the double jump)

1) Fact (for the first jump):
A not Δ^0_2 implies that there is
a 1-random set B s.t. $A \oplus B \equiv_T B'$
where $B \leq_T A \oplus \emptyset'$

2) working conjecture:
A not Δ^0_2 implies that there is
a 1-random B s.t. $A \oplus B \equiv_T B''$
where $B \leq_T A \oplus \emptyset''$

(Here we would need an adaptable construction for fat Π^0_1 classes).

Class : Δ^0_2 and $n = 1$ (the first jump)

case: K-trivials.

(Nies) some K-trivials A cannot be joined above $\emptyset'$ by any incomplete 1-random set,

i.e. $A \oplus B \not\geq_T \emptyset'$ for all 1-random $B \not\geq_T \emptyset'$ and thus, $A \oplus B \not\geq_T B'$ for all 1-random B

Theorem (Nies) If Y is 1-random and Δ^0_2 then there is a (necessarily) K-trivial A s.t.
$A \oplus Z \geq_T Y$ implies $Z \geq_T Y$
for all 1-random Z.

Question: All K-trivials ?

case: not K-trivials (still Δ^0_2)

Then for such A

$A \oplus B \equiv_T A'$ for some 1-random B.

Question: Can for any such A for some 1-random B hold

$A \oplus B \equiv_T B'$? (and $B <_T \emptyset'$?)

Definition A degree $\mathbf{a}$ is cuppable (resp. **-cuppable) if for every $\mathbf{b}$, $\mathbf{a} > \mathbf{b} > \mathbf{0}$ there is a degree $\mathbf{c}$ (resp. a **-degree $\mathbf{c}$) with $\mathbf{c} < \mathbf{a}$ s.t. $\mathbf{b} \cup \mathbf{c} = \mathbf{a}$.

By the same argument (as for the first jump) we can get.

Theorem Every $\mathbf{a} \geq \mathbf{0}'$ is PA-cuppable, moreover, for any nonzero $\mathbf{c}$ below $\mathbf{a}$ there is a PA degree $\mathbf{b}$ s.t. $\mathbf{c} \cup \mathbf{b} = \mathbf{b}' = \mathbf{a}$

But (due to very slim Π^0_1 classes, see pages 7,8)

Fact No PA $\mathbf{a} \not\geq \mathbf{0}'$ is PA-cuppable.

Question Which incomplete PA degrees are (at least) cuppable?
(Posner: Every sufficiently high degree is cuppable).

Theorem (K)
Any degree $\mathbf{a} >> \mathbf{0}$ has the cupping property, i.e. can be (nontrivially) joined to any greater degree (even by a degree $>> \mathbf{0}$).

Fact

For any Δ^0_2 1-random Z, $Z = Z_0 \oplus Z_1$, there is always a K-trivial B below both Z_0, Z_1.

Question

If B is K-trivial, is it always possible to (recursively) split above B any Δ^0_2 1-random Z s.t. $B \leq_T Z$?

1-randoms arise by a special diagonalization of (some) Σ^0_1 objects (similarly for $n > 1$).

Proposition

For every n-random set A there is a n-DNR function f s.t. $f \leq_T A$ (or $\equiv_T$).

Hint. use a binary representation of $\varphi_x(x)$ or $\varphi_x^{\emptyset^{(n)}}(x)$ for an approximation in measure.

Remark

Similarly with n-FPF functions instead of n-DNR, where n-FPF functions are defined as follows:

$n = 0 \qquad W_{f(x)} \neq W_x$

$n = 1 \qquad W_{f(x)} \neq^* W_x$

$n \geq 2 \qquad W^{(n-2)}_{f(x)} \not\equiv_T W^{(n-2)}_x.$

Definition

A Scott set is a nonempty set $F \subseteq 2^\omega$ s.t. whenever $T \subseteq 2^{<\omega}$ is an infinite tree recursive in a finite join of elements of F, then $[T] \cap F \neq \emptyset$.

Remark

Sets representable in a complete extension of PA form a Scott set (Scott).

Question (H. Friedman)

Given a Scott set F and nonrecursive set $X \in F$, is there $Y \in F$ s.t. X and Y are T-incomparable ?

Theorem (Slaman, K, based on others)
The positive answer.

It is based on the following.

Proposition
For any $\mathbf{b} \neq \mathbf{0}$ there is an infinite $\mathbf{b}$-recursive tree $T \subseteq 2^{<\omega}$ s.t. every infinite path through T has degree incomparable with $\mathbf{b}$.

Idea - two cases (given $X \in \mathbf{b}$):
i) X is not a basis for 1-Rand : use sets 1-random in X to avoid both lower and upper cone of X
ii) X is a basis for 1-Rand: by deep results of Nies, Hirschfeldt and others such X is Δ^0_2 and low for K. Slaman produced the desired tree for this case (finite injury for avoiding an upper cone and making all paths not low for K).

THANK YOU!