ON DOWNEY'S CONJECTURE

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Abstract. We prove that the degree structures of the d.c.e. and the 3-c.e. Turing degrees are not elementarily equivalent, thus refuting a conjecture of Downey. More specifically, we show that the following statement fails in the former but holds in the latter structure: There are degrees $\mathbf{f} > \mathbf{e} > \mathbf{d} > \mathbf{0}$ such that any degree $\mathbf{u} \leq \mathbf{f}$ is either comparable with both $\mathbf{e}$ and $\mathbf{d}$, or incomparable with both.

1. The Theorems

In 1965, Putnam [Pu65] defined the n-c.e. sets as a generalization of the c.e. (or computably enumerable) sets:

Definition 1. Given an integer $n > 0$, we call a set $A \subseteq \omega$ *n-c.e.* if there is a computable sequence of sets $\{A_s\}_{s\in\omega}$ such that for all $x \in \omega$,

$$A_0(x) = 0,$$
$$A(x) = \lim_s A_s(x), \text{ and}$$
$$\{s \in \omega \mid A_s(x) \neq A_{s+1}(x)\}.$$

(Note that a c.e. set is thus simply a 1-c.e. set; and a 2-c.e. set is a *d.c.e.* set, i.e., a difference of two c.e. sets.)

These sets were first extensively studied (and extended to the α-c.e. sets for computable ordinals α) by Ershov [Er68a, Er68b, Er70] and are nowadays often said to form the *Ershov hierarchy*, which stratifies the Δ^0_2-sets.

The *n-c.e. degrees*, i.e., the Turing degrees of the n-c.e. sets, were first investigated by Lachlan (late 1960's, unpublished), who showed that for any n-c.e. degree $\mathbf{d} > \mathbf{0}$, there is a c.e. degree $\mathbf{a}$ with $\mathbf{d} > \mathbf{a} > \mathbf{0}$, and

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by Cooper [Co71], who showed that there is a *properly d.c.e. degree*, i.e., a Turing degree containing a d.c.e. set, but no c.e. set.

The d.c.e. and, more generally, the n-c.e. Turing degrees form an intermediate degree structure between the computably enumerable and the Δ^0_2-degrees. It is interesting to compare the n-c.e. degrees with both of them, since they share some of the properties of either structure.

By the above-mentioned result of Lachlan and the existence of a minimal Δ^0_2-degree (Sacks [Sa61]), the n-c.e. degrees and the Δ^0_2-degrees do not form elementarily equivalent degree structures. The first elementary difference between the c.e. and the d.c.e. degrees was found by Arslanov [Ar88] (see also Cooper, Lempp, Watson [CLW89]), who showed that every nonzero d.c.e. degree $\mathbf{d}$ *cups to* $\mathbf{0}'$, i.e., that there is a d.c.e. degree $\mathbf{e} < \mathbf{0}'$ with $\mathbf{d} \cup \mathbf{e} = \mathbf{0}'$, whereas Yates and Cooper (1973, unpublished, cf. D. Miller [Mi81]) showed this to fail for the c.e. degrees. Further elementary differences were found by Downey [Do89], who showed that the Lachlan Nondiamond Theorem [La66] for the c.e. degrees fails for the d.c.e. degrees; and by Cooper, Harrington, Lachlan, Lempp, and Soare [CHLLS91], who showed that for all $n > 1$, the n-c.e. degrees are not dense, in contrast to the Sacks Density Theorem [Sa64] for the c.e. degrees.

Given that no one was able to find, or even conceive of, an elementary difference between the n-c.e. degrees for various $n > 1$, Downey formulated the following

Conjecture 2 (Downey [Do89]). *For any $m, n > 1$, the degree structures of the m-c.e. and the n-c.e. degrees are elementarily equivalent.*

In this paper, we will refute this conjecture:

Theorem 3. *The degree structures of the d.c.e. and the 3-c.e. degrees are not elementarily equivalent. (More precisely, there is an $\forall\exists$-sentence in the language of partial orderings only on which they differ.)*

This theorem will immediately follow by Corollary 6 and Theorem 7 below. We conjecture that our proof can be extended as follows:

Conjecture 4. *For any distinct $m, n \geq 1$, the degree structures of the m-c.e. and the n-c.e. degrees are not elementarily equivalent.*

We are now ready to state the technical results establishing Theorem 3:

Theorem 5. *Let E and D be d.c.e. sets and X a c.e. set such that $X \leq_T E$, $E \not\leq_T D$, $D \not\leq_T X$, and both D and E are c.e. in X. Then there exists a d.c.e. set U such that $X \leq_T U \leq_T E$ and $U \mid_T D$.*

This theorem will be proved in section 2. From this theorem, we obtain

Corollary 6. *There are no d.c.e. degrees* $\mathbf{f} > \mathbf{e} > \mathbf{d} > \mathbf{0}$ *such that any d.c.e. degree* $\mathbf{u} \leq \mathbf{e}$ *is comparable with* $\mathbf{d}$*, and any d.c.e. degree* $\mathbf{u}$ *with* $\mathbf{d} \leq \mathbf{u} \leq \mathbf{f}$ *is comparable with* $\mathbf{e}$.

Proof. For the sake of a contradiction, assume that such degrees $\mathbf{f} > \mathbf{e} > \mathbf{d} > \mathbf{0}$ exist. Note that by Robinson [Ro71] there is no c.e. degree $\mathbf{x}$ such that $\mathbf{d} < \mathbf{x} \leq \mathbf{e}$ or $\mathbf{e} < \mathbf{x} \leq \mathbf{f}$.

Let $F \in \mathbf{f}$, $E \in \mathbf{e}$ and $D \in \mathbf{d}$ be d.c.e. sets. By Lachlan (unpublished), we can choose a c.e. set $X \leq_T E$ such that D and E are c.e. in X. Then by Theorem 5 we have $D \leq_T X$. Since X is c.e. and $X \leq_T E$, the case $D <_T X$ is not possible. Thus, $X \in \mathbf{d}$ and $\mathbf{d}$ is a c.e. degree.

Let $Y \leq_T F$ be a c.e. set such that E and F are c.e. in Y. Without loss of generality we can assume that $D \leq_T Y$ (otherwise we can replace Y by $X \oplus Y$). Since Y is c.e. and $Y \leq_T F$, we cannot have $E \leq_T Y$. Then by Theorem 5 we must have a d.c.e. set U such that $Y \leq_T U \leq_T F$ and $U \mid_T E$. This contradicts our assumption since $D \leq_T U$. $\square$

On the other hand, we will show that Corollary 6 fails in the 3-c.e. degrees:

Theorem 7. *There are 3-c.e. degrees* $\mathbf{f} > \mathbf{e} > \mathbf{d} > \mathbf{0}$ *such that any 3-c.e. degree* $\mathbf{u} \leq \mathbf{e}$ *is comparable with* $\mathbf{d}$*, and any 3-c.e. degree* $\mathbf{u}$ *with* $\mathbf{d} \leq \mathbf{u} \leq \mathbf{f}$ *is comparable with* $\mathbf{e}$*. (In fact,* $\mathbf{e}$ *and* $\mathbf{d}$ *can be chosen d.c.e. and c.e., respectively.)*

This theorem will be proved in section 3. We obtain immediately

Corollary 8. *There are d.c.e. degrees* $\mathbf{e} > \mathbf{d} > \mathbf{0}$ *such that any d.c.e. degree* $\mathbf{u} \leq \mathbf{e}$ *is comparable with* $\mathbf{d}$.

Note that from the proof of Corollary 6, it follows that if the degrees $\mathbf{e}$ and $\mathbf{d}$ are as in Corollary 8, then the degree $\mathbf{d}$ must be c.e. This gives a first example of an (infinite) definable class in the d.c.e. degrees consisting only of c.e. degrees. Note that this class cannot be equal to the class of all c.e. degrees by the existence of non-isolating c.e. degrees shown by Arslanov, Lempp, Shore [ALS96].

Note also that all known sentences in the language of partial ordering, which are true in the n-c.e. degrees and false in the $(n+1)$-c.e. degrees for some $n \geq 1$ (including the sentence from Theorem 3 for $n = 2$ and the sentence from Corollary 8 for $n = 1$), belong to the level $\forall\exists$ or higher. We conjecture that for all $n \geq 1$, the $\exists\forall$-theory of the n-c.e. degrees is a subtheory of the $\exists\forall$-theory of the $(n+1)$-c.e. degrees.

The rest of this paper will be devoted to the proofs of Theorem 5 and Theorem 7.

2. The proof of Theorem 5

2.1. **The requirements for Theorem 5.** To prove Theorem 5 we can suppose that $\overline{E}$ is not c.e. in D (otherwise we can consider the set $E \oplus X$ instead of E since if $\overline{E} \oplus \overline{X}$ is c.e. in D then $X \leq_T D$ and hence $E \leq_T D$).

Let Y be the "Lachlan" set for $E \oplus D$, i.e., if $E \oplus D = E^1 \oplus D^1 - E^2 \oplus D^2$ for some c.e. sets E^1, D^1, E^2, and D^2, where $E_2 \subseteq E_1$ and $D_2 \subseteq D_1$, then $Y = h^{-1}(E^2 \oplus D^2)$, where h is a 1-1 computable function such that $E^1 \oplus D^1 = \mathrm{rng}(h)$. Then, obviously, $Y \leq_T X$. Hence we can suppose that X is a set of the form $Y \oplus Z$ for some c.e. set Z.

Let $\{E^1_s\}_{s\in\omega}$, $\{E^2_s\}_{s\in\omega}$, $\{D^1_s\}_{s\in\omega}$, and $\{D^2_s\}_{s\in\omega}$ be computable enumerations of the c.e. sets E^1, E^2, D^1 and D_2, respectively, such that $E^2_s \subseteq E^1_s$ and $D^2_s \subseteq D^1_s$. Let $E_s = E^1_s - E^2_s$ and $D_s = D^1_s - D^2_s$. Let $\{X_s\}_{s\in\omega}$ be a computable enumeration of X.

Let f and g be partial computable functions such that $f(y) = 2h^{-1}(2y)$ and $g(y) = 2h^{-1}(2y+1)$. (So $f(\omega \oplus \emptyset) = E \oplus \emptyset$ and $g(\emptyset \oplus \omega) = \emptyset \oplus D$.)

To prove Theorem 5 it is sufficient now to construct a d.c.e. set $U \leq_T E \oplus X \equiv_T E$ meeting for all Turing functionals Γ and Λ the following requirements:

$$\mathcal{P}_\Gamma : U = \Gamma(D) \rightarrow \overline{E} = \mathrm{dom}(\Phi(D));$$
$$\mathcal{N}_\Lambda : D = \Lambda(U \oplus X) \rightarrow D = \Psi(X).$$

Here, the functionals Φ and Ψ will be built during the construction below.

We first consider basic modules for $\mathcal{P}$- and $\mathcal{N}$-requirements in isolation.

2.2. **The Basic Module for the $\mathcal{P}$-requirement in isolation.** To satisfy the $\mathcal{P}$-requirement we proceed by an ω-sequence of cycles. For an arbitrary cycle $k \in \omega$ we

(1) pick a witness $u_k \notin U$ and wait for $\Gamma(D; u_k)[s] \downarrow = 0$ at a stage s,
(2) then open cycle $k+1$, define $\Phi(D; k) = 0$ with $\varphi(k) = \gamma(u_k)$, and
(3) wait for k to enter into E or $D \upharpoonright \gamma(u_k)$ to change. In the latter case go to Step 1 with the same witness u_k. In the former case,
(4) enumerate u_k into U, stop all cycles $> k$, and
(5) wait for a stage s_1 when $\Gamma(D; u_k)[s_1] \downarrow = 1$. After that

(6) reopen cycles $> k$, and leave $\Phi(D;k)$ undefined. (Note that $\Phi(D;k)$ must now be undefined even though D is only a d.c.e. set since we never define such a computation unless there is a corresponding computation $\Gamma(D;u_k)$ with use at most that of $\Phi(D;k)$.)
(7) If later k leaves E then we define $\Phi(D;k) = 0$ with $\varphi(k) = 0$, and extract u_k from U (for the sake of $U \leq_T E$).

There are the following possible outcomes of this strategy.

A. There are only finitely many cycles opened. This means that some cycle k waits at Step 1 or 5 forever. We denote this outcome by ω.
B. For some cycle k there is an infinite loop from Step 3 to Step 1. We denote this outcome by k for the least such $k \in \omega$.
C. Otherwise. In this case we will have $\overline{E} = \mathrm{dom}(\Phi(D))$, which is impossible.

2.3. **The Basic Module for the $\mathcal{N}$-requirement in isolation.** Again we proceed by an ω-sequence of cycles. For each cycle l,

(1) wait for $D \restriction (l+1) = \Lambda(U \oplus X) \restriction (l+1))$, and
(2) define $\Psi(X;l) = D(l)$ with $\psi(l) = \lambda(l)$, and restrain $U \restriction \lambda(l)$. Open the cycle $l+1$.
(3) Wait for $D \restriction (l+1)$ or $X \restriction \psi(l)$ to change. In the latter case go to Step 1. In the former case we stop the cycles $> l$ and
(4) wait for $D \restriction (l+1) = \Lambda(U \oplus X) \restriction (l+1)$ again.
(5) Redefine $\Psi(X)(l) = D(l)$, reopen the cycles $> l$.

There are the following possible outcomes of this strategy.

A. There are only finitely many cycles opened. This means that some cycle l waits at Step 1 or 4 forever. We denote this outcome by ω.
B. For some cycle l there is an infinite loop from Step 3 to Step 1. We denote this outcome by l for the least such $l \in \omega$.
C. Otherwise. In this case we will have $D = \Psi(X)$, which is impossible.

Note that for the sake of the requirement $U \leq_T E$ we may need to extract some u_k from U, injuring some $\mathcal{N}$-requirement. This can only be because of some k leaving E and, therefore, $f(k)$ entering $X = Y \oplus Z$. To avoid this difficulty, it is enough to define $\psi(l)$ in Step 2 to be greater than all $f(k)$ for any $k \in E$ such that there is a witness $u_k \in U$ (of some $\mathcal{P}$-strategy) such that $u_k < \delta(l)$).

