Eliminating concepts: K-trivial equals low for random

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July 18, 2005

K-trivial = low for ML-random

- A set A is K-trivial if each initial segment has minimal prefix free complexity. That is, there is $c \in \mathbb{N}$ such that

$$\forall n\ K(A \upharpoonright n) \leq K(n) + c$$

(Chaitin, 1975).

- A is low for ML-random if each ML-random set is already ML-random relative to A (Zambella, 1990).

In these lectures we will see that the two concepts are equivalent.

More concepts

Further concepts have been introduced:

- Basis for ML-randomness (Kučera , APAL 1993)
- Low for K (Muchnik jr, in a Moscow seminar, 1999).

They will also be eliminated, by showing equivalence with K-triviality. However, they all show different aspects of the same notion.

- Low for K, low for random, basis for ML-randomness mean computationally weak
- K-trivial means far from random

Existence

In three cases, the researcher who introduced the property (Solovay, Kučera, Muchnik) showed the existence of a non-computable set with that property.

For low for ML-random, this was only shown later (Kučera, Terwijn 1997).

All examples were c.e., except for Solovay's example of a K-trivial, which was only Δ^0_2. Later it was improved to a c.e. example (Kummer unpubl., Calude & Coles 1999).

Part I: Close to computable

Low for K

In general, adding an oracle A to the computational power of the universal machine decreases $K(y)$. A is low for K if this is not so. In other words,

$$\forall y\ K(y) \overset{+}{\leq} K^A(y).$$

Let $\mathcal{M}$ denote this class. It was introduced by Andrej A. Muchnik (1999), who proved that there is a c.e. noncomputable $A \in \mathcal{M}$.

Proposition 1 *If A is low for K, then A is GL_1*

Proof:

- If $e \in A'_t$, then $K(t) \leq^+ K^A(t) \leq^+ 2 \log e$.
- $\emptyset'$ can compute $s = \max\{U(\sigma) : |\sigma| \leq 2 \log e + c$. Then $e \in A' \Leftrightarrow e \in A'_s$.

Low for ML-random

Let MLRand denote the class of Martin-Löf-random sets.

- Because an oracle A increases the power of tests, $\mathsf{MLRand}^A \subseteq \mathsf{MLRand}$.
- A is low for ML-random if $\mathsf{MLRand}^A = \mathsf{MLRand}$ (Zambella, 1990). $\mathrm{Low}(\mathsf{MLRand})$ denotes this class.

Theorem 2 (Kucera and Terwijn, 1997) *There is a non-computable c.e. set that is low for ML-random.*

Easy: $\mathcal{M} \subseteq \text{Low}(\mathsf{MLRand})$

Schnorr's Theorem relativizes:

Z is Martin-Löf random relative to A iff

for some c, $\forall n\ K^A(Z \upharpoonright n) \geq n - c$.

Apply this:

- MLRand can be defined in terms of K, and
- MLRand^A in terms of K^A.

So low for K implies low for ML-random.

Bases for ML-randomness

Kucera (APAL, 1993) studied sets A such that

$$A \leq_T Z \text{ for some } Z \in \mathsf{MLRand}^A.$$

That is, A can be computed from a set random relative to it.

We will call such a set a basis for ML-randomness. (Kučera called them "basis for 1-RRA".)

- Each low for ML-random set A is a basis for ML-randomness.
- For, by the Kučera-Gács Theorem there is a ML-random Z such that $A \leq_T Z$.
- Then Z is ML-random relative to A.

Two theorems

I discuss two theorems:

1. There is a c.e. non-computable basis for ML (Kučera1993)

2. Each basis for ML is low for K. (HNS ta)

Two concepts gone

We have already seen the easy inclusions

$$\text{low for } K \Rightarrow \text{low for ML} \Rightarrow \text{basis for ML},$$

Then, by Theorem 2., all three classes are the same.

And K-triviality?

- low for K $\Rightarrow$ K-trivial is immediate
- The converse K-trivial $\Rightarrow$ low for K is hard. The proof is separate from all of the above.
- (Chaitin 1976) proved that K-trivial implies Δ^0_2, by an elegant argument involving the coding theorem.

