Locally countable orderings

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The basic Refs are Downey and Hirschfeldt's book, Lerman's book, Moschovakis' book and Sacks' book.
http://www.mcs.vuw.ac.nz/~downey/

Notations

We say $\mathbb{P} = \langle P, \leq \rangle$ is a *partial order* if $\leq$ is a self-reflexive transitive binary relation on P.

Definition

A partial order $\mathbb{P} = \langle P, \leq \rangle$ is *locally countable* if for every $p \in P$, $|\{q \in P | q \leq p\}| \leq \aleph_0$.

An antichain in a partial order is a non-empty set in which any two different elements are incomparable.

Motivation

Counting the antichains.

Theorem (Sacks)

Every maximal chain in the Turing degrees has size $\aleph_1$ and is of measure 0.

Theorem (folklore)

Every maximal antichain in the Turing degrees has size $2^{\aleph_0}$

Question

How big can be a size of a chain and an antichain in the Turing degrees? Is there an antichain in the Turing degrees which is not of measure 0?

Main Question

Question

For any locally countable Σ_1^1 partial order $\mathbb{P} = \langle 2^\omega, \leq_P \rangle$, does there exist a nonmeasurable antichain in $\mathbb{P}$?

Higher recursion theory I

$x \leq_h y$ if x is $\Delta^1_1(z)$ definable. $x \leq_h y$ is a Π^1_1-relation.

Theorem (Harrison)

For any real z and countable $\Sigma^1_1(z)$ set $Z \subset 2^\omega$, if $x \in Z$, then $x \leq_h z$.

Corollary

If $\mathbb{P} = \langle 2^\omega, \leq_P \rangle$ is a Σ^1_1 locally countable partial order, then for any $x, y \in 2^\omega$, $x \leq_P y$ implies $x \leq_h y$.

Chains

Lemma (Sacks)

If x is not Δ^1_1, then $\mu(\{y | x \leq_h y\}) = 0$.

Hence we have the following proposition.

Proposition

For any locally countable Σ^1_1 partial order $\mathbb{P} = \langle 2^\omega, \leq_P \rangle$, every chain in $\mathbb{P}$ has measure 0.

Proposition

1. *Assume* $ZFC + V = L$*, there is a locally countable* Δ^1_2 *partial order* $\mathbb{P} = \langle 2^\omega, \leq_P \rangle$ *for which there is a chain of measure 1.*
2. *Assume* $ZFC + MA_{\aleph_1}$*, for any locally countable partial order* $\mathbb{P} = \langle 2^\omega, \leq_P \rangle$*, every chain has measure 0.*

Randomness theory I

Definition (Martin-Löf)

(i) Given a real x, a $\Sigma^0_n(x)$ Martin-Löf test is a computable collection $\{V_n : n \in \mathbb{N}\}$ of $\Sigma^0_n(x)$ sets such that $\mu(V_n) \leq 2^{-n}$.

(ii) Given a real x, a real y is said to pass the $\Sigma^0_n(x)$ Martin-Löf test if $y \notin \bigcap_{n\in\omega} V_n$.

(iii) Given a real x, a real y is said to be n-x-random if it passes all $\Sigma^0_n(x)$ Martin-Löf tests.

Definition

Given a string $\sigma \in 2^{<\omega}$, a real x and a set $S \subseteq 2^{<\omega}$.

1. $\sigma \Vdash x \in S$ if $\sigma \prec x$ and $\sigma \in S$.
2. $\sigma \Vdash x \notin S$ if $\sigma \prec x$ and $\forall \tau \succeq \sigma(\tau \notin S)$.

Definition

Given reals x, y and a number $n \geq 1$, x is n-y-generic if for every $\Sigma^0_n(y)$ set $S \subseteq 2^{<\omega}$, there is a string $\sigma \prec x$ so that either $\sigma \Vdash x \in S$ or $\sigma \Vdash x \notin S$.

Randomness theory II

Theorem (Miller and Yu)

For every $z \in 2^\omega$, every 1-random real Turing reducible to a 1-z-random real is also 1-z-random.

Theorem (van Lambalgen)

For any number $n > 0$ and real $x = x_0 \oplus x_1$ (or $x = x_1 \oplus x_0$), x is n-random if and only if x_0 is n-random and x_1 is n-x_0-random.

So if $x = x_0 \oplus x_1$ is n-random, then $x_0 <_T x$.

Theorem (Kurtz and Kautz)

For every 2-random real x, there is a 1-generic real y so that $y <_T x$.

0-law

Proposition

If $X \subseteq 2^\omega$ and $\mu(X) > 0$, then there are two reals $x, y \in X$ so that $x <_T y$.

Proof.

We use Jockusch' proof.
$\mu(X \cap [\sigma]) > 2^{-|\sigma|-1}$.
$Y = \{x \in [\sigma] | x = x_0 \oplus x_1 \ \& \ x_0 \in X \cap [\sigma]\}$. □

So we have the following proposition.

Corollary

If X is an antichain in the Turing degrees (or h-degrees) and measurable, then $\mu(X) = 0$.