2.4. **$\mathcal{P}$- and $\mathcal{N}$-strategies together.** Now let us consider an $\mathcal{N}$-strategy below the k_0-outcome of a $\mathcal{P}$-strategy and the following interplay of strategies:

1. At a stage s in cycle k_0, we see $U(u_{k_0}) = \Gamma(D; u_{k_0})\downarrow$, open a cycle $k > k_0$, and define u_k.
2. Later, at a stage s_1, we may have $\Gamma(D; k_0)\uparrow$. In this case we go to the $\mathcal{N}$-strategy, and
3. suppose that the $\mathcal{N}$-strategy restrains u_k (via its cycle l, say),
4. then we see that at a stage s_2, again $U(u_{k_0}) = \Gamma(D; u_{k_0})\downarrow$, and
5. returning to the $\mathcal{P}$-strategy, we may need to enumerate u_k into U, injuring the $\mathcal{N}$-restraint.

There are two possibilities.

1) At Step 5, we return to the $\mathcal{P}$-strategy with a D-use which is different from that which we had at stage s. In this case we can simply redefine $\Phi(D; k_0)$, without enumeration of u_k into U.
2) At Step 5 we return to the $\mathcal{P}$-strategy with the same D-use which we had at stage s.

By our strategy this can be only if for some $t < \gamma(k_0)$ we have $t \notin D[s]$, $t \in D[s_1]$ and $t \notin D[s_2]$.

To avoid this difficulty, it is enough to change the $\mathcal{N}$-strategy so that at stage s_1 the use function $\psi(l)$ is defined to be greater than all $g(x)$ for any $x \in D \upharpoonright \gamma(k_0)$. Then, after stage s_2, we have no need to keep the restraint imposed at stage s_1 since $X \upharpoonright \psi(l)$ has changed by stage s_2.

2.5. **The tree of strategies.** In the formal construction below we will use the tree of strategies $T = (\omega+1)^{<\omega}$. Let $<_L$ and $<$ be the standard relations on T under the ordered alphabet $\{0 < 1 < 2 < \cdots < \omega\}$.

We also fix some effective listing of all $\mathcal{P}$- and $\mathcal{N}$-requirements such that each $\mathcal{P}$-requirement has an even index in this listing and each $\mathcal{N}$-requirement has an odd index. For the tree T assign the nth requirement in this listing to all nodes $\xi \in T$ of length n. Let $T = \mathrm{EVN} \cup \mathrm{ODN}$ where EVN and ODN are the sets of nodes of T of even and odd length, respectively.

Let $L_\sigma = \{\langle \tau, k\rangle \in \mathrm{ODN} \times \omega : \tau\hat{\ }k \le \sigma\}$. It is easy to see that if $\tau_1\hat{\ }k_1 <_L \tau_2\hat{\ }k_2$ and $\langle \tau_2, k_2\rangle \in L_\sigma$, then $\langle \tau_1, k_1\rangle \in L_\sigma$.

2.6. **The construction.** *Stage* $s = 0$. Let $U_0 = \emptyset$ and $\delta_0 = \lambda$ (where λ is the empty string), set $r_0(\tau, k) = 0$ for all k and all $\tau \in \mathrm{ODN}$, and call no k marked by any $\tau \in \mathrm{ODN}$ at stage $s = 0$. Also, no witness is assigned to any $\sigma \in \mathrm{EVN}$. The functionals $\Phi_{\sigma,0}$ and $\Psi_{\tau,0}$ are totally undefined for all $\sigma \in$ EVN and all $\tau \in$ ODN. (Therefore, their use functions $\varphi_{\sigma,0}$ and $\psi_{\sigma,0}$ are equal to 0 at all arguments.)

Stage $s+1$ consists of several parts:

Part I) For all $\tau \in$ ODN and for all k, define $r'_s(\tau,k) = r_s(\tau,k)$ if $X_{s+1} \upharpoonright \psi_{\tau',s}(k') = X_s \upharpoonright \psi_{\tau',s}(k')$ for all $\langle \tau', k'\rangle$ such that $\tau'^\frown k' \leq_L \tau^\frown k$. (Here we define $\sigma \leq_L \tau$ iff $\sigma = \tau$ or $\sigma <_L \tau$.) Otherwise, set $r'_s(\tau,k) = 0$.

Part II) Define $U_{s+1} = (U_s - \{u : u = \langle \sigma, k, \alpha, t\rangle \ \&\ k \notin E_{s+1}\}) \bigcup \{u : u = \langle \sigma, k, \alpha, t\rangle$ is a witness & $k \in E_{s+1}$ & $\alpha \subseteq D_{s+1}$ & $R_s(\sigma) \leq u)\}$, where $R_s(\sigma) = \max\{\rho_s(\sigma,\tau,k) : \langle \tau, k\rangle \in L_\sigma\}$ and

$$\rho_s(\sigma,\tau,k) = \begin{cases} +\infty & \text{if } \tau^\frown k \subseteq \sigma \text{ and } r'_s(\tau,k) > 0 \\ r'_s(\tau,k) & \text{otherwise} \end{cases}$$

(Here $\rho_s(\sigma,\tau,k)$ is the restraint $r'_s(\tau,k)$ imposed on the positive σ-strategy. If $r'_s(\tau,k) > 0$ for $\tau^\frown k \subseteq \sigma$ we must set $\rho_s(\sigma,\tau,k) = \infty$ since if $r'_s(\tau,k) > 0$ then it is possible that $r'_s(\tau,k') > 0$ for many $k' > k$. This is a formalization of the fact that if $r'_s(\tau,k) > 0$ then any witness below $\tau^\frown k$ cannot enter into U; such witnesses can enter only if we have a window $r'_s(\tau,k) = 0$; of course, such witnesses must be greater than restraints $r'_s(\tau,k')$, $k' < k$).

Part III) In this part we define δ_{s+1} of length $s+1$ by induction: Let $\delta_{s+1} \upharpoonright n = \sigma$ be already defined and let $n < s+1$. Consider the following cases:

Case 1: $\sigma \in$ EVN and σ is assigned to a requirement $\mathcal{P}_\Gamma$.

Subcase 1a: There exists a witness $y = \langle \sigma, k, \alpha, t\rangle$ such that $U_{s+1}(y) \neq \Gamma_{s+1}(D_{s+1}; y)\downarrow$: Then define $\delta_{s+1} \upharpoonright (n+1) = (\delta_{s+1} \upharpoonright n)^\frown \omega$.

Subcase 1b: There exist $k \in \omega$ and a witness $y = \langle \sigma, k, \alpha, t\rangle$ such that either $\Gamma_{s+1}(D_{s+1}; y)$ is undefined or, for every $t \leq s$ such that $\sigma \subseteq \delta_t$, we have $D_{s+1} \upharpoonright \gamma_{s+1}(y) \neq D_t \upharpoonright \gamma_t(y)$: In this subcase we let k_0 be least such k and define $\delta_{s+1} \upharpoonright (n+1) = (\delta_{s+1} \upharpoonright n)^\frown k_0$.

Subcase 1c: Otherwise: Let k_0 be the least k such that there is no witness $\langle \sigma, k, \alpha, t\rangle$ with $\alpha \subseteq D_{s+1}$, and let

$$\alpha_0 = D_{s+1} \upharpoonright \max\{\gamma_{s+1}(D_{s+1}, \langle \sigma, k, \alpha, t\rangle) : k < k_0 \ \&\ \alpha \subseteq D_{s+1}\}.$$

Assign $\langle \sigma, k_0, \alpha_0, s+1\rangle$ as a new witness, and set $\delta_{s+1} \upharpoonright (n+1) = (\delta_{s+1} \upharpoonright n)^\frown k_0$. Note that since Subcase 1c is the unique place in the construction where new witnesses are assigned and since we assume that the use-function $\gamma_t(D_t, y)$ is non-decreasing in t (unless an old definition applies again since D has changed back), we have that, for each set Z and for all k and τ, there is at most one witness of the form $\langle \tau, k, \alpha, t\rangle$ where $\alpha \subseteq Z$ and $t \in \omega$.

For any $k < k_0$ such that $k \notin E_{s+1}$ and $\Phi_{\sigma,s}(D_{s+1}; k)$ is undefined, we define $\Phi_{\sigma,s+1}(D_{s+1}; k) = 0$ and, for the witness $\langle \sigma, k, \alpha, t\rangle$ such that $\alpha \subseteq D_{s+1}$, set the use $\varphi_{\sigma,s+1}(k) = \max(|\alpha|, \gamma_{s+1}(\langle \sigma, k, \alpha, t\rangle))$.

Case 2: $\sigma \in$ ODN and σ is assigned to a requirement $\mathcal{N}_\Lambda$: This case is divided into two subcases.

Subcase 2a: There exists k marked by the node σ such that $D_{s+1}(k) \neq \Lambda_{s+1}(U_{s+1} \oplus X_{s+1}; k)\downarrow$: Then define $\delta_{s+1} \restriction (n+1) = (\delta \restriction n)^\frown\omega$.

Subcase 2b: Otherwise: Let k_0 be the least k such that $r'_{s+1}(\sigma, k) = 0$. Define $\delta_{s+1} \restriction (n+1) = (\delta_{s+1} \restriction n)^\frown k_0$. If k_0 is not yet marked then we now declare k_0 marked by the node σ.

Set $r_{s+1}(\sigma, k_0) = 0$ and $\psi_{\sigma,s+1}(k) = 0$ if either $\Lambda_{s+1}(U_{s+1} \oplus X_{s+1}; k_0)$ is undefined or if $\Lambda_{s+1}(U_{s+1} \oplus X_{s+1}; k_0)\downarrow\neq D_{s+1}(k_0)$. Otherwise, set $r_{s+1}(\sigma, k_0) = 1 + \lambda_{s+1}(k_0)$ and $\Psi_{\sigma,s+1}(X_s; k_0) = D_{s+1}(k_0)$ with use

$$\begin{gathered}\psi_{\sigma,s+1}(k_0) = 1 + \max(\{\lambda_{s+1}(k_0)\} \cup \\ \{f(k) : (\exists u < \lambda_{s+1}(k_0))[u = \langle \tau, k, \alpha, t\rangle \ \&\ u \in U_{s+1})]\} \cup \\ \{g(p) : (\exists u < \lambda_{s+1}(k_0))[u = \langle \tau, k, \alpha, t\rangle \text{ is a witness } \ \& \\ u \notin U_{s+1} \ \ \& \ (\exists k' < k)[\tau^\frown k' \subseteq \sigma] \ \&\ p \in D_{s+1} \ \&\ \alpha(p)\downarrow= 0]\} \cup \\ \{\psi_{s,\tau}(k) : \tau^\frown k <_L \sigma^\frown k_0\}).\end{gathered}$$

Part IV) Define $r_{s+1}(\sigma, k) = r'_s(\sigma, k)$ if $\sigma \in$ ODN and $\sigma^\frown k \not\subseteq \delta_{s+1}$. Also for all σ and k such that $\sigma \in$ ODN and $\sigma^\frown k \not\subseteq \delta_{s+1}$, we define $\psi_{\sigma,s+1}(k) = \psi_{\sigma,s}(k)$ if $r_{s+1}(\sigma, k) > 0$, and $\psi_{\sigma,s+1}(k) = 0$ if $r_{s+1}(\sigma, k) = 0$.

Part V) Initialize all $\sigma \in$ EVN such that $\delta_{s+1} <_L \sigma$. The initialization of σ at stage $s+1$ means that all current witnesses of σ cease to be witnesses.

End of the construction.

2.7. **The verification.** Let $F(\sigma, n, \alpha, s, t)$ be the predicate

$$(\forall v \geq t)[\langle \sigma, n, \alpha, s\rangle < R_v(\sigma)].$$

Lemma 9. *There is an X-c.e. predicate $\widetilde{F}(\sigma, n, \alpha, s, t)$ such that for all σ, n, α, s, and t,*

1. *$\widetilde{F}(\sigma, n, \alpha, s, t)$ implies $F(\sigma, n, \alpha, s, t)$, and*
2. *$F(\sigma, n, \alpha, s, t)$ implies $\widetilde{F}(\sigma, n, \alpha, s, t)$ if $\delta_v \not<_L \sigma$ for all $v \geq t$.*

Proof. Let $\delta_v \not<_L \sigma$ for all $v \geq t$. To prove the lemma, it is sufficient to prove that if $F(\sigma, n, \alpha, s, t)$ then there is a stage $v_0 \geq t$ such that

1) $\langle \sigma, n, \alpha, s\rangle < R_v(\sigma)$ for all v with $t \leq v < v_0$, and
2) $\langle \sigma, n, \alpha, s\rangle < \rho_{v_0}(\sigma, \tau_0, k_0)$ for some $\langle \tau_0, k_0\rangle \in L_\sigma$ such that $X_{v_0} \restriction \psi_{\tau,t}(k) = X \restriction \psi_{\tau,t}(k)$ for every $\langle \tau, k\rangle$ such that $\tau^\frown k \leq_L \tau_0{}^\frown k_0$,

since the converse obviously holds for arbitrary σ, n, α, s, and t. (Recall that we defined $\sigma \leq_L \tau$ iff $\sigma = \tau$ or $\sigma <_L \tau$.)

Suppose that $\delta_v \not<_L \sigma$ for all $v \geq t$ and $F(\sigma, n, \alpha, s, t)$. Since $\delta_v \not<_L \sigma$ for all $v \geq t$ we have $r'_v(\tau, k) \geq r'_{v+1}(\tau, k)$ for all $v \geq t$ and all $\langle \tau, k \rangle \in L_\sigma$ such that $\tau^\frown k \not\subseteq \sigma$ (since the construction can set a new restraint $r'_v(\tau, k)$ only if $\tau^\frown k \subseteq \delta_v$ and since $\tau^\frown k <_L \sigma$).

First we consider the case when there is $\langle \tau_0, k_0 \rangle \in L_\sigma$ such that $\tau_0{}^\frown k_0 \not\subseteq \sigma$ and $r'_v(\tau_0, k_0) > \langle \sigma, n, \alpha, s \rangle$ for all $v \geq t$. By the definition of r'_s from the construction, we have $X_t \upharpoonright \psi_{\tau,t}(k) = X \upharpoonright \psi_{\tau,t}(k)$ for all $\langle \tau, k \rangle$ such that $\tau^\frown k \leq_L \tau_0{}^\frown k_0$ so that we have claims 1) and 2) for $v_0 = t$.