  A direct way to show, say, low for ML-random implies Δ^0_2 is much harder. This was the original open question of Kučera -Terwijn.

A basis for ML

Theorem 3 (Kučera 1993) *There is a c.e. non-computable A which is a basis for ML.*

This follows from the next two results. This proof is a bit different from Kučera's original one.

Theorem 4 (Kučera 1985) *Let Z be Δ^0_2 and diagonally non-computable. Then there is a c.e. non-computable set $A \leq_T Z$.*

Apply this to a low ML-random set (say), and use that

Lemma 5 (Hirschfeldt.N.Stephan ta) *If $Z <_T \emptyset'$ is ML-random and $A \leq_T Z$ is c.e., then Z is already ML-random relative to A.*

A special case of Kučera's 1985 Theorem

Theorem *Let Z be Δ^0_2 and ML-random. Then there is a simple set $A \leq_T Z$.*

This can be proved without using the Recursion Theorem. Kučera had first thought of this special case. Then he generalized it to d.n.c. Z, where the Recursion Theorem is needed.

Recall that a Solovay test $\mathcal{G}$ is given by an effective enumeration of strings $\sigma_0, \sigma_1, \ldots$, such that

$$\sum_i 2^{-|\sigma_i|} < \infty.$$

If Z is ML-random, then for almost all i, $\sigma_i \not\preceq Z$.

We want to meet the requirement

$$S_e : \ |W_e| = \infty \Rightarrow A \cap W_e \neq \emptyset.$$

Construction. At stage s, if S_e is not satisfied yet, see if there is an x, $2e \leq x < s$, such that

$$x \in W_{e,s} - W_{e,s-1} \ \ \& \ \ \forall t_{x<t<s} Z_t \upharpoonright e = Z_s \upharpoonright e.$$

If so, put x into A. Put the string $\sigma = Z_s \upharpoonright e$ into $\mathcal{G}$. Declare S_e satisfied.

- Clearly A is (promptly) simple.
- For almost all $\sigma \in \mathcal{G}$, $\sigma \not\preceq Z$. In that case, after stage s, $Z \upharpoonright e$ has to take a value not seen before. Since $e < x$, this shows $A \leq_T Z$.

The enumeration into A is heavily restrained. If $Z \upharpoonright e$ changes another time at stage s, then no $x < s$ can be enumerated after s for the sake of S_e. Z can do that as late and as often as it wants.

Base for ML $\Rightarrow$ low for K

(...the hungry sets theorem.)

Theorem 6 (Hirschfeldt, Nies, Stephan, 2004) *If A is a basis for ML-randomness, then A is low for K.*

There, we actually only proved the conclusion that A is K-trivial, but it's the same proof.

The KC-Theorem

We use this tool:

- A c.e. set $W \subseteq \mathbb{N} \times 2^{<\omega}$ is a Kraft-Chaitin set (KC set) if the weight

$$\sum\{2^{-r} : \langle r, y\rangle \in W\} \leq 1.$$

- From a Kraft-Chaitin set L, one can effectively obtain a prefix free machine M such that

$$\forall r, y[\langle r, y\rangle \in L \;\Leftrightarrow\; \exists w\ (|w| = r \;\&\; M(w) = y)].$$

We enumerate polite requests $\langle r, y\rangle$ ("please describe y with length r"). If their total weight is at most 1, then for each request (also called axiom), there actually is an M description of length r.

Proving the hungry sets theorem

(...and why it is called that way.)

Theorem. *If A is a basis for ML-randomness then A is low for K.*

Given a Turing functional Φ, we define a ML test $(V_d^X)_{d \in \mathbb{N}^+}$ relative to oracle X (later, we use this test for $X = A$).

Suppose $A = \Phi^Z$. Goal: If $Z \notin V_d^A$ then A is low for K, with constant $d + \mathcal{O}(1)$.