Higher randomness theory

Definition

Given a real z and a number $n \geq 1$,

1. A real $x \in 2^\omega$ is $\Pi^1_n(z)$-random if $x \notin A$ for each $\Pi^1_n(z)$ set A for which $\mu(A) = 0$.
2. A real $x \in 2^\omega$ is $\Sigma^1_n(z)$-random if $x \notin A$ for each $\Sigma^1_n(z)$ set A for which $\mu(A) = 0$.
3. A real $x \in 2^\omega$ is $\Delta^1_n(z)$-random if $x \notin A$ for each $\Delta^1_n(z)$ set A for which $\mu(A) = 0$.

Calculating the complexity I

Proposition (Sacks)

The predicate, $\mu(A) > r$, is Π^1_1, where A ranges over Π^1_1 sets and r ranges over rationals.

A set C is $d - \Sigma^1_1$ if it is a difference of two Σ^1_1 sets.

Corollary

The predicate, $\mu(A \cap B) > r$, is Δ^1_2, where A ranges over Π^1_1 sets, B ranges over Σ^1_1 and r ranges over rationals. In other words, the predicate, $\mu(C) > r$, is Δ^1_2, where C ranges over $d - \Sigma^1_1$ sets, r ranges over rationals.

Calculating the complexity II

Given two predicates $P(z,i), Q(z,i)$ for which $\forall z \forall i \neg(P(z,i)\ \&\ Q(z,i))$ and a real x (or a string $\sigma \in 2^{<\omega}$) , we use $\Sigma(P,Q,z,i) \leftrightarrow x(i)$ (or $\Sigma(P,Q,z,i) \leftrightarrow \sigma(i)$) to denote:

$$(x(i) = 0 \rightarrow P(z,i))\ \&\ (x(i) = 1 \rightarrow Q(z,i)))$$

(or

$$(\sigma(i) = 0 \rightarrow P(z,i))\ \&\ (\sigma(i) = 1 \rightarrow Q(z,i))).$$

Note $\{z | \Sigma(P,Q,z,i) \leftrightarrow \sigma(i)\}$ is uniformly $d - \Sigma^1_1$ when σ ranges over $2^{<\omega}$ and i ranges over ω if both P and Q are $d - \Sigma^1_1$.

Attacking the question I

Lemma

For any reals $x \leq_h z$*, there is a* Π^1_1 *predicate* $P(z, i)$ *and* $d - \Sigma^1_1$*-predicate* $Q(z, i)$ *so that* $\Sigma(P, Q, z, i) \leftrightarrow x(i)$ *and* $\forall z \forall i \neg(P(z, i) \;\&\; Q(z, i))$ *.*

Proof.

Since $x \leq_h z$, there are two Π^1_1 predicates $R(z, i), S(z, i)$ so that $x(i) = 0$ iff $R(z, i)$ iff $\neg S(z, i)$. Define $P = R$ and $Q = S \wedge \neg R$. $\square$

Attacking the question II

Lemma

If $x \in 2^\omega$ is Δ^1_2-random, then for any Π^1_1 predicate $P(z,i)$ and $d - \Sigma^1_1$ predicate $Q(z,i)$ which satisfies $\forall z \forall i \neg (P(z,i) \ \&\ Q(z,i))$, there is a constant c so that

$$\forall n(\mu(\{z \in 2^\omega | \forall i \leq n(\Sigma(P,Q,z,i) \leftrightarrow x(i))\}) \leq 2^{-n+c}).$$

Proof.

Define a Δ^1_2-sequnece
$\mathcal{V}_\sigma = \{z \in 2^\omega \mid \forall i \leq |\sigma|(\Sigma(P,Q,z,i) \leftrightarrow \sigma(i))\}$.

$$F_i = \{\sigma \in 2^{<\omega} \mid \mu(\mathcal{V}_\sigma) > 2^{-|\sigma|+i}\}$$

D is a maximal prefix-free subset of F_i. □

continued.

Then

$$\mu(\{z \in 2^\omega \mid \exists \sigma \in D \forall i \leq n(\Sigma(P, Q, z, i) \leftrightarrow \sigma(i))\}) = \sum_{\sigma \in D} \mu(\mathcal{V}_\sigma) > \sum_{\sigma \in D} 2^{-|\sigma|+i} = 2^i \sum_{\sigma \in D} 2^{-|\sigma|} = 2^i \mu([D]).$$

So $\mu([D]) < 2^{-i}$. □

Attacking the question III

Lemma

For any real z and Δ^1_2-random real x, if x is not 1-z-random and $y \geq_h x$, then y is not $\Delta^1_2(z)$-random.

Proof.

Define a $d - \Sigma^1_1$-sequence

$$F_\sigma = \{z \in 2^{<\omega} | \forall i \leq |\sigma| (\Sigma(P, Q, z, i) \leftrightarrow \sigma(i))\}.$$

Define a Δ^1_2-sequence $\{G_\sigma\}$:
$G_\sigma = F_\sigma$ if $\mu(F_\sigma) \leq 2^{-|\sigma|+c}$ and $G_\sigma = \emptyset$ otherwise.
A computable collection of $\Sigma^0_1(z)$ sets $\{V_i\}_{i\in\omega}$ so that $\mu(V_i) \leq 2^{-i}$ and $x \in \bigcap_{i\in\omega} V_i$. Fix a c.e. collections of z-c.e. prefix free sets $\{\hat{V}_i\}_{i\in\omega}$ so that $[\hat{V}_i] = V_i$ for each i. □

continued.