Thus we can assume that there is a stage $v_1 \geq t$ such that $r'_v(\tau_0, k_0) \leq \langle \sigma, n, \alpha, s \rangle$ for all $v \geq v_1$ and all $\langle \tau_0, k_0 \rangle \in L_\sigma$ with $\tau_0{}^\frown k_0 \not\subseteq \sigma$.

Note also that if there are a stage $w \geq v_1$ and $\langle \tau_0, k_0 \rangle \in L_\sigma$ such that $\tau_0{}^\frown k_0 \subseteq \sigma$ and $r'_v(\tau_0, k_0) > 0$ for all $v \geq w$, then we have claims 1) and 2) for $v_0 = w$.

Finally, it remains to show that it is not possible that for every $w \geq v_1$ and every $\langle \tau_0, k_0 \rangle$ with $\tau_0{}^\frown k_0 \subseteq \sigma$, there is a stage $v \geq w$ such that $r'_v(\tau_0, k_0) = 0$.

Indeed, in that case, there would be some $\langle \tau_0, k_0 \rangle$ such that $\tau_0{}^\frown k_0 \subseteq \sigma$ and $\tau_0{}^\frown k_0 \subseteq \delta_v$ for infinitely many stages v. Otherwise, if for any such $\langle \tau_0, k_0 \rangle$ there are only finitely many stages v such that $\tau_0{}^\frown k_0 \subseteq \delta_v$ then there is a stage $v' \geq v_1$ such that $r'_v(\tau_0, k_0) = 0$ for any $v \geq v'$ so that $\langle \sigma, n, \alpha, s \rangle \leq R_v(\sigma)$ which contradicts with $F(\sigma, n, \alpha, s, t)$.

Choose the $\subseteq$-greatest node τ_0 such that $\tau_0{}^\frown k_0 \subseteq \sigma$ for some $k_0 \in \omega$ and $\tau_0{}^\frown k_0 \subseteq \delta_v$ for infinitely many stages v.

By the choice of $\langle \tau_0, k_0 \rangle$, for each $\langle \tau_1, k_1 \rangle$ such that $\tau_0{}^\frown k_0 \subset \tau_1{}^\frown k_1 \subseteq \sigma$, we have $\tau_1{}^\frown k_1 \subseteq \delta_v$ for finitely many v. Therefore, there is a stage $v_2 \geq v_1$ such that $r'_v(\tau_1, k_1) = 0$ for all $v \geq v_2$ and all $\langle \tau_1, k_1 \rangle$ with $\tau_0{}^\frown k_0 \subset \tau_1{}^\frown k_1 \subseteq \sigma$.

Let $v_3 \geq v_2$ be any stage such that $\tau_0{}^\frown k_0 \subseteq \delta_{v_3+1}$. By Subcase 2b of the construction, we have $r'_{v_3}(\tau_1, k_1) = 0$ for each $\langle \tau_1, k_1 \rangle$ with $\tau_1{}^\frown k_1 \subseteq \tau_0{}^\frown k_0$. Thus, by the choice of the stages v_1 and v_2, we have $R_{v_3}(\sigma) \leq \langle \sigma, n, \alpha, s \rangle$. This contradicts $F(\sigma, n, \alpha, s, t)$.

Note that by the part II of the construction, if $F(\sigma, n, \alpha, s, t)$ holds then $U_t(\langle \sigma, n, \alpha, s \rangle) \geq U_v(\langle \sigma, n, \alpha, s \rangle)$ for all $v \geq t$, i.e., the number $\langle \sigma, n, \alpha, s \rangle$ cannot enter U after stage t.

Lemma 10. *$U \leq_T E \oplus X \equiv_T E$.*

Proof. Let $u = \langle \sigma, k, \alpha, s_0 \rangle$. By construction, if at stage s_0, u is not declared a witness then $u \notin U$. Suppose that u is chosen as a witness at stage s_0 (and so we have $\alpha \subseteq D_{s_0}$). Obviously, if $k \notin E$

then $u \notin U$. Suppose that $k \in E$ and let $s_1 \geq s_0$ be the least stage such that $k \in E_{s_1}$. Then we check whether α was initialized at a stage s with $s_0 < s < s_1$. If so, then $u \notin U$ (since u ceases to be a witness at stage s). To compute $U(u)$ in the case when α is not initialized at a stage s with $s_0 < s < s_1$, we wait for a stage $s \geq s_1$ such that one of the following clauses holds:

1) $u \in U_s$,
2) $\delta_s <_L \sigma$,
3) $\widetilde{F}(\sigma, k, \alpha, s_0, s)$,
4) $(\exists p)[\alpha(p) \downarrow= 1 \ \& \ p \notin D_s]$, or
5) $(\exists p)[\alpha(p) \downarrow= 0 \ \& \ p \in D_s \ \& \ g(p) \notin X]$.

If such stage s exists then we have $u \in U$ iff $u \in U_s$. Indeed, if clause 3) holds then by Lemma 9 (1) we have $F(\sigma, k, \alpha, s_0, s)$, so $U(u) = U_s(u)$. If one of the clauses 4) or 5) holds then α will not be compatible with D_v for all $v \geq s$ so that by Part II of the construction, u cannot enter U at any stage $> s$, i.e., $U(u) = U_s(u)$.

It remains to argue that there is such stage s. Indeed, if $u \notin U$ and $\delta_s \not<_L \sigma$ for all $s \geq s_1$ then by Part II of the construction, this is possible only because at every stage $s \geq s_1$, either α is not compatible with D_s, or $u < R_s(\sigma)$. If $\alpha \not\subseteq D$ then at some stage $s \geq s_1$, either clause 4) or clause 5) must hold. If $\alpha \subseteq D$ then from some stage $s \geq s_1$, we always will have $u < R_s(\sigma)$, i.e., $F(\sigma, k, \alpha, s_0, s)$. Thus, by Lemma 9 (2), we have $\widetilde{F}(\sigma, k, \alpha, s_0, s)$, i.e., clause 3).

Lemma 11. $\delta = \liminf_s \delta_s$ *exists.*

Proof. Suppose that $\liminf_s \delta_s \upharpoonright n = \sigma$ exists but $\liminf_s \delta_s \upharpoonright (n+1)$ does not exist.

We will consider two possibilities.

Case 1: $\sigma \in \mathrm{EVN}$ and σ is assigned to a requirement $\mathcal{P}_\Gamma$. Note that $\liminf_s R_s(\sigma) < +\infty$. Let s_0 be the least stage such that $\delta_s \not<_L \sigma$ for all $s \geq s_0$.

We prove by induction that, for all k, there exists a permanent witness $\langle \sigma, k, \alpha, t \rangle$ such that $\alpha \subseteq D$. Suppose that for any $k < k_0$, this is true. To prove this for k_0, we first note that by construction, for any k there can be at most one such witness $\langle \sigma, k, \alpha, t \rangle$. Let us denote it by u_k (if it exists).

It is easy to see that $(\forall k < k_0)[\Gamma(D; u_k) \downarrow= U(u_k)]$. Indeed, otherwise either $\liminf_s \delta_s \upharpoonright (n+1) = \sigma\widehat{\ }\omega$, or $\liminf_s \delta_s \upharpoonright (n+1) = \sigma\widehat{\ }k$ for some $k < k_0$.

By the construction, we choose, at some stage s, a permanent witness $\langle \sigma, k_0, \alpha, s \rangle$ with $\alpha = D \upharpoonright \max\{\gamma_e(D, u_k) : k < k_0\}$.

As $\Gamma(D;u_k)\downarrow= U(u_k)=0$ for all $k\notin E$, we have $\overline{E}\subseteq \mathrm{dom}(\Phi_\sigma(D))$. Suppose now that $k\in E$ and $\Phi_\sigma(D;k)$ is defined. So there exists a witness $\langle\sigma,k,\alpha,t\rangle$ with $\alpha\subseteq D$ such that $\Gamma_e(D;\langle\sigma,k,\alpha,t\rangle)\downarrow=0$. But this means that either $\langle\sigma,k,\alpha,t\rangle<\liminf_s R_s(\sigma)$ or $\langle\sigma,k,\alpha,t\rangle$ was initialized so that the witness $\langle\sigma,k,\alpha,t\rangle$ was assigned before stage s_0. Therefore, $\overline{E}=^* \mathrm{dom}(\Phi_\sigma(D))$, which is impossible.

Case 2: $\sigma\in$ ODN and σ is assigned to a requirement $\mathcal{N}_\Lambda$. Fix a stage s_0 such that $\delta_s\not<_L\sigma$ and $U_s(u)=U_{s_0}(u)$ for all $s\geq s_0$ and all $u\in S$ where

$$S=\{\langle\tau,k,\alpha,t\rangle:\tau\in\mathrm{EVN}\ \&\\ \langle\tau,k,\alpha,t\rangle \text{ is a witness at stage } t\ \&\ [\tau^\frown k\leq\sigma]\}.$$

Note that the set S is finite. Indeed, it is clear that the set of all witnesses $\langle\tau,k,\alpha,t\rangle$ with $\tau^\frown k<_L\sigma$ is finite. Furthermore, the set of all witnesses $\langle\tau,k',\alpha,t\rangle$ with $\tau^\frown k'\subseteq\sigma$ is also finite, otherwise, by Subcase 1c) of the construction, $\Gamma(D;k)$ is undefined for some $k<k'$ where Γ is such that τ is assigned to $\mathcal{P}_\Lambda$, and this is not possible since $\liminf_s\delta_s\restriction n=\sigma$.

To obtain a contradiction, we will show that $D(k)=\Psi_\sigma(X;k)$ for all numbers k which are not marked by the node σ at stage s_0.

Indeed, for all k, $\Psi_\sigma(X;k)$ is defined (otherwise $\liminf_s\delta_s\restriction(n+1)$ exists). Therefore, for any k, the construction, at some stage s_k, imposes a restraint $r_{s_k}(\sigma,k)$ such that $r_{s_k}(\sigma,k)=r_s(\sigma,k)=r'_s(\sigma,k)>0$ for all $s\geq s_k$ and which supports the equation $D(k)=\Lambda(U\oplus X;k)=\Psi_\sigma(X;k)$.

Suppose that k is not marked by node σ at stage s_0. Then the restraint $U\restriction r_{s_k}(\sigma,k)$ cannot be injured by witnesses from S. Also it cannot be injured by witnesses entering U via nodes $\tau>\sigma$. Therefore, only the following possibilities to injure the restraint remain:

1) u leaves $U\restriction r_{s_k}(\sigma,k)$ at a stage $s'>s_k$, or

2) u enters into $U\restriction r_{s_k}(\sigma,k)$ at a stage $s'>s_k$ and $\tau^\frown k'\subseteq\sigma$ and $u=\langle\tau,k'',\alpha,t\rangle$ for some $k'<k''$ and $\alpha\subseteq D_{s'}$.

From 1) and the definition of Ψ_σ at stage s_k, it easily follows that $X_{s'}\restriction\psi_{\sigma,s'}(k)\neq X_{s_k}\restriction\psi_{\sigma,s_k}(k)$, which is impossible.

It follows from 2) that $t\leq s_k$ and $\alpha\subseteq D_t$. Therefore, there exists a witness $u'=\langle\tau,k',\alpha',t'\rangle$ with $\alpha'\subseteq\alpha$, $t'\leq t$, and $U_t(u')=\Gamma(D;\alpha,u')\downarrow$. But then $\alpha\not\subseteq D_{s_k}$.

Now it follows from the definition of $\Psi_\sigma(X;k)$ at stage s_k that $X_{s'}\restriction\psi_{\sigma,s'}(k)\neq X_{s_k}\restriction\psi_{\sigma,s_k}(k)$, which is impossible. Therefore, $D=^*\Psi_\sigma(X)$.

Lemma 12. *For all Turing functionals Λ, $D\neq\Lambda(U\oplus X)$.*

Proof. Suppose that $D = \Lambda(U \oplus X)$. Let $\sigma \subseteq \delta$ be assigned to the requirement $\mathcal{N}_\Lambda$. If $\sigma\widehat{\ }\omega \subseteq \delta$ then, obviously, $D \neq \Lambda(U)$. Therefore, $\sigma\widehat{\ }k_0 \subseteq \delta$ for some k_0. Let s_0 be a stage such that for all $s \geq s_0$ the following conditions holds:

(1) $r_{s_0}(\tau, k) = r_s(\tau, k) = r'_s(\tau, k) > 0$ for all $\langle \tau, k\rangle$ such that $\tau\widehat{\ }k <_L \sigma\widehat{\ }k_0$,
(2) $D_{s_0}(k_0) = D_s(k_0) = \Lambda_{e,s_0}(U_{s_0} \oplus X_{s_0}; k_0)$,
(3) $U_{s_0} \upharpoonright \lambda_{s_0}(k_0) = U_s \upharpoonright \lambda_{s_0}(k_0)$, and
(4) $X \upharpoonright a = X_{s_0} \upharpoonright a$,

where

$$\begin{gathered} a = 1 + \max(\{\lambda_{s_0}(k_0)\} \cup \\ \{f(k) : (\exists u < \lambda_{s_0}(k_0))[u = \langle \tau, k, \alpha, t\rangle \text{ is a witness at stage } t \;\&\; u \in U_{s_0}]\} \\ \cup \{g(p) : (\exists u < \lambda_{s_0}(k_0))[u = \langle \tau, k, \alpha, t\rangle \text{ is a witness at stage } t \;\&\; u \notin U_{s_0} \\ \& \; (\exists k' > k)[\tau\widehat{\ }k' \subseteq \sigma] \;\&\; \alpha(p) \downarrow = 0 \;\&\; p \in D_{s_0}]\} \cup \\ \{\psi_{\tau,s_0}(k) : \tau\widehat{\ }k <_L \sigma\widehat{\ }k_0\}). \end{gathered}$$

Now let $s_1 = \mu s \geq s_0(\sigma\widehat{\ }k_0 \subseteq \delta_{s_1})$. By the construction, at stage s_1, we set the restraint $r_{s_1}(\sigma, k_0) = r_s(\sigma, k_0) = r'_s(\sigma, k_0) > 0$ for all $s \geq s_1$. This contradicts $\sigma\widehat{\ }k_0 \subseteq \delta$.