To realise this goal, we also build a uniformly c.e. sequence $(L_d)_{d \in \mathbb{N}^+}$ of KC sets. For each computation

$$\boldsymbol{U}^{\eta}(\sigma) = y \text{ where } \eta \preceq A,$$

that is, when y has a description σ with oracle A, we want to ensure there is a description without an oracle that is only a constant longer. Thus we want to put an axiom

$$\langle |\sigma| + d + 1, y \rangle$$

into L_d. The problem is that we don't know A, so we don't know which η's to take. This may blow up L_d. To avoid this, the description $\boldsymbol{U}^{\eta}(\sigma) = y$ first has to prove itself worthy.

We build an auxiliary ML-test (V_d) relative to A. If $Z \notin V_d$ then L_d works. To do so we effectively enumerate open sets $C^\eta_{d,\sigma}$ and let

$$V_d^A = \bigcup_{\eta \prec A} C^\eta_{d,\sigma}.$$

While $\mu(C^\eta_{d,\sigma}) < 2^{-|\sigma|-d}$, $C^\eta_{d,\sigma}$ is hungry. We feed it with fresh oracle strings α, where $\eta \prec \Phi^\alpha$.

All the open sets $[C^\eta_{d,\sigma}]^\preceq$ are disjoint. When $\mu(C^\eta_{d,\sigma})$ reaches $2^{-|\sigma|-d-1}$, we put the required axiom $\langle |\sigma| + d + 1, y\rangle$ into L_d. This will happen in the relevant case $\eta \preceq A$.

Fome Zero Lemma. *Suppose $Z \notin V_d$. Then for each description $\boldsymbol{U}^\eta(\sigma) = y$, where $\eta \prec A$, $\mu(C^\eta_{d,\sigma}) = 2^{-|\sigma|-d}$. (It is hungry no more at the end of time.)*

We account the weight of those axioms against the measure of the sets $C^\eta_{d,\sigma}$. So L_d is a KC set. QED

Discussion

Given a Turing functional Φ. For each A, let

$$S_A = \{Z : A = \Phi^Z\}.$$

Also let $S_{A,n} = [\{\sigma : A \upharpoonright n = \Phi^\sigma\}]^{\preceq}$ (an open set u.c.e. in A).

- Sacks: If A is non-computable, then $\mu S_A = 0$.

  So $(S_{A,n})_{n \in \mathbb{N}^+}$ is a weak ML-test relative to A.

- Miller, Yu: If A is ML-random, then there is c such that

$$\forall n \, \mu S_{A,n} \leq 2^{-n+c}.$$

- Hungry sets: Based on Φ, one can build a relative ML-test (V_d) such that A not low for K $\Rightarrow S_A \subseteq \bigcap_d V_d^A$. Then, since there is a universal ML test, the whole class $\{Z : A \leq_T Z\}$ is ML-null relative to A. (The converse holds as well: if A is low for K then $\Omega \geq_T A$ is ML-random in A, so the class is not.)

Martingales

Next we study lowness for randomness notions implied by ML.

A martingale is a function $M : \{0,1\}^* \mapsto \mathbb{R}_0^+$ such that

$$M(x0) + M(x1) = 2M(x)$$

M succeeds on Z if

$$\limsup_n M(Z \upharpoonright n) = \infty.$$

CRand and KLRand

- Z is computably random (CRand) if no computable martingale M succeeds on Z. That is, $M(Z \upharpoonright n)$ is bounded.
- While a martingale always bets on the next position, a non-monotonic betting strategy can choose some position that has not been visited yet.
- Z is Kolmogorov-Loveland random (KLRand) if no non-monotonic betting strategy succeeds on Z.

$$\mathsf{MLRand} \subseteq \mathsf{KLRand} \subset \mathsf{CRand}.$$

It is a major open problem if the first inclusion is proper, too.

Lowness notions

The following is a further improvement of the original result (Nies 2002) that Low(MLRand) implies low for K.

Theorem 7 *If* $\mathsf{MLRand} \subseteq \mathrm{CRand}^A$ *then* A *is low for* K*.*

(The converse implication holds trivially.)