Define a $\Delta^1_2(z)$-sequence H_i so that $H_i = \bigcup_{\sigma \in \hat{V}_{i+c}} G_\sigma$. Then

$$\mu(H_i) \leq \sum_{\sigma \in \hat{V}_{i+c}} \mu(G_\sigma)$$

$$\leq \sum_{\sigma \in \hat{V}_{i+c}} 2^{-|\sigma|+c} = 2^c \cdot \sum_{\sigma \in \hat{V}_{i+c}} 2^{-|\sigma|} = 2^c \cdot \mu(V_{i+c}) \leq 2^{-i}.$$

Since $\{H_i\}_{i \in \omega}$ is a $\Delta^1_2(z)$ sequence of $d - \Sigma^1_1$ sets, $H = \bigcap_{i \in \omega} H_i$ is a $\Delta^1_2(z)$ set and $\mu(H) = 0$. But for each i, there is a $\sigma \in \hat{V}_i$ for which $\sigma \prec x$ and so $F_\sigma \subseteq H_i$. Hence $y \in F_\sigma \subseteq H_i$ for each i. Thus $y \in H$. So y is not $\Delta^1_2(z)$-random. □

Given a set $X \subseteq 2^\omega$, define $\mathcal{U}_h(X) = \{y | \exists x \in X(x \leq_h y)\}$.

Lemma

Suppose $X \subset 2^\omega$ *which only contains* Δ^1_2*-random reals, if* $\mu(X) = 0$*, then* $\mu(\mathcal{U}_h(X)) = 0$*.*

Proof.

Define $\mathcal{R} = \{x | x \text{ is } \Delta^1_2\text{-random.}\}$. Take a maximal set $X \subset \mathcal{R}$ so that $\forall x \in X \forall y \in X (x \neq y \implies \forall z(z \leq_h x \;\&\; z \leq_h y \implies z \text{ is not } \Delta^1_2\text{-random}))$. □

Lemma

There exists a nonmeasurable antichain in the h-degrees.

Solving the question

Theorem

For any locally countable Σ^1_1 partial order $\mathbb{P} = \langle 2^\omega, \leq_P \rangle$, there exists a nonmeasurable antichain in $\mathbb{P}$.

Beyond recursion theory

Proposition

Assume ZFC $+ V = L$, there is a locally countable Δ^1_2 partial order on 2^ω in which every antichain has size 1.

Theorem

Assume ZFC $+ MA_{\aleph_1}$. If a partial order $\mathbb{P} = \langle 2^\omega, \leq \rangle$ is locally countable, then there exists a nonmeasurable antichain in $\mathbb{P}$.

Applications to some specific orders

Corollary

There exists a nonmeasurable antichain in the Turing degrees.

Corollary

There exists a nonmeasurable antichain in the K-degrees.

Possible generalizations I

Given a set $X \subseteq 2^\omega$, define $\mathcal{U}(X) = \{y | \exists x \in X (x \leq_T y)\}$.

Proposition

There exists an antichain X for which $\mu(X) = 0$ and $\mathcal{U}(X)$ is not of measure 0.

Can $\mathcal{U}(X)$ be measurable? or non-measurable?.

Question (Jockusch)

Does there exist an antichain X in the Turing degrees for which $\mu(X) = 0$ and $\mu(\mathcal{U}(X)) = 1$?

Proposition

For any antichain X in the Turing degrees, if $\mu(\mathcal{U}(X)) > 0$, then $\mu(X) = 0$.

Possible generalizations II

We say that a set $X \subset 2^\omega$ is a *quasi-antichain* in the Turing degrees if it satisfies the following properties:

1. $\exists x \in X \exists y \in X (x \not\equiv_T y)$.
2. $\forall x \in X \forall y (x \equiv_T y \to y \in X)$.
3. $\forall x \in X \forall y \in X (x \not\equiv_T y \to x \not\leq_T y)$.

It is not hard to see that there is a nonmeasurable quasi-antichain in the Turing degrees.

Question (Jockusch)

Is every maximal quasi-antichain in the Turing degrees nonmeasurable?

Possible generalizations III

We say that a partial order $\mathbb{P} = \langle 2^\omega, \leq_P \rangle$ is *locally null* if for every x, $\mu(\{y | y \leq_P x\}) = 0$. We have the following proposition.

Proposition

For any measurable locally null partial order $\mathbb{P} = \langle 2^\omega, \leq_P \rangle$, every chain in $\mathbb{P}$ has measure 0.

Corollary

Every chain in any Π^1_1 locally countable partial order is of measure 0.

Question

Is it true that for any locally countable Π^1_1 partial order $\mathbb{P} = \langle 2^\omega, \leq_P \rangle$, there exists a nonmeasurable antichain in $\mathbb{P}$?

Thanks