Lemma 13. *For all Turing functionals Γ, $U \neq \Gamma(D)$.*

Proof. Let $\sigma \subseteq \delta$ be assigned to the requirement $\mathcal{P}_\Gamma$. Suppose that $U = \Gamma(D)$. It follows that we cannot have $\sigma\widehat{\ }\omega \subseteq \delta$. Therefore, $\sigma\widehat{\ }k_0 \subseteq \delta$ for some k_0. This means that $\Gamma(D; u)$ is undefined for some permanent witness $u = \langle \sigma, k_0, \alpha, t\rangle$, a contradiction.

3. The proof of Theorem 7

3.1. **The requirements for Theorem 7.** We will construct a 3-c.e. set F, a d.c.e. set E, and a c.e. set D such that the Turing degrees $\mathbf{f} = \deg(F \oplus E \oplus D)$, $\mathbf{e} = \deg(E \oplus D)$, and $\mathbf{d} = \deg(D)$ satisfy Theorem 7. We need to meet, for all Turing functionals Φ, Ψ, Π, Σ, and Ω, the following requirements:

$$\begin{aligned} \mathcal{R}_{\Phi,U} &: U = \Phi(F \oplus E \oplus D) \rightarrow \\ &\qquad \exists \Gamma\,(U = \Gamma(E \oplus D)) \text{ or } \exists \Delta\,(E = \Delta(U \oplus D)), \\ \mathcal{S}_{\Psi,V} &: V = \Psi(E \oplus D) \rightarrow \exists \Theta\,(V = \Theta(D)) \text{ or } \exists \Lambda\,(D = \Lambda(V)), \end{aligned}$$

$\mathcal{P}^F_\Pi : F \neq \Pi(E \oplus D)$,

$\mathcal{P}^E_\Sigma : E \neq \Sigma(D)$,

$\mathcal{P}^D_\Omega : D \neq \Omega$.

Here, the functionals Γ, Δ, Θ, and Λ will be built by us, i.e., by strategies on our tree of strategies as explained below.

3.2. **The intuition for the strategies for Theorem 7.**

3.2.1. *Strategies for $\mathcal{R}$ and $\mathcal{S}$ in isolation.* In the absence of other strategies, an $\mathcal{R}$-strategy will simply build its functional Γ as it sees $\Phi(F \oplus E \oplus D)$ compute U; it will ensure that $\Gamma(E \oplus D)$ correctly computes U by changing $E \oplus D$ whenever U changes (after an F-change). We proceed similarly for an $\mathcal{S}$-strategy in isolation.

3.2.2. *Strategies for $\mathcal{P}$ in isolation.* In the absence of other strategies, a $\mathcal{P}$-strategy is simply a Friedberg-Muchnik strategy: E.g., a $\mathcal{P}^F$-strategy picks a witness x targeted for F, waits for a computation $\Pi(E\oplus D; x) = 0$, and then enumerates x into F and protects the computation $\Pi(E \oplus D; x) = 0$ from now on by restraining E.

3.2.3. *$\mathcal{P}^F$-strategies below an $\mathcal{R}$-strategy.* There is an obvious conflict between the above-described $\mathcal{R}$- and $\mathcal{P}^F$-strategies: The former strategy may need to change $E\oplus D$ each time F changes (since the F-change allows a U-change) whereas the latter strategy is trying to change F while restraining $E \oplus D$. In the fashion of an infinite-injury priority argument, the solution is to cancel Γ whenever some $\mathcal{P}^F$-strategy has enumerated another witness x into F and found that it resulted in a change of U via some number u_x entering U, say. The $\mathcal{R}$-strategy will then start building its Turing functional Δ and generate an infinite "stream" of numbers y_x such that strategies below the Δ-outcome of the $\mathcal{R}$-strategy will be restricted to enumerating numbers from that stream into E. For numbers in the stream, we then have that $y_x \in E$ iff $x \notin F$ iff $u_x \notin U$, where the latter equivalence can be ensured by restraining F on all other numbers $\leq \varphi(u_x)$.

More precisely, a $\mathcal{P}^F$-strategy (below the Γ-outcome of an $\mathcal{R}$-strategy) proceeds as follows:

(1) Pick a witness x targeted for F.
(2) Wait for a computation $\Pi(E \oplus D; x) = 0$ at a stage s_x, say.
(3) Enumerate x into F and protect the computation $\Pi(E\oplus D; x) = 0$ by restraining $E \oplus D$.
Now the $\mathcal{R}$-strategy takes over and proceeds as follows:

(4) Wait for $\Phi(F \oplus E \oplus D; u)$ to be defined again for all u for which $\Gamma(E \oplus D; u)$ is currently defined (at a stage s'_x, say).
(5) If $\Phi(F \oplus E \oplus D; u) = \Gamma(E \oplus D; u)$ for all these u, then the $\mathcal{P}^F$-strategy's diagonalization succeeded in a finitary way, and the $\mathcal{P}^F$-strategy stops.
(6) Otherwise, cancel Γ and let u_x be the least u with $\Phi(F \oplus E \oplus D; u) \neq \Gamma(E \oplus D; u) \downarrow$. Set $\Delta(U \oplus D; y_x) = E(y_x)$ with use $\delta(y_x) = u_x$ for some number y_x bigger than any number mentioned so far. For all $y < y_x$ for which $\Delta(D \oplus U; y)$ is currently undefined, define $\Delta(D \oplus U; y) = E(y)$ with use 0 (since $E(y)$ will not change any more for those y). Create a *triple* $\langle x, u_x, y_x \rangle$.
(7) Allow the strategies below the Δ-outcome to be eligible to act once and go back to Step 1.

Note that the $\mathcal{R}$-strategy has three possible outcomes:

finite: There are only finitely many expansionary stages: Then $\Phi(F \oplus E \oplus D) \neq U$.

Γ: There are infinitely many expansionary stages, but Γ is canceled only finitely often by a $\mathcal{P}$-strategy below the Γ-outcome of the $\mathcal{R}$-strategy, and so all these $\mathcal{P}^F$-strategies eventually wait at Step 2 forever or stop at Step 5: Then $\Gamma(E \oplus D)$ can compute U correctly, and all $\mathcal{P}^F$-requirements can be satisfied.

Δ: There are infinitely many expansionary stages, and Γ is canceled infinitely often: Then $\Delta(U \oplus D)$ is defined infinitely often and, by the way the uses are chosen, is a total function. Every time the parameter s'_x is set, the set $E \upharpoonright (y_x+1)$ can change only at a number of the form y_x from now on. For such numbers, we will ensure for all future expansionary stages s:

$$\begin{aligned} x \in F_s &\implies \Phi(F \oplus E \oplus D; u_x)[s] = \Phi(F \oplus E \oplus D; u_x)[s'_x] \\ &\implies \Delta(U \oplus D; y_x)[s] = E_s(y_x); \text{ and} \\ x \notin F_s &\implies \Phi(F \oplus E \oplus D; u_x)[s] = \Phi(F \oplus E \oplus D; u_x)[s_x] \\ &\implies \Delta(U \oplus D; y_x)[s] = E_s(y_x). \end{aligned}$$

Thus $\Delta(U \oplus D)$ correctly computes E.

A $\mathcal{P}^F$-strategy below the Δ-outcome can act as if in isolation since it does not have to worry about Γ-correction injuring its $E \oplus D$-restraint.

3.2.4. *$\mathcal{P}^E$-strategies below an $\mathcal{R}$-strategy.* A $\mathcal{P}^E$-strategy below the Γ-outcome of an $\mathcal{R}$-strategy can act as if in isolation since it does not have to worry about Γ-correction injuring its D-restraint.

A $\mathcal{P}^E$-strategy below the Δ-outcome of an $\mathcal{R}$-strategy has to use witnesses of the form y_x from triples supplied by the $\mathcal{R}$-strategy. When the $\mathcal{P}^E$-strategy is ready to enumerate y_x (since $\Sigma(D; y_x) = 0$), it will simultaneously extract x from F to ensure Δ-correction as explained in section 3.2.3, namely, we can reset $\Delta(U \oplus D; y_x) = 1$ with use $\delta(y_x) = u_x$.

3.2.5. *$\mathcal{P}^D$-strategies below an $\mathcal{R}$-strategy.* There is no conflict between these strategies.

3.2.6. *$\mathcal{P}$-strategies below an $\mathcal{S}$-strategy.* This situation is exactly like the situation for $\mathcal{P}^F$- and $\mathcal{P}^E$-strategies below an $\mathcal{R}$-strategy, with E, D, V, and Ψ in place of F, E, U, and Φ, respectively. (And there is never a conflict between $\mathcal{P}^F$-strategies and an $\mathcal{S}$-strategy.)

We now analyze some typical conflicts between $\mathcal{P}$-strategies below more than one $\mathcal{R}$- or $\mathcal{S}$-strategy.

3.2.7. *$\mathcal{P}^E$-strategies below an $\mathcal{S}$-strategy below the Δ-outcome of an $\mathcal{R}$-strategy.* A $\mathcal{P}^E$-strategy below the Θ-outcome of such an $\mathcal{S}$-strategy has to deal with two problems simultaneously when enumerating a number y into E: It has to ensure that $\Delta(U \oplus D)$ can compute this, and that any V-change allowed by this E-change can be computed by $\Theta(D)$ in spite of the D-restraint imposed by the $\mathcal{P}^E$-strategy. We combine the ideas from sections 3.2.3 and 3.2.4 to let this $\mathcal{R}$-strategy act as follows:

(1) Pick a witness $y = y_x$ targeted for E associated with a triple $\langle x, u_x, y_x \rangle$ supplied by a $\mathcal{P}^F$-strategy below the Γ-outcome of the $\mathcal{R}$-strategy.
(2) Wait for a computation $\Sigma(D; y) = 0$ at a stage s_y, say.
(3) Enumerate y into E, extract x from F, and protect the computation $\Sigma(D; y) = 0$ by restraining D.
Now the $\mathcal{S}$-strategy takes over and proceeds as follows:
(4) Wait for $\Psi(E \oplus D; v)$ to be defined again for all v for which $\Theta(D; v)$ is currently defined (at a stage s'_y, say).
(5) If $\Psi(E \oplus D; v) = \Theta(D; v)$ for all these v, then the $\mathcal{P}^E$-strategy's diagonalization succeeded in a finitary way, and we stop.
(6) Otherwise, cancel Θ and let v_y be the least v with $\Psi(E \oplus D; v) \neq \Theta(D; v)$. Set $\Lambda(V; z_y) = D(z_y)$ with use $\lambda(z_y) = v_y$ for some number z_y bigger than any number mentioned so far. For all $z < z_y$ for which $\Lambda(V; z)$ is currently undefined, define $\Lambda(V; z) = D(z)$ with use 0 (since $D(z)$ will not change any more for those z). Create a *quintuple* $\langle x, u_x, y_x, v_y, z_y \rangle$.
(7) Allow the strategies below the Λ-outcome of the $\mathcal{S}$-strategy to be eligible to act once and go back to Step 1.

Note that the $\mathcal{S}$-strategy has three possible outcomes:

finite: There are only finitely many expansionary stages: Then $\Psi(E \oplus D) \neq V$.

Θ**:** There are infinitely many expansionary stages, but Θ is canceled only finitely often by a $\mathcal{P}^E$-strategy below the Θ-outcome of the $\mathcal{S}$-strategy, and so all these $\mathcal{P}^E$-strategies eventually wait at Step 2 forever or stop at Step 5: Then $\Theta(D)$ can compute V correctly, and all $\mathcal{P}^E$-requirements can be satisfied.

Λ**:** There are infinitely many expansionary stages, and Θ is canceled infinitely often: Then $\Lambda(V)$ is defined infinitely often and, by the way the uses are chosen, is a total function. Every time the parameter s'_y is set, the set $D \upharpoonright (z_y + 1)$ can change only at a number of the form z_y from now on. For such numbers, we will ensure for all future expansionary stages s:

$$\begin{aligned} y \in E_s &\implies \Psi(E \oplus D; v_y)[s] = \Psi(E \oplus D; v_y)[s'_y] \\ &\implies \Lambda(V; z_y)[s] = D_s(z_y); \text{ and} \\ y \notin E_s &\implies \Psi(E \oplus D; v_y)[s] = \Psi(E \oplus D; v_y)[s_y] \\ &\implies \Lambda(V; z_y)[s] = D_s(z_y). \end{aligned}$$

Thus $\Lambda(V)$ correctly computes D.

A $\mathcal{P}^E$-strategy below the Λ-outcome can act as if being only below an $\mathcal{R}$-strategy since it does not have to worry about Θ-correction injuring its D-restraint.

3.2.8. *$\mathcal{P}^D$-strategies below an $\mathcal{S}$-strategy below the Δ-outcome of an $\mathcal{R}$-strategy.* A $\mathcal{P}^D$-strategy below the Θ-outcome of such an $\mathcal{S}$-strategy can act as if in isolation since it does not impose any restraint.

A $\mathcal{P}^D$-strategy below the Λ-outcome of an $\mathcal{S}$-strategy has to use witnesses of the form z_y from quintuples supplied by the $\mathcal{S}$-strategy. When the $\mathcal{P}^D$-strategy is ready to enumerate z_y (since $\Omega(z_y) = 0$), it will simultaneously extract y from E and re-enumerate x into F to ensure Λ- and Δ-correction as explained in sections 3.2.4 and 3.2.6.