A lemma

Let R be any r.e. open set such that $\mu R < 1$ and $\textsf{Non-MLRand} \subseteq R$ We will define a martingale functional L. If $\textsf{MLRand} \subseteq \mathrm{CRand}^A$ then $\mathrm{Success}(L^A) \subseteq \textsf{Non-MLRand}$, and we may apply the following lemma to $N = L^A$.

Lemma 8 *Let N be any martingale such that $\mathit{Success}(N) \subseteq \textsf{Non-MLRand}$. Then there are $v \in 2^{<\omega}$ and $m \in \mathbb{N}$ such that $v \notin R$, and if $N(x)$ is large then $x \succeq v$ is not a random string, namely*

$$\forall x \succeq v[N(x) \geq 2^m \Rightarrow x \in R]. \tag{1}$$

Proving the Theorem

Theorem. If $\mathsf{MLRand} \subseteq \mathrm{CRand}^A$ then A is low for K.

Let us assume we know the witnesses of the lemma v, m. Thus

$$\forall x \succeq v[L^A(x) \geq 2^m \Rightarrow x \in R].$$

As before, if $\boldsymbol{U}^\eta(\sigma) = y$ where $\eta \preceq A$, we want a corresponding set C^η_σ such that $\mu \boldsymbol{C}^\eta_\sigma \geq 2^{-|\sigma|-c}$, c some constant (no parameter d occurs here).

Feed the set C^η_σ as follows:

- at stage s pick a clopen set D, $\mu D = \epsilon$ of long strings $x \succeq v$, $D \cap R_s = \emptyset$. Define $L^\eta(x) \geq 2^m$ for each $x \in D$.
- If really $\eta \prec A$, then D has to go into R. Once this happens repeat with a new set of measure ϵ, as long as C^η_σ is hungry.

Low(**KLRand**)

If A is low for KLRand, then

$$\mathsf{MLRand} \subseteq \mathsf{KLRand} = \mathsf{KLRand}^A \subseteq \mathsf{CRand}^A.$$

Corollary 9 *Each low for KLRand set is low for K.*

Of course, it is open if the two classes are the same.

Low(CRand)= computable

Earlier result:

Theorem 10 *[with B. Bedregal, Porto Alegre] Each Low(CRand) set is hyper-immune free.*

But also, by Theorem 7 each Low(CRand) set is K-trivial, hence Δ^0_2. Since the only hyper-immune free Δ^0_2 are the computable sets, we have

Theorem 11 *If A is Low(CRand) then A is computable.*

This answers **Question 4.8** in Ambos-Spies/Kucera (1999) in the negative. Conjectured this way by Downey.

Low(SchnorrRand)

Being computably traceable is a strengthening of A being hyper-immune free: the value $f(x)$ of each $f \leq_T A$ is in a small effectively given set $D_{g(x)}$. Here g is a computable function depending on f, and $|D_{g(x)}| \leq h(x)$ for a fixed bound h.

Theorem 12 (**Terwijn and Zambella 2000**) *A is low for Schnorr tests $\Leftrightarrow$ A is computably traceable.*

Theorem 13 (**Kjos-Hanssen, Nies, Stephan**) *Each computably random set is Schnorr random relative to A $\Leftrightarrow$ A is computably traceable.*

One ingredient is the result with Bedregal, which works already if each computably random set is Schnorr random relative to A.

Part II: Far from random

Definition of K-trivial

- For a string y, up to constants, $K(|y|) \le K(y)$, since we can compute $|y|$ from y (here we write numbers in binary).
- A set A is K-trivial if, for some $b \in \mathbb{N}$

$$\forall n \; K(A \upharpoonright n) \le K(n) + b,$$

namely, the K complexity of all initial segments is minimal.

This is opposite to ML-randomness:

- Z is ML-random if all complexities $K(Z \upharpoonright n)$ are near the upper bound $n + K(n)$, while
- Z is K-trivial if they have the minimal possible value $K(n)$ (all within constants).

Why prefix free complexity?

If one defined K-triviality using the usual Kolmogorov complexity C instead of K, then one obtained only the computable sets (Chaitin, 1975).

(Chaitin, 1976) still managed to prove that the K-trivial sets are Δ_2^0.

Solovay (1975) was the first to construct a non-computable K-trivial A , which was Δ_2^0 as expected but not c.e.

Constructions

[Downey, Hirschfeldt, Nies, Stephan 2001] gave a short "definition" of a simple K-trivial set, which had been anticipated by various researchers (Kummer, Zambella). We use the "cost function"

$$\boxed{c(x,s) = \sum_{x<y\leq s} 2^{-K_s(y)}}.$$

This determines a non-computable set A:

$A_s = A_{s-1} \cup \{x : \exists e$

$W_{e,s} \cap A_{s-1} = \emptyset$	we haven't met e-th simplicity requirement
$x \in W_{e,s}$	we can meet it, via x
$x \geq 2e$	make A co-infinite
$c(x,s) \leq 2^{-(e+2)}\}$	ensure A is K-trivial.

This needs $I\Sigma_1$

Though the construction is very simple, induction over Σ_1 formulas is needed to verify it, because of work of Chong, Slaman, ... :

- There is a saturated $\mathcal{M} \models I\Delta_1$ (even $\mathcal{M} \models B\Sigma_1$) with a Σ_1 cofinal f whose domain is the standard part
- In such an $\mathcal{M}$, each regular c.e. set A is computable. Here A is regular if for each $n \in \mathcal{M}$, $A \upharpoonright n$ is a string of $\mathcal{M}$ (i.e., encoded by an element of $\mathcal{M}$)
- Each K-trivial set $A \subseteq \mathcal{M}$ is regular, since $A \upharpoonright n$ has a prefix free description in $\mathcal{M}$, for each n.

Necessity of the cost function method, c.e. case

- Suppose the c.e. set A is K-trivial via a constant b.
- For each s, one can effectively determine an $f(s) > s$ such that $\forall n < s\ K(A \upharpoonright n) \leq K(n) + b\ [f(s)]$
- Let $s(0) = 0$ and $s(i+1) = f(s(i))$.

Proposition 14 *Let A be c.e. and K-trivial via b. Then*

$$\sum\{c(x, s(i)) : x < s(i) \textit{ is minimal s.t. } A_{s(i)}(x) \neq A_{s(i+1)}(x)\} \leq 2^b.$$

Bounding the cost

- Let $x_i < s(i)$ be minimal such that $A_{s(i)}(x) \neq A_{s(i+1)}(x)$.
- For each y, $x_i < y \leq s(i)$, by definition there is at stage $s(i+1)$ a prefix free description of $A \restriction y$ of length $\leq K_{s(i+1)}(y) + b$. Let D_i be the open set generated by such descriptions.
- Since A is c.e., the strings $A \restriction y$ described at different stages $s(i+1)$ are distinct, so that $D_i \cap D_j = \emptyset$ for $i \neq j$. Hence $\sum_i \mu D_i \leq 1$.

Since

$$\sum_{x_i < y \leq s(i)} 2^{-K_{s(i)}(y)} \leq \sum_{x_i < y \leq s(i)} 2^{-K_{s(i+1)}(y)} \leq 2^b \mu D_i,$$

this shows $\sum_i c(x_i, s(i)) \leq 2^b$, as required.

A question

We have by now seen two ways to obtain a non-computable c.e. K trivial set A. Both are injury free.

(i) Take an incomplete ML-random Z, and build $A \leq_T Z$ using Kučera's methods. Then A is low for K, and hence K-trivial

(ii) The cost function construction, which is necessary by the above.

Can each K-trivial set be obtained via (i)? That is

Question 15 *If A is K-trivial, is there an incomplete ML-random Z above A ?*

Later we will see that each K trivial set is below a c.e. K-trivial set. So we don't have to require that the given K-trivial set A is c.e.

wtt-incompleteness

Showing the downward closure of $\mathcal{K}$ under $\leq_{wtt}$ is an easy exercise. Since the *wtt*-complete set Ω is not K–trivial, no K–trivial set A satisfies $\emptyset' \leq_{wtt} A$. To introduce some new techniques, I give a direct proof of this. Suppose $\emptyset' \leq_{wtt} A$ for K-trivial A.