3.2.9. *$\mathcal{P}^E$-strategies below an $\mathcal{S}$-strategy below the Δ-outcomes of an $\mathcal{R}_1$- and an $\mathcal{R}_0$-strategy.* A $\mathcal{P}^E$-strategy below the Θ-outcome of such an $\mathcal{S}$-strategy has to deal with three problems simultaneously when enumerating a number y into E: It has to ensure that the $\mathcal{R}_i$-strategy's $\Delta_i(U_i \oplus D)$ can compute this (for $i = 0, 1$), and that any V-change allowed by this E-change can be computed by $\Theta(D)$ in spite of the D-restraint imposed by the $\mathcal{P}^E$-strategy. The extra difficulty in this situation arises from the need not to let the two $\mathcal{R}$-strategies' numbers x_0 and x_1 (supplied by $\mathcal{P}^F_0$-strategies below the Γ-outcome of the $\mathcal{R}_0$-strategy, and $\mathcal{P}^F_1$-strategies below the Γ-outcome of the $\mathcal{R}_1$-strategy, respectively, working as described in section 3.2.3) interfere with each other and destroy the F-uses controlling U_i at $u_i = u_{x_i}$ (for $i = 0, 1$). We overcome this problem by picking the witnesses in reverse order and then enumerating them into F in the other order so that the uses for $u_1 = u_{x_1}$ are not affected by x_0:

(1) Wait for a $\mathcal{P}_1^F$-strategy below the Γ-outcome of the $\mathcal{R}_1$-strategy to have picked a witness x_1 targeted for E for which there is now a computation $\Pi_1(E \oplus D; x_1) = 0$ at a stage s_{x_1}, say. Do not let this $\mathcal{P}_1^F$-strategy enumerate x_1 into F yet.
(2) Wait for a $\mathcal{P}_0^F$-strategy below the Γ-outcome of the $\mathcal{R}_0$-strategy to have picked a witness x_0 targeted for E which is greater than all numbers mentioned by stage s_{x_1} and for which there is now a computation $\Pi_0(E \oplus D; x_0) = 0$ at a stage s_{x_0}, say.
(3) Let this $\mathcal{P}_0^F$-strategy enumerate x_0 into F and check whether there is some least number $u_0 = u_{x_0}$ with $\Gamma_0(E \oplus D; u_0) \neq \Phi_0(F \oplus E \oplus D; u_0)$. If not, then cancel x_0 and return to the beginning of Step 2.
(4) Otherwise, let the above $\mathcal{P}_1^F$-strategy enumerate x_1 into F. Now check whether this results in U_0 changing at the above number u_0 again. If so, then we force a permanent diagonalization for $\mathcal{R}_0$ as follows: Extract x_1 from F to force a third U_0-change at u_0. When (and if) this occurs, then extract x_0 from F to force an (impossible) fourth U_0-change at u_0. Once we have this permanent diagonalization, we stop.
(5) Otherwise, check whether there is some least number $u_1 = u_{x_1}$ with $\Gamma_1(E \oplus D; u_1) \neq \Phi_1(F \oplus E \oplus D; u_1)$. If not, then cancel x_1 and return to the beginning of Step 1.
(6) Otherwise, pick a single witness $y = y_{x_0} = y_{x_1}$ targeted for E bigger than any number mentioned so far, associate y with the triples $\langle x_0, u_0, y\rangle$ and $\langle x_1, u_1, y\rangle$, and define $\Delta_i(U_i \oplus D; y) = 0$ with use $\delta_i(y) = u_i$ (for $i = 0, 1$).
(7) Wait for a computation $\Sigma(D; y) = 0$ at a stage s_y, say.
(8) Enumerate y into E, extract x_0 and x_1 from F, and protect the computation $\Sigma(D; y) = 0$ by restraining D.
(9) Wait for $\Psi(E \oplus D; v)$ to be defined again for all v for which $\Theta(D; v)$ is currently defined (at a stage s'_y, say).
(10) If $\Psi(E \oplus D; v) = \Theta(D; v)$ for all these v, then the $\mathcal{P}^E$-strategy's diagonalization succeeded in a finitary way, and we stop.
(11) Otherwise, cancel Θ and let v_y be least such that $\Psi(E \oplus D; v_y) \neq \Theta(D; v_y)$. Set $\Lambda(V; z_y) = D(z_y)$ with use $\lambda(z_y) = v_y$ for some number z_y bigger than any number mentioned so far. For

all $z < z_y$ for which $\Lambda(V; z)$ is currently undefined, define $\Lambda(V; z) = D(z)$ with use 0 (since $D(z)$ will not change any more for those z). Create a *tuple* $\langle x_0, u_0, x_1, u_1, y, v_y, z_y\rangle$.

(12) Allow the strategies below the Λ-outcome of the $\mathcal{S}$-strategy to be eligible to act once and go back to Step 1.

3.2.10. *$\mathcal{P}^D$-strategies below an $\mathcal{S}$-strategy below the Δ-outcomes of an $\mathcal{R}_0$- and an $\mathcal{R}_1$-strategy.* A $\mathcal{P}^D$-strategy below the Λ-outcome of an $\mathcal{S}$-strategy has to use witnesses of the form z_y from tuples supplied by the $\mathcal{S}$-strategy. When the $\mathcal{P}^D$-strategy is ready to enumerate z_y (since $\Omega(z_y) = 0$), it will simultaneously extract y from E and re-enumerate x_0 and x_1 into F to ensure Λ- as well as Δ_0- and Δ_1-correction as explained in sections 3.2.4 and 3.2.6 .

3.2.11. *$\mathcal{P}^D$-strategies below the Λ-outcomes of an $\mathcal{S}_1$- and an $\mathcal{S}_0$-strategy below the Δ-outcome of an $\mathcal{R}$-strategy.* A $\mathcal{P}^D$-strategy below the Λ-outcome of such an $\mathcal{S}$-strategy has to deal with two kinds of problems simultaneously when enumerating a number z into D: It has to ensure that the $\mathcal{S}_j$-strategy's $\Lambda_j(V_j)$ can compute this (for $j = 0, 1$), and that the extraction of y_j from E due to Λ_j-correction is accompanied by the re-enumeration of x_j into F to allow $\Delta_j(U_j)$ to compute this E-change. The extra difficulty in this situation arises from the need not to let the two $\mathcal{S}$-strategies' numbers y_0 and y_1 (supplied by a $\mathcal{P}_0^E$-strategy below the Θ-outcome of the $\mathcal{S}_0$-strategy, and a $\mathcal{P}_1^E$-strategy below the Θ-outcome of the $\mathcal{S}_1$-strategy, respectively, working as described in section 3.2.6) as well as the $\mathcal{R}$-strategy's two numbers x_0 and x_1 (supplied by a $\mathcal{P}_0^F$-strategy and a $\mathcal{P}_1^F$-strategy below the Γ-outcome of the $\mathcal{R}$-strategy, respectively, working as described in section 3.2.4) interfere with each other and destroy the E-uses controlling V_j at $v_j = v_{y_j}$ (for $j = 0, 1$) or the F-uses controlling U at $u_j = u_{x_j}$ (for $j = 0, 1$). We overcome this problem by picking the witnesses in reverse order and then enumerating them into F and E in the other order so that the uses for $u_1 = u_{x_1}$ are not affected by x_0, the uses for $u_0 = u_{x_0}$ are not affected by y_1, and the uses for $v_1 = v_{y_1}$ are not affected by y_0:

(1) Wait for a $\mathcal{P}_1^F$-strategy below the Γ-outcome of the $\mathcal{R}$-strategy to have picked a witness x_1 targeted for F for which there is a computation $\Pi_1(E \oplus D; x_1) = 0$ at a stage s_{x_1}, say.

(2) Let this $\mathcal{P}_1^F$-strategy enumerate x_1 into F and check whether there is some least number $u_1 = u_{x_1}$ with $\Gamma(E \oplus D; u_1) \neq \Phi(F \oplus E \oplus D; u_1)$ (at a stage s'_{x_1}, say). If not, then cancel x_1 and return to the beginning of Step 1. Otherwise, reserve this x_1 and let it only be associated with a triple from Step 5.

(3) Wait for a $\mathcal{P}_0^F$-strategy below the Γ-outcome of the $\mathcal{R}$-strategy to have picked a witness x_0 targeted for F which is greater than all numbers mentioned by stage s'_{x_1} and for which there is a computation $\Pi_0(E \oplus D; x_0) = 0$ at a stage s_{x_0}, say.

(4) Let this $\mathcal{P}_0^F$-strategy enumerate x_0 into F and check whether there is some least number $u_0 = u_{x_0}$ with $\Gamma(E \oplus D; u_0) \neq \Phi(F \oplus E \oplus D; u_0)$ (at a stage s'_{x_0}, say). If not, then cancel x_0 and return to the beginning of Step 3. Otherwise, reserve this x_0 and let it only be associated with a triple from Step 6.

(5) Wait for a $\mathcal{P}_1^E$-strategy below the Θ-outcome of the $\mathcal{S}_1$-strategy to have picked a witness $y_1 = y_{x_1}$ targeted for E and associated with the triple $\langle x_1, u_1, y_1 \rangle$ which is greater than all numbers mentioned by stage s'_{x_0} and for which there is now a computation $\Sigma_1(D; y_1) = 0$ at a stage s_{y_1}, say. Define $\Delta(U \oplus D; y_1) = 0$ with use $\delta(y_1) = u_1$. Do not let the $\mathcal{P}_1^E$-strategy enumerate y_1 into E yet.

(6) Now wait for a $\mathcal{P}_0^E$-strategy below the Θ-outcome of the $\mathcal{S}_0$-strategy to have picked a witness $y_0 = y_{x_0}$ targeted for E and associated with the triple $\langle x_0, u_0, y_0 \rangle$ which is greater than all numbers mentioned by stage s_{y_1} and for which there is now a computation $\Sigma_0(D; y_0) = 0$ at a stage s_{y_0}, say. Define $\Delta(U \oplus D; y_0) = E(y_0)$ with use $\delta(y_0) = u_0$.

(7) Now let the $\mathcal{P}_0^E$-strategy enumerate y_0 into E and extract x_0 from F.

(8) Check whether there is some least $v_0 = v_{y_0}$ with $\Theta_0(D; v_0) \neq \Psi_0(E \oplus D; v_0)$ (at a stage s'_{y_0}, say). (Note that the enumeration of x_0 into F does *not* allow V_1 to change at v_1 since the oracle of Ψ_1 includes only E and D, and not F.) If not, then cancel x_0, y_1, and y_0, and return to Step 3.

(9) Otherwise, let the $\mathcal{P}_1^E$-strategy enumerate y_1 into E and extract x_1 from F.

(10) Now first check whether this results in V_0 changing at the above number v_0 again. If so, then we force a permanent diagonalization for $\mathcal{S}_0$ as follows: Extract y_1 from E and re-enumerate x_1 into F to force a third V_0-change at v_0. When (and if) this occurs, then extract y_0 from E and re-enumerate x_0 into F to force an (impossible) fourth V_0-change at v_0. Once we have this permanent diagonalization, we stop.

(11) Otherwise, check whether there is some least $v_1 = v_{y_1}$ with $\Theta_1(D; v_1) \neq \Psi_1(E \oplus D; v_1)$ (at a stage s'_{y_1}, say). If not, then cancel x_1, x_0, y_1, and y_0, and return to Step 1.

(12) Otherwise, set $\Lambda_j(V_j; z^*) = 0$ with use $\lambda_j(z^*) = v_j$ (for $j = 0, 1$) for some number z^* greater than all numbers mentioned by stage s'_{y_1}. For all $z < z^*$ and $j = 0, 1$ for which $\Lambda_j(V_j; z)$ is currently undefined, define $\Lambda_j(V_j; z) = D(z)$ with use 0 (since $D(z)$ will not change any more for those z). Create a *tuple*

$$\langle x_0, u_0, x_1, u_1, y_0, v_0, y_1, v_1, z^* \rangle.$$

(13) Wait for a computation $\Omega(z^*) = 0$.

(14) Enumerate z^* into D, extract y_0 and y_1 from E, re-enumerate x_0 and x_1 into F, and stop.

3.2.12. *$\mathcal{P}^D$-strategies below two $\mathcal{S}$- and four $\mathcal{R}$-strategies.* In our next intuitive example, we will combine the techniques from sections 3.2.9 and 3.2.11 and consider a $\mathcal{P}^D$-strategy below the Λ-outcome of an $\mathcal{S}_1$-strategy which is located below, in increasing order of priority, the Δ-outcomes of an $\mathcal{R}_3$- and an $\mathcal{R}_2$-strategy, the Λ-outcome of an $\mathcal{S}_0$-strategy, and the Δ-outcomes of an $\mathcal{R}_1$- and an $\mathcal{R}_0$-strategy.

A $\mathcal{P}^D$-strategy below the Λ-outcome of such an $\mathcal{S}$-strategy again has to deal with two kinds of problems simultaneously when enumerating a number z into D: It has to ensure that the $\mathcal{S}_j$-strategy's $\Lambda_j(V_j)$ can compute this (for $j = 0, 1$), and that the extraction of y_j from E due to Λ_j-correction is accompanied by re-enumeration into F to allow $\Delta_i(U_i)$ to compute this E-change. The extra difficulty in this situation arises from the need not to let the two $\mathcal{S}$-strategies' numbers y_0 and y_1 (supplied by $\mathcal{P}_0^E$-strategies below the Θ-outcome of the $\mathcal{S}_0$-strategy, and $\mathcal{P}_1^E$-strategies below the Θ-outcome of the $\mathcal{S}_1$-strategy, respectively, working as described in section 3.2.6) as well as the $\mathcal{R}_i$-strategy's numbers (supplied by a $\mathcal{P}^F$-strategy below the Γ-outcome of the $\mathcal{R}_i$-strategy, for $i = 0, 1, 2, 3$, working as described in section 3.2.6) interfere with each other and destroy the E-uses controlling V_j at $v_j = v_{y_j}$ (for $j = 0, 1$) or the F-uses controlling U_i (for $i = 0, 1, 2, 3$). We overcome this problem by picking the witnesses in reverse order and then enumerating them into F and E in the other order; we also pick two numbers each targeted for F to generate two numbers each in U_i (for $i = 0, 1$) for y_0 and y_1, respectively:

(1) For $i = 3, 2, 1, 0$ (in decreasing order):
   (a) Wait for a $\mathcal{P}_i^F$-strategy below the Γ-outcome of the $\mathcal{R}_i$-strategy to have picked a witness x_i targeted for F which is greater than all numbers mentioned by stage $s_{x_{i+1}}$ (if $i < 3$) and for which there is a computation $\Pi_i(E \oplus D; x_i) = 0$ at a stage s_{x_i}, say. Do not let this $\mathcal{P}_i^F$-strategy enumerate x_i into F yet.

(2) For $i = 0, 1, 2, 3$ (in increasing order):
   (a) Let this $\mathcal{P}_i^F$-strategy enumerate x_i into F.
   (b) For $k = 0, \ldots, i-1$ (in increasing order), check whether the enumeration of x_i into F has allowed another U_k-change at the number u_k found before. If so, then we force a permanent diagonalization for $\mathcal{R}_k$ as follows: Extract x_i from F to force a third U_k-change at u_k. When (and if) this occurs, then extract x_l for all $l \in (i, k]$ from F to force an (impossible) fourth U_k-change at u_k. Once we have this permanent diagonalization, we stop.
   (c) Otherwise check whether there is some least $u_i = u_{x_i}$ with $\Gamma_i(E \oplus D; u_i) \neq \Phi_i(F \oplus E \oplus D; u_i)$ (at a stage s'_{x_i}, say). If not, then cancel x_k (for $k \leq i$) and return to the beginning of Step 1 with this i.
   (d) Otherwise, reserve this x_i and let it only be associated with a tuple from Step 5.

(3) For $l = 1, 0$ (in decreasing order):
   (a) Wait for a $\mathcal{P}_l'^F$-strategy below the Γ-outcome of the $\mathcal{R}_l$-strategy to have picked a witness x'_l targeted for F which is greater than all numbers mentioned by stage s'_{x_0} (if $l = 1$) or stage $s_{x'_1}$ (if $= 0$) and for which there is a computation $\Pi'_l(E \oplus D; x'_l) = 0$ at a stage $s_{x'_l}$, say. Do not let this $\mathcal{P}_l'^F$-strategy enumerate x'_l into F yet.

(4) For $l = 0, 1$ (in increasing order):
   (a) Let this $\mathcal{P}_l'^F$-strategy enumerate x'_l into F.
   (b) If $l = 1$ then check whether the enumeration of x'_1 into F has allowed another U_0-change at the number u'_0 found before. If so, then we force a permanent diagonalization for $\mathcal{R}_0$ as follows: Extract x'_1 from F to force a third U_0-change at u'_0. When (and if) this occurs, then extract x'_0 from F to force an (impossible) fourth U_0-change at u'_0. Once we have this permanent diagonalization, we stop.
   (c) Otherwise check whether there is some least $u'_l = u'_{x'_l}$ with $\Gamma_l(E \oplus D; u'_l) \neq \Phi_l(F \oplus E \oplus D; u'_l)$ (at a stage $s'_{x'_l}$, say). If not, then cancel x'_m (for $m \leq l$) and return to the beginning of Step 3 with this l.
   (d) Otherwise, reserve this x'_l and let it only be associated with a tuple from Step 6.

(5) Wait for a $\mathcal{P}_1^E$-strategy below the Θ-outcome of the $\mathcal{S}_1$-strategy to have picked a witness y_1 targeted for E and associated with

the tuple

$$\langle x_0, u_0, x_1, u_1, x_2, u_2, x_3, u_3, y_1\rangle$$

which is greater than all numbers mentioned by stage $s'_{x'_0}$ and for which there is now a computation $\Sigma_1(D; y_1) = 0$ at a stage s_{y_1}, say. For all $i \le 3$, define $\Delta_i(U_i \oplus D; y_1) = 0$ with use $\delta_i(y_1) = u_i$. Do not let the $\mathcal{P}^E_1$-strategy enumerate y_1 into E yet.

(6) Now wait for a $\mathcal{P}^E_0$-strategy below the Θ-outcome of the $\mathcal{S}_0$-strategy to have picked a witness y_0 targeted for E and associated with the triple

$$\langle x'_0, u'_0, x'_1, u'_1, y_0\rangle$$

which is greater than all numbers mentioned by stage s_{y_1} and for which there is now a computation $\Sigma_0(D; y_0) = 0$ at a stage s_{y_0}, say. For all $l \le 1$, define $\Delta_l(U_l \oplus D; y_0) = 0)$ with use $\delta_l(y_0) = u'_l$.

(7) Now let the $\mathcal{P}^E_0$-strategy enumerate y_0 into E and extract x'_0 and x'_1 from F.

(8) Check whether there is some least $v_0 = v_{y_0}$ with $\Theta_0(D; v_0) \ne \Psi_0(E \oplus D; v_0)$ (at a stage s'_{y_0}, say). (Note that the enumeration of x'_0 and x'_1 into F does *not* allow V_1 to change at v_1 since the oracle of Ψ_1 includes only E and D, and not F.) If not, then cancel x'_1, x'_0, y_1, and y_0, and return to Step 3.

(9) Otherwise, let the $\mathcal{P}^E_1$-strategy enumerate y_1 into E and extract x_i (for all $i \le 3$) from F.

(10) Now check first whether this results in V_0 changing at the above number v_0 again. If so, then we force a permanent diagonalization for $\mathcal{S}_0$ as follows: Extract y_1 from E and re-enumerate x_i (for all $i \le 3$) into F to force a third V_0-change at v_0. When (and if) this occurs, then extract y_0 from E and re-enumerate x'_0 and x'_1 into F to force an (impossible) fourth V_0-change at v_0. Once we have this permanent diagonalization, we stop.

(11) Otherwise, check whether there is some least $v_1 = v_{y_1}$ with $\Theta_1(D; v_1) \ne \Psi_1(E \oplus D; v_1)$ (at a stage s'_{y_1}, say). If not, then cancel all x_i, x'_k, and y_l, and return to Step 1.

(12) Otherwise, set $\Lambda_l(V_l; z^*) = 0$ with use $\lambda_l(z^*) = v_l$ (for $l = 0, 1$) for some number z^* greater than all numbers mentioned by stage s'_{y_1}. Create a *tuple*

$$\langle x_0, u_0, x_1, u_1, x_2, u_2, x_3, u_3, x'_0, u'_0, x'_1, u'_1, y_0, v_0, y_1, v_1, z^*\rangle.$$

(13) Wait for a computation $\Omega(z^*) = 0$.

(14) Enumerate z^* into D, extract y_0 and y_1 from E, re-enumerate x_i and x'_l into F (for all $i \le 3$ and $l \le 1$), and stop.

3.2.13. *$\mathcal{P}^F$-strategy between a $\mathcal{P}^E$- and an $\mathcal{R}$-strategy.* We next need to analyze the interaction between an $\mathcal{R}$-strategy and a $\mathcal{P}^F$-strategy below its Δ-outcome in the case that some $\mathcal{P}^E$-strategy below the Δ-outcome but of lower priority than the $\mathcal{P}^F$-strategy has already enumerated a number. The problem is that in that case, the late enumeration of a witness x, say, by the $\mathcal{P}^F$-strategy may destroy the setup for the definition of $\Delta(U \oplus D; y)$ for the $\mathcal{P}^E$-strategy's witness y (where y was picked after x, and so x may be less than the number u_y in the triple $\langle x_y, u_y, y\rangle$ used by the $\mathcal{P}^E$-strategy) so that we lose control over the computation $\Delta(U \oplus D; y)$. However, in this case, we can easily diagonalize for $\mathcal{R}$.

More specifically, the $\mathcal{P}^F$-strategy proceeds as follows:

(1) Pick a witness x.
(2) Wait for a computation $\Pi(E \oplus D; x) = 0$.
(3) Enumerate x into F at a stage s'_x, say.
Now the $\mathcal{R}$-strategy takes over and proceeds as follows:
(4) Wait for $\Phi(F \oplus E \oplus D; u)$ to be defined again for all $u \leq$ any number mentioned by stage s'_x.
(5) Check if $\Delta(U \oplus D; y) \downarrow\neq E(y)$ for some y. If not, then the $\mathcal{P}^F$-strategy's diagonalization succeeded in a finitary way, and the $\mathcal{P}^F$-strategy stops.
(6) Otherwise, let $i = \Delta(U \oplus D; y)$ for the least such y. Then there is are earlier stages $s_0 < s_1$, say, at which we set $\Delta(U \oplus D; y) = i$ and $\Delta(U \oplus D; y) = 1 - i$, respectively. So at stage s_1, we must have changed $E(y)$ from value i to value $1-i$, and, in order to correct $\Delta(U \oplus D; y)$, for some numbers x_y and u_y, we must have changed $F(x_y)$ to force $U(u_y)$ to change for at least the second time, while, since now again $\Delta(U \oplus D; y) = i$, $U(u_y)$ has now changed for at least the third time. We now force a permanent diagonalization for $\mathcal{R}$ by extracting x from F and stop.

A $\mathcal{P}^F$-strategy below the Δ-outcome of an $\mathcal{R}$-strategy is thus still finitary in any case, but it has the potential to force a finite outcome for the $\mathcal{R}$-strategy.

3.2.14. *$\mathcal{P}^E$-strategy between a $\mathcal{P}^D$- and an $\mathcal{S}$-strategy.* An analogous situation arises in the interaction between an $\mathcal{S}$-strategy and a $\mathcal{P}^E$-strategy below its Θ-outcome in the case that some $\mathcal{P}^D$-strategy below the Θ-outcome but of lower priority than the $\mathcal{P}^E$-strategy has already enumerated a number.

We are now ready to describe the construction formally.

3.3. **The tree of strategies for Theorem 7.** We fix an arbitrary computable priority ordering of all $\mathcal{R}$-, $\mathcal{S}$-, and $\mathcal{P}$-requirements of order type ω. Let

$$O = \{\Lambda <_O \Theta <_O \Delta <_O \Gamma <_O \mathit{fin}\}$$

be the *set of outcomes* of our strategies, with the priority ordering $<_O$ as stated.

For the tree of strategies $T \subseteq O^{<\omega}$ to be defined, assign the nth requirement from our above listing of all requirements to all nodes $\xi \in T$ of length n.

The *tree of strategies* is now defined by recursion as follows: Given a node (or "strategy") ξ already on T, we let the set of immediate successors of ξ be

$$\begin{array}{ll} \{\xi\frown\langle\Delta\rangle, \xi\frown\langle\Gamma\rangle, \xi\frown\langle \mathit{fin}\rangle\}, & \text{if } \xi \text{ is an } \mathcal{R}\text{-strategy;} \\ \{\xi\frown\langle\Lambda\rangle, \xi\frown\langle\Theta\rangle, \xi\frown\langle \mathit{fin}\rangle\}, & \text{if } \xi \text{ is an } \mathcal{S}\text{-strategy;} \\ \{\xi\frown\langle \mathit{fin}\rangle\}, & \text{if } \xi \text{ is a } \mathcal{P}\text{-strategy.} \end{array}$$

We now fix some notation for our tree: Given any strategy $\eta \in T$, we let

$$X(\eta) = \{\xi \mid \xi \text{ is an } \mathcal{R}\text{-strategy and } \xi\frown\langle\Delta\rangle \subseteq \eta\}, \text{ and}$$
$$Y(\eta) = \{\xi \mid \xi \text{ is an } \mathcal{S}\text{-strategy and } \xi\frown\langle\Theta\rangle \subseteq \eta\}$$

be the sets of $\mathcal{R}$- and $\mathcal{S}$-strategies with Δ- and Θ-outcome above η, respectively. (These will be the strategies to which η may have to make *x-requests* or *y-requests*, respectively, as described in the next section.)

3.4. **The full strategies for Theorem 7.** In this section, we describe in full detail the action of the individual strategies, while in the next section, we will give the full construction in terms of the description of the strategies.

3.4.1. *The general $\mathcal{R}$-strategy.* An $\mathcal{R}_{\Phi,U}$-strategy $\xi \in T$ acts as follows at substage t of stage s at which it is eligible to act:

(1) Check whether the *length of agreement* $\ell(\Phi(F \oplus E \oplus D), U)$ between $\Phi(F \oplus E \oplus D)$ and U now exceeds any number mentioned by the end of the previous ξ-expansionary stage, which we will call s'. (Here, 0 is always a *ξ-expansionary stage.*) If not, then end the substage and let $\xi\frown\langle \mathit{fin}\rangle$ be *eligible to act next*.
(2) Otherwise, call s a *ξ-expansionary stage* and check whether there is some number y such that $\Delta(U \oplus D; y) \downarrow\neq E(y)$. If so, then let y be least such and fix $i = \Delta(U \oplus D; y)$. There must be stages $s_0 < s_1 < s$ such that at stage s_0, we set $\Delta(U \oplus D; y) = i$,

and at stage s_1, we set $\Delta(U \oplus D; y) = 1 - i$, but now $(U \oplus D) \upharpoonright (\delta_{s_0}(y) + 1) = (U \oplus D)_{s_0} \upharpoonright (\delta_{s_0}(y) + 1)$. Also, at or just before stage s_1, $E(y)$ must have been changed from value i to value $1-i$ by a $\mathcal{P}^E$- or $\mathcal{S}$-strategy $\zeta \supseteq \xi^\frown\langle\Delta\rangle$ (otherwise, ξ would have been initialized since stage s_0, or $\Delta(U \oplus D; y)$ would not have been redefined at stage s_1), and this $E(y)$-change must have been accompanied by an F-change at some number x_y, which forced a U-change at some number u_y (this being at least the second $U(u_y)$-change). Now, since stage s_1, there must have been a third $U(u_y)$-change (with a corresponding $\Phi(F \oplus E \oplus D; u_y)$-change). This $\Phi(F \oplus E \oplus D; u_y)$-change must be due to the action of some strategy $\zeta' \supseteq \xi^\frown\langle\Delta\rangle$ of higher priority than ζ. We will prove in Lemma 15 that ζ' must be a $\mathcal{P}^F$-strategy which enumerates a number x, say, into F. We now permanently diagonalize for ξ by extracting x from F and *end the stage*. (Unless ξ is initialized later, this will be the last ξ-expansionary stage.)