- We build an r.e. set B, and by the Recursion Theorem we can assume we are given a total *wtt*-reduction Γ such that $B = \Gamma^A$, whose use is bounded by a computable function g.

- We build a KC-set L. Thus we enumerate polite requests $\langle r, n\rangle$ and have to ensure $\sum_r 2^{-r}$ is at most 1. By the recursion theorem, we may assume the coding constant d for L is given in advance. Then, putting $\langle r, n\rangle$ into L causes $K(n) \leq r + d$ and hence $K(A \upharpoonright n) \leq r + b + d$, where b is the triviality constant.

- Let
$$k = 2^{b+d+1}$$
- Let $n = g(k)$ (the use bound). We wait till $\Gamma^A(k)$ converges, and put the single request $\langle r, n \rangle$ into L, where $r = 1$. Our total investment is $1/2$.
- Each time the opponent has a U-description of $A \upharpoonright n$ of length $\leq r + b + d$, we force $A \upharpoonright n$ to change, by putting into B the largest number $\leq k$ which is not yet in B.
- If we reach $k + 1$ such changes, then his total investment is
$$(k + 1)2^{-(1+b+d)} > 1,$$
contradiction.

Turing-incompleteness

- Consider the more general result that K-trivial sets are T-incomplete (Downey, Hirschfeldt, N, Stephan, 2003), mostly H... There is no recursive bound on the use of $\Gamma^A(k)$.
- The problem now is that the opponent might, before giving a description of $A_s \upharpoonright n$, move this use beyond n, thereby depriving us of the possibility to cause further changes of $A \upharpoonright n$.
- The solution is to carry out many attempts in parallel, based on computations $\Gamma^A(m)$ for different m.
- Each time the use of such a computation changes, the attempt is cancelled. What we placed in L for this attempt now becomes garbage. We have to ensure that the weight of the garbage does not build up too much, otherwise L is not KC.

More details: j-sets

The following is a way to keep track of the number of times the opponent had to give new descriptions of strings $A_s \upharpoonright n$. We only consider *stages* where A looks K-trivial with constant b.

- At *stage* t, a finite set E is a j-set if for each $n \in E$ first we put a request $\langle r_n, n \rangle$ into L, and then j times at *stages* $s < t$ the opponent had to give new descriptions of $A_s \upharpoonright n$ of length $r_n + b + d$.
- A c.e. set with an enumeration $E = \bigcup E_t$ is a j-set if E_t is a j-set at each *stage* t.
- Recall that the weight $wt(E)$ is $\sum \{2^{-r_n} : n \in E\}$

The weight of a k-set

- If the c.e. set E is a k-set, $k = 2^{b+d+1}$ as defined above, then

$$wt(E) \leq 1/2,$$

because k times the opponent has to match our description of n, which has length r_n, by a description of a string $A_s \upharpoonright n$ that is at most $b + d$ longer.

Procedures

Assume A is K-trivial and Turing complete. As in the wtt-case, we attempt to build a k-set F_k of weight $> 1/2$ and reach a contradiction.

The procedure P_j $(2 \leq j \leq k)$ enumerates a j-set F_j. The construction begins calling P_k, which calls P_{k-1} many times, and so on down to P_2, which enumerates L (and F_2).

Each procedure P_j has rational parameters $q, \beta \in [0, 1]$.

- The goal q is the weight it wants F_j to reach
- The garbage quota β is how much it garbage it can produce.

When the procedure reaches its goal it returns.

Decanter model

We visualize this construction by a machine similar to Lerman's pinball machine. However, since we enumerate rational quantities instead of single objects, we replace the balls there by amounts of a precious liquid, namely 1983 Brunello wine.

Our machine consists of decanters $F_k, F_{k-1}, \ldots, F_0$. At any *stage* F_j is a j set. F_{j-1} can be emptied into F_j.