(3) Otherwise, check whether ξ has previously, at a stage s'', say, fulfilled an x-request of some $\mathcal{P}^E$- or $\mathcal{S}$-strategy η (where $\xi \in X(\eta)$) with a pair $\langle x, u\rangle$ such that
  (a) η has not been initialized since stage s'';
  (b) η had also made an x-request to some $\mathcal{R}$-strategy $\hat{\xi} \in X(\eta)$ with $\hat{\xi} \supset \xi$;
  (c) $U(u)$ has changed since the last ξ-expansionary stage s'; and either
  (d) $\hat{\xi}$ enumerated a number $\hat{x}$ into F at stage s'; or
  (e) ξ started diagonalization via the pair $\langle x, u\rangle$ at stage s' (as defined below), and $\hat{\xi}$ enumerated a number $\hat{x}$ into F at the last ξ-expansionary stage before stage s'.

If Case 3d holds, then
- extract $\hat{x}$ from F;
- if η is a $\mathcal{P}^E$-strategy then cancel η's x-requests to all $\hat{\xi} \in X(\eta)$;
- if η is an $\mathcal{S}$-strategy then
  - let ζ be the $\mathcal{P}^D$-strategy such that $\eta \in Y(\zeta)$ and ζ made the y-request to η, causing η making the x-request to ξ;
  - cancel ζ's y-requests to all $\hat{\eta} \in Y(\zeta)$; and
  - for all $\hat{\eta} \in Y(\zeta)$, cancel $\hat{\eta}$'s x-requests to all $\hat{\xi} \in X(\hat{\eta})$;
- initialize all strategies $\geq \xi^\frown\langle \mathit{fin}\rangle$; and
- *end the stage.*

(In this case, we say ξ has *started diagonalization via the pair* $\langle x, u\rangle$.)
If Case 3e holds, then
- extract $\tilde{x}$ from F for all $\mathcal{R}$-strategies $\tilde{\xi} \in X(\eta)$ with $\xi \subseteq \tilde{\xi} \subset \hat{\xi}$ (where $\tilde{x}$ is the number last enumerated by $\tilde{\xi}$ into F);
- initialize all strategies $\geq \xi^\frown\langle fin\rangle$; and
- *end the stage.*

(Unless ξ is initialized later, this will be the last ξ-expansionary stage.)

(4) Otherwise, check whether there is some (highest-priority) $\mathcal{P}^E$- or $\mathcal{S}$-strategy η such that
  (a) $\xi \in X(\eta)$;
  (b) all of η's x-requests to strategies $\hat{\xi} \in X(\eta)$ with $\xi \subset \hat{\xi} \subset \eta$ have been started (as defined below);
  (c) if η is an $\mathcal{S}$-strategy and so some $\mathcal{P}^D$-strategy $\zeta \supseteq \eta^\frown\langle\Lambda\rangle$ has made a y-request to η, then for all $\mathcal{S}$-strategies $\hat{\eta} \in Y(\zeta)$ with $\hat{\eta} \supset \eta$, all of $\hat{\eta}$'s x-requests have already been fulfilled (as defined below); and either
  (d) η has made an x-request to ξ, and that request has not yet been started by ξ (as defined below); or
  (e) η has made an x-request to ξ, and that request has been started by ξ but not yet fulfilled by ξ (as defined below).

(5) If there is no such η satisfying Case 4e, then check whether there is some u such that $\Phi(F \oplus E \oplus D; u) \downarrow \neq U(u)$.
  (a) If there is no such u, then
    - define $\Gamma(E \oplus D; u') = U(u')$ for all $u' \leq \ell(\Phi(F \oplus E \oplus D), U)$ for which $\Gamma(E \oplus D; u')$ is currently undefined, with old use $\gamma(u')$ (if defined previously), or new use $\gamma(u')$ bigger than any number mentioned before (otherwise);
    - end the substage; and
    - let $\xi^\frown\langle\Gamma\rangle$ be *eligible to act next.*

  If later during this stage, some $\mathcal{P}^F$-strategy $\zeta \supseteq \xi^\frown\langle\Gamma\rangle$ wants to enumerate a number x into F but is delayed, then we say that ξ has *started η's x-request* (where η is as above satisfying Case 4d).
  (b) Otherwise, i.e., if there is such u, then
    - cancel Γ;
    - define $\Delta(U \oplus D; y) = E(y)$ for all $y \leq s$ for which $\Delta(U \oplus D; y)$ is currently undefined, with old use $\delta(y)$

(if defined previously), or new use $\delta(y)$ bigger than any number mentioned before (otherwise);
- end the substage; and
- let $\xi^\frown\langle\Delta\rangle$ be *eligible to act next.*

(6) If there is such η satisfying Case 4e, and if ξ has not enumerated a number x into F at the previous ξ-expansionary stage s', then there is a $\mathcal{P}^F$-strategy $\zeta \supseteq \xi^\frown\langle\Gamma\rangle$ which found a computation $\Pi(E\oplus D; x) = 0$ at the last substage of stage s' (but was delayed in the enumeration of x into F). Now ξ
- enumerates this x into F;
- initializes all strategies $\geq \xi^\frown\langle\Gamma\rangle$; and
- *ends the stage.*

(7) Otherwise, ξ has enumerated a number x into F at the last substage of the previous ξ-expansionary stage s'. Check whether there is some u such that $\Phi(F \oplus E \oplus D; u) \downarrow\neq U(u)$.
   (a) If there is no such u, then ξ's attempt at fulfilling η's x-request has failed. So
   - let η again *make x-requests* (which have not been started) to all $\mathcal{R}$-strategies $\hat{\xi} \in X(\eta)$ with $\hat{\xi} \subseteq \xi$;
   - initialize all strategies $\geq \xi^\frown\langle\Gamma\rangle$; and
   - *end the stage.*

   (b) If there is a (least) such u, then we say that ξ has *fulfilled η's x-request* with the pair $\langle x, u\rangle$. Now ξ continues as in Step 5b.

3.4.2. *The general $\mathcal{S}$-strategy.* An $\mathcal{S}_{\Psi,V}$-strategy $\xi \in T$ acts as follows at substage t of stage s at which it is eligible to act:

(1) Check whether the *length of agreement* $\ell(\Psi(E\oplus D), V)$ between $\Psi(E \oplus D)$ and V now exceeds any number mentioned by the end of the previous ξ-expansionary stage, which we will call s'. (Here, 0 is always a *ξ-expansionary stage.*) If not, then end the substage and let $\xi^\frown\langle fin\rangle$ be *eligible to act next.*

(2) Otherwise, call s a *ξ-expansionary stage* and check whether there is some number z such that $\Lambda(V; z) \downarrow\neq D(z)$. If so, then let z be least such. Since D is c.e., there must be stages $s_0 < s_1 < s$ such that at stage s_0, we set $\Lambda(V; z) = 0$, and at stage s_1, we set $\Lambda(V; z) = 1$, but now $V \restriction (\lambda_{s_0}(z)+1) = V_{s_0} \restriction (\lambda_{s_0}(z)+1)$. Also, at or just before stage s_1, z must have been enumerated into D by a $\mathcal{P}^D$-strategy $\zeta \supseteq \xi^\frown\langle\Lambda\rangle$ (otherwise, ξ would have been initialized since stage s_0, or $\Lambda(V; z)$ would not have been redefined at stage s_1), and this $D(z)$-change must have been accompanied by an E-change at some number y_z, which forced

a V-change at some number v_z (this being at least the second $V(v_z)$-change). Now, since stage s_1, there must have been a third $V(v_z)$-change (with a corresponding $\Psi(E \oplus D; v_z)$-change). This $\Psi(E \oplus D; v_z)$-change must be due to the action of some strategy $\zeta' \supseteq \xi^\frown\langle\Lambda\rangle$ of higher priority than ζ. We will prove in Lemma 16 that ζ' must be a $\mathcal{P}^E$-strategy which enumerates a number y, say, into E. We now permanently diagonalize for ξ by extracting y from E and re-enumerating into F all x which were extracted when y was enumerated into E, and *end the stage*. (Unless ξ is initialized later, this will be the last ξ-expansionary stage.)

(3) Otherwise, check whether ξ has previously, at a stage s'', say, fulfilled a y-request of some $\mathcal{P}^D$-strategy η (such that $\xi \in Y(\eta)$) with a pair $\langle y, v\rangle$ such that
   (a) η has not been initialized since stage s'';
   (b) η had also made a y-request to some $\mathcal{S}$-strategy $\hat{\xi} \in Y(\eta)$ with $\hat{\xi} \supset \xi$;
   (c) $V(v)$ has changed since the previous ξ-expansionary stage s'; and either
   (d) $\hat{\xi}$ enumerated a number $\hat{y}$ into E at stage s'; or
   (e) ξ started diagonalization via the pair $\langle y, v\rangle$ at stage s' (as defined below, and so $\hat{\xi}$ enumerated a number $\hat{y}$ into E at the last ξ-expansionary stage before stage s').

If Case 3d holds, then
- extract $\hat{y}$ from E;
- re-enumerate into F all $\hat{x}$ which were extracted from F at the time $\hat{y}$ was enumerated into E;
- cancel η's y-requests to all $\hat{\xi} \in Y(\eta)$;
- for all $\hat{\xi} \in Y(\eta)$, cancel $\hat{\xi}$'s x-requests to all $\zeta \in X(\hat{\xi})$;
- initialize all strategies $\geq \xi^\frown\langle fin\rangle$; and
- *end the stage.*

(In this case, we say ξ has *started diagonalization via the pair* $\langle y, v\rangle$.)

If Case 3e holds, then
- extract from E all numbers $\tilde{y}$ for all $\mathcal{S}$-strategies $\tilde{\xi} \in Y(\eta)$ with $\xi \subseteq \tilde{\xi} \subset \hat{\xi}$ (where $\tilde{y}$ is the number last enumerated into E by $\tilde{\xi}$);
- re-enumerate into F all numbers $\tilde{x}$ which were extracted from F when any of these $\tilde{y}$ was enumerated into E;
- initialize all strategies $\geq \xi^\frown\langle fin\rangle$; and
- *end the stage.*

(Unless ξ is initialized later, this will be the last ξ-expansionary stage.)

(4) Otherwise, check whether there is some (highest-priority) $\mathcal{P}^D$-strategy $\eta \supseteq \xi^\frown\langle\Delta\rangle$ such that
   (a) $\xi \in Y(\eta)$;
   (b) η's y-requests to all strategies $\hat{\xi} \in Y(\eta)$ with $\xi \subset \hat{\xi}$ have been started (as defined below);
   (c) all x-requests of all $\mathcal{S}$-strategies $\hat{\xi} \in Y(\eta)$ have been fulfilled; and either
   (d) η has made a y-request to ξ, and that request has not yet been started by ξ (as defined below); or
   (e) η has made a y-request to ξ, and that request has been started by ξ but not yet fulfilled by ξ (as defined below).

(5) If there is no such η satisfying Case 4e, then check whether there is some v such that $\Psi(E \oplus D; v)\downarrow\neq V(v)$.
   (a) If there is no such v, then
      - define $\Theta(D; v') = V(v')$ for all $v' \leq \ell(\Psi(E \oplus D), V)$ for which $\Theta(D; v')$ is currently undefined, with old use $\vartheta(v')$ (if defined previously), or new use $\vartheta(v')$ bigger than any number mentioned before (otherwise);
      - end the substage; and
      - let $\xi^\frown\langle\Theta\rangle$ be *eligible to act next.*

      If later during this stage, some $\mathcal{P}^E$-strategy $\zeta \supseteq \xi^\frown\langle\Theta\rangle$ wants to enumerate a number y into E but is delayed, then we say that ξ has *started η's y-request* (where η is as above satisfying Case 4d).
   (b) Otherwise, i.e., if there is such v, then
      - cancel Θ;
      - define $\Lambda(V; z) = D(z)$ for all $z \leq s$ for which $\Lambda(V; z)$ is currently undefined, with old use $\vartheta(z)$ (if defined previously), or new use $\vartheta(z)$ bigger than any number mentioned before (otherwise);
      - end the substage; and
      - let $\xi^\frown\langle\Lambda\rangle$ be *eligible to act next.*

(6) If there is such η satisfying Case 4e, and ξ has not enumerated a number y into E at the previous ξ-expansionary stage s', then there is a $\mathcal{P}^E$-strategy $\zeta \supseteq \xi^\frown\langle\Theta\rangle$ which found a computation $\Sigma(D; y) = 0$ at the last substage of stage s' (but was delayed in the enumeration of y into E). Now ξ
   - enumerates this y into E;

- extracts from F all x such that some $\mathcal{R}$-strategy $\zeta \subset \xi$ has fulfilled ξ's x-request via a pair $\langle x, u\rangle$;
- initializes all strategies $\geq \xi^\frown\langle\Theta\rangle$; and
- *ends the stage.*

(7) Otherwise, ξ has enumerated a number y into E at the last substage of the previous ξ-expansionary stage s'. Check whether there is some v such that $\Psi(E \oplus D; v)\downarrow\neq V(v)$.

   (a) If there is no such v, then ξ's attempt at fulfilling η's y-request has failed. So
   - let η again *make y-requests* (which have not been started) to *all* $\mathcal{S}$-strategies $\hat{\xi} \in Y(\eta)$;
   - let each $\hat{\xi} \in Y(\eta)$ with $\hat{\xi} \subseteq \xi$ again *make x-requests* (which have not been started) to all $\mathcal{R}$-strategies $\zeta \in X(\hat{\xi})$;
   - initialize all strategies $\geq \xi^\frown\langle\Theta\rangle$; and
   - *end the stage.*

   (b) If there is a (least) such v, then we say that ξ has *fulfilled η's y-request* with the pair $\langle y, v\rangle$. Now ξ continues as in Case 5b.

3.4.3. *The general $\mathcal{P}^F$-strategy.* A $\mathcal{P}^F_\Pi$-strategy $\xi \in T$ acts as follows at substage t of stage s at which it is eligible to act:

(1) If this is the first stage at which ξ is eligible to act since ξ was last initialized (and so ξ does not have a witness), then ξ chooses a witness x bigger than any number mentioned before and *ends the stage.*

(2) Otherwise, let x be ξ's current witness. Next check whether there is a computation $\Pi(E \oplus D; x)\downarrow= 0$. If not, then end the substage and let $\xi^\frown\langle fin\rangle$ be *eligible to act next.*

(3) Otherwise, check whether ξ has already enumerated x into F. If so, then end the substage and let $\xi^\frown\langle fin\rangle$ be *eligible to act next.*

(4) Otherwise, check whether there is an x-request from a strategy η to an $\mathcal{R}$-strategy $\zeta \in X(\eta)$ with $\zeta^\frown\langle\Gamma\rangle \subseteq \xi$. If not, then
   - enumerate x into F;
   - initialize all strategies $> \xi$; and
   - *end the stage.*

(5) Otherwise, fix the highest-priority (i.e., shortest) such ζ and the corresponding η. We say ξ *is delayed in the enumeration of x*, and we *end the stage.* (At the next ζ-expansionary stage, ζ will then enumerate x into F unless it has been initialized before then.)

3.4.4. *The general $\mathcal{P}^E$-strategy.* A $\mathcal{P}^E_\Sigma$-strategy $\xi \in T$ acts as follows at substage t of stage s at which it is eligible to act:

(1) If this is the first stage at which ξ is eligible to act since ξ was last initialized, then check whether $X(\xi) \neq \emptyset$. If so, then make x-requests to all $\zeta \in X(\xi)$ and *end the stage.*
(2) Otherwise, check whether ξ has a witness y. If not, then ξ chooses a witness y bigger than any number mentioned before and *ends the stage.*
(3) Otherwise, let y be ξ's current witness. Next check whether there is a computation $\Sigma(D; y) \downarrow= 0$. If not, then end the substage and let $\xi^\frown\langle fin\rangle$ be *eligible to act next.*
(4) Otherwise, check whether ξ has already enumerated y into E. If so, then end the substage and let $\xi^\frown\langle fin\rangle$ be *eligible to act next.*
(5) Otherwise, check whether there is a y-request from a strategy η to an $\mathcal{S}$-strategy $\zeta \in Y(\eta)$ with $\zeta^\frown\langle\Theta\rangle \subseteq \xi$. If not, then
   - enumerate y into E;
   - extract x from F for all $\eta \in X(\xi)$ such that η fulfilled ξ's x-request via a pair $\langle x, u\rangle$;
   - initialize all strategies $> \xi$; and
   - *end the stage.*
(6) Otherwise, fix the highest-priority (i.e., shortest) such ζ and the corresponding η. We say ξ *is delayed in the enumeration of* y, and we *end the stage.* (At the next ζ-expansionary stage, ζ will then enumerate y into E unless it has been initialized before then.)

3.4.5. *The general $\mathcal{P}^D$-strategy.* A $\mathcal{P}^D_\Omega$-strategy $\xi \in T$ acts as follows at substage t of stage s at which it is eligible to act:

(1) If this is the first stage at which ξ is eligible to act since ξ was last initialized then check whether $Y(\xi) \neq \emptyset$. If so, then
   - make y-requests to all $\eta \in Y(\xi)$;
   - have all $\eta \in Y(\xi)$ make x-requests to all $\zeta \in X(\eta)$; and
   - *end the stage.*
(2) Otherwise, check whether ξ has a witness z. If not, then ξ chooses a witness z bigger than any number mentioned before and *ends the stage.*
(3) Otherwise, let z be ξ's current witness. Next check whether there is a computation $\Omega(z) \downarrow= 0$. If not, then end the substage and let $\xi^\frown\langle fin\rangle$ be *eligible to act next.*

(4) Otherwise, check whether ξ has already enumerated z into D. If so, then end the substage and let $\xi^\frown\langle fin\rangle$ be *eligible to act next*.

(5) Otherwise,
   - enumerate z into D;
   - extract y from E for all $\eta \in Y(\xi)$ such that η fulfilled ξ's y-request via a pair $\langle y, v\rangle$;
   - re-enumerate x into F for all $\zeta \in X(\eta)$ such that $\eta \in Y(\xi)$, and ζ fulfilled η's x-request via a pair $\langle x, u\rangle$; and
   - *ends the stage*.

This concludes the description of the individual strategies.

3.5. **The construction for Theorem 7.** Before describing the full construction, we need to define some more terms.

A strategy $\xi \in T$ is *initialized* by

- making all its witnesses and restraints undefined;
- making ξ's functionals totally undefined (if ξ is an $\mathcal{R}$- or $\mathcal{S}$-strategy); and
- canceling all x- and y-requests made by or made to ξ.

The construction now proceeds in stages s. At stage 0, we initialize all strategies $\xi \in T$.

Each stage $s > 0$ consists of substages t. At a substage t, a strategy $\xi \in T$ of length t will be eligible to act. (Thus the unique strategy $\langle\rangle \in T$ of length 0 will always be eligible to act at substage 0.) Each strategy, when eligible to act, will act depending on the requirement assigned to it and as specified in section 3.4 and then either end the stage (if the strategy requires this or if $s = t$), or else determine the strategy to be eligible to act at substage $t + 1$. At the end of stage s, we initialize all strategies $>_L f_s$ where f_s is the strategy eligible to act at the last substage of stage s.

3.6. **The verification for Theorem 7.** Our first lemma checks some routine facts, the proofs of which we leave to the reader.

Lemma 14. (1) *The* true path *of the construction exists, i.e., there exists an infinite path f through T such that each strategy $\xi \subset f$ is eligible to act infinitely often but each strategy $\xi <_L f$ is eligible to act only finitely often.*

(2) *Each strategy $\xi \in T$ is initialized only finitely often.*

(3) *F, E, and D are a 3-c.e. set, a d.c.e. set, and a c.e. set, respectively.* □

Lemma 15. *Each $\mathcal{R}$-requirement is satisfied.*

Proof. If the hypothesis $U = \Phi(F \oplus E \oplus D)$ of an $\mathcal{R}_{\Phi,U}$-strategy $\xi \subset f$ holds, then, since ξ is eligible to act infinitely often by Lemma 14, there are infinitely many ξ-expansionary stages. We now distinguish two cases:

Case 1: ξ cancels its functional Γ only finitely often: Then Step 5a of the $\mathcal{R}$-strategy will apply infinitely often, and by the way Γ is defined and the uses are chosen, $\Gamma(E \oplus D)$ will be a total function. Finally, after the last time Step 5b applies, there cannot be any argument at which $\Gamma(E \oplus D)$ and U disagree since this would cause a disagreement between $\Phi(F \oplus E \oplus D)$ and U.

Case 2: ξ cancels its functional Γ infinitely often: Then Step 5b of the $\mathcal{R}$-strategy will apply infinitely often, and by the way Δ is defined and the uses are chosen, $\Delta(U \oplus D)$ will be a total function. We need to verify that $\Delta(U \oplus D)$ computes the set E correctly.

For the sake of a contradiction, let s be the first stage at which ξ defines a Δ-computation $\Delta(U \oplus D; y) \downarrow \neq E(y)$ (with true oracle, for some least y). Then at stage s, every strategy $>_L \xi\widehat{\ }\langle\Delta\rangle$ is initialized and thus cannot cause or allow $\Delta(U \oplus D; y)$ or $\Phi(F \oplus E \oplus D) \upharpoonright (\delta(y)+1)$ to change. Similarly, no strategy $<_L \xi$ will be eligible to act after stage s. Finally, if a strategy $\zeta \subset \xi$ allows $\Delta(U \oplus D; y)$ or $\Phi(F \oplus E \oplus D) \upharpoonright (\delta(y) + 1)$ to change, then, since ξ is not initialized after stage s, ζ must be an $\mathcal{R}$- or $\mathcal{S}$-strategy, and $\zeta\widehat{\ }\langle\Delta\rangle \subseteq \xi$ or $\zeta\widehat{\ }\langle\Lambda\rangle \subseteq \xi$, respectively, and the change must be due to a request by a $\mathcal{P}$-strategy $\eta \supseteq \xi\widehat{\ }\langle\Delta\rangle$. Thus, in any case, any change allowing $\Delta(U \oplus D; y)$ or $\Phi(F \oplus E \oplus D) \upharpoonright (\delta(y) + 1)$ to change must be due to a request by a $\mathcal{P}$-strategy $\eta \supseteq \xi\widehat{\ }\langle\Delta\rangle$, which we now fix. Also fix the $\mathcal{P}$-strategy η' such that a request by η' caused $E(y)$ to change first. By initialization, we must have $\eta \leq \eta'$. If $\eta = \eta'$ then as explained in section 3.2, this conflict cannot arise by the mechanism of requests; so assume that $\eta < \eta'$.

Now, if η is a $\mathcal{P}^D$-strategy, then η's action will permanently change D and destroy the computation $\Delta(U \oplus D; y)$. If η is a $\mathcal{P}^E$-strategy, then η will have made an x-request to ξ before η' did, and so the U-change initiated by η will destroy the computation $\Delta(U \oplus D; y)$. Finally, if η is a $\mathcal{P}^F$-strategy then the mechanism of Step 2 will force a diagonalization for ξ. $\square$

Lemma 16. *Each $\mathcal{S}$-requirement is satisfied.*

Proof. This proof is very similar to that of Lemma 15, the main difference being that Ψ has only oracle $E \oplus D$.

If the hypothesis $V = \Psi(E \oplus D)$ of an $\mathcal{S}_{\Psi,V}$-strategy $\xi \subset f$ holds, then, since ξ is eligible to act infinitely often by Lemma 14, there are infinitely many ξ-expansionary stages. We now distinguish two cases:

Case 1: ξ cancels its functional Θ only finitely often: Then Step 5a of the $\mathcal{S}$-strategy will apply infinitely often, and by the way Θ is defined and the uses are chosen, $\Theta(D)$ will be a total function. Finally, after the last time Step 5b applies, there cannot be any argument at which $\Theta(D)$ and V disagree since this would cause a disagreement between $\Psi(E \oplus D)$ and V.

Case 2: ξ cancels its functional Θ infinitely often: Then Step 5b of the $\mathcal{S}$-strategy will apply infinitely often, and by the way Λ is defined and the uses are chosen, $\Lambda(V)$ will be a total function. We need to verify that $\Lambda(V)$ computes the set D correctly.

For the sake of a contradiction, let s be the first stage at which ξ defines a Λ-computation $\Lambda(V; z) \downarrow \neq D(z)$ (with true oracle, for some least z). Then at stage s, every strategy $>_L \xi\widehat{\ }\langle\Lambda\rangle$ is initialized and thus cannot cause or allow $\Lambda(V; z)$ or $\Psi(E \oplus D) \upharpoonright (\lambda(z) + 1)$ to change. Similarly, no strategy $<_L \xi$ will be eligible to act after stage s. Finally, if a strategy $\zeta \subset \xi$ allows $\Lambda(V; z)$ or $\Psi(E \oplus D) \upharpoonright (\lambda(z) + 1)$ to change, then, since ξ is not initialized after stage s, ζ must be an $\mathcal{R}$- or $\mathcal{S}$-strategy, and $\zeta\widehat{\ }\langle\Delta\rangle \subseteq \xi$ or $\zeta\widehat{\ }\langle\Lambda\rangle \subseteq \xi$, respectively, and the change must be due to a request by a $\mathcal{P}$-strategy $\eta \supseteq \xi\widehat{\ }\langle\Lambda\rangle$. Thus, in any case, any change allowing $\Lambda(V; z)$ or $\Psi(E \oplus D) \upharpoonright (\lambda(z) + 1)$ to change must be due to a request by a $\mathcal{P}$-strategy $\eta \supseteq \xi\widehat{\ }\langle\Lambda\rangle$, which we now fix. Also fix the $\mathcal{P}$-strategy η' such that a request by η' caused $D(z)$ to change first. By initialization, we must have $\eta \leq \eta'$. If $\eta = \eta'$ then as explained in section 3.2, this conflict cannot arise by the mechanism of requests; so assume that $\eta < \eta'$.

Now, if η is a $\mathcal{P}^D$-strategy, then η will have made a y-request to ξ before η' did, and so the V-change initiated by η will destroy the computation $\Lambda(V; z)$. If η is a $\mathcal{P}^E$-strategy, then the mechanism of Step 2 will force a diagonalization for ξ. Finally, if η is a $\mathcal{P}^F$-strategy then η cannot affect $\Lambda(V; z)$ or $\Psi(E \oplus D) \upharpoonright (\lambda(z) + 1)$. $\square$

Lemma 17. *Each $\mathcal{P}^F$-requirement is satisfied.*

Proof. Fix a $\mathcal{P}^F_\Pi$-strategy $\xi \subset f$. By the construction, ξ must eventually have a fixed witness x, say, targeted for F such that there are no x-requests to an $\mathcal{R}$-strategy η with $\eta\widehat{\ }\langle\Gamma\rangle \subseteq \xi$ delaying the enumeration of x. Then ξ succeeds in meeting its requirement in the usual Friedberg-Muchnik fashion. $\square$

Lemma 18. *Each $\mathcal{P}^E$-requirement is satisfied.*

Proof. Fix a $\mathcal{P}^E_\Sigma$-strategy $\xi \subset f$. By the construction, ξ must eventually have a fixed witness y, say, targeted for E such that there are no y-requests to an $\mathcal{S}$-strategy η with $\eta\widehat{\ }\langle\Theta\rangle \subseteq \xi$ delaying the enumeration of y. Furthermore, for each $\mathcal{R}$-strategy η with $\eta\widehat{\ }\langle\Delta\rangle \subseteq \xi$, since $\eta\widehat{\ }\langle\Delta\rangle \subseteq \xi \subset f$, ξ's x-request must eventually be satisfied (namely, by the next stage at which ξ is eligible to act after ξ has picked its witness y). Now ξ succeeds in meeting its requirement in the usual Friedberg-Muchnik fashion. $\square$

Lemma 19. *Each $\mathcal{P}^D$-requirement is satisfied.*

Proof. Fix a $\mathcal{P}^D_\Omega$-strategy $\xi \subset f$. By the construction, ξ must eventually have a fixed witness z, say, targeted for D. Furthermore, for each $\mathcal{S}$-strategy η with $\eta\widehat{\ }\langle\Lambda\rangle \subseteq \xi$, since $\eta\widehat{\ }\langle\Lambda\rangle \subseteq \xi \subset f$, ξ's y-request must eventually be satisfied (namely, by the next stage at which ξ is eligible to act after ξ has picked its witness x). Now ξ succeeds in meeting its requirement in the usual Friedberg-Muchnik fashion. $\square$

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