The procedure $P_j(q, \beta)$ wants to add weight q to F_j. It fills F_{j-1} up to q and then returns, by emptying it into F_j. The emptying is done by enumerating B and hence adding one more A-change.

The emptying device is a hook (the $\gamma^A(m)$-marker), which besides being used on purpose may go off finitely often by itself (premature A-change). When F_{j-1} is emptied into F_j then $F_{j-2}, \ldots, F_0$ are spilled on the floor.

Though the recursion starts by calling P_k with goal 1, wine is first poured into the highest decanter F_0, and thereby into the left domain of L. We want to ensure that at least half the wine we put into F_0 reaches F_k. Recall that the parameter β is the amount of garbage $P_j(q, \beta)$ allows. If v is the number of times an emptying device has gone off by itself, then P_j lets P_{j-1} fill F_{j-1} in portions of $2^{-v}\beta$. Then when F_{j-1} is emptied into F_j, at most $2^{-v}\beta$ can be lost because of being in higher decanters $F_{j-2}, \ldots, F_0$.

This uses a key idea: when we have to cancel a run $P_j(q, \beta)$ because of a premature A-change, what becomes garbage is not F_{j-1}, but rather what the sub-procedures called by this run were working on. The set F_{j-1} already is a $j-1$-set, so all we need is another A-change, which is provided here by the cancellation itself as opposed to being caused actively once the run reaches its goal.

Who enumerates L?

The bottom procedure $P_2(q, \beta)$, which is where the recursion reaches ground. It puts axioms $\langle r_n, n\rangle$ into L and the top decanter F_0, where $2^{-r_n} = 2^{-v}\beta$. Once it sees the corresponding $A \upharpoonright n$ description, it empties F_0 into F_1. However, if the hook $\gamma^A(m)$ belonging to P_2 moves before that, then F_0 is spilled on the floor, while F_1 is emptied into F_2.

So much for the discussion of Turing incompleteness.

Each K-trivial set is low

Recall that $J^A(e)$ is the jump.

- A procedure $P_j(q, \beta)$ is started when $J^A(e)$ newly converges. The goal q is $\alpha 2^{-e}$, where α is the garbage quota of the procedure of type P_{j+1} that called it (assuming $j < k$).

- For different e they run in parallel. So we now have a tree of decanters. (This could be avoided by letting a new convergence $J^A(e')$ ($e' < e$) cancel a run for $J^A(e)$.)

- We cannot change A actively any more (and we are happy if it doesn't). However, this creates a new type of garbage, where $P_j(q, \beta)$ reaches its goal, but no A change happens after that would allow to empty F_{j-1} into F_j. The total is $\leq \sum_e 2^{-e}\alpha$, which is ok.

The man with the golden run

- The initial procedure P_k never returns, since it has goal 1, while a k-set has weight at most $1/2$.
- So there must be a golden run of a procedure $P_{j+1}(q,\alpha)$: it doesn't reach its goal, but all the subprocedures it calls either reach their goals or are cancelled by premature A change.
- The golden run shows that A is low: When the run of the subprocedure P_j based on a computation $J^A(e)$ returns, then we guess that $J^A(e)$ converges. If A changes then P_{j+1} receives the fixed quantity $2^{-e}\alpha$. In this case we change the guess back to "divergent". This can only happen so many times (where so= $2^e q/\alpha$), else P_{j+1} reaches its goal.

Comments

- This actually shows that A is super-low: the number of mind changes in the approximation of A' is computably bounded.
- The lowness index is not obtained uniformly: we needed to know which run is golden. This non-uniformity is necessary (DHNS, 2003).

The full result

The final result is

$$K\text{-trivial} = \text{low for } K.$$

This was obtained joint with Hirschfeldt, via a modification of my result that the K-trivial sets are closed downward under $\leq_T$. It implies lowness, as

- we have seen an easy proof that each low for K set is GL_1, and
- each K-trivial set is Δ^0_2.

Proving the full result

- A procedure $P_j(q, \beta)$ is started when $U^A(\sigma) = y$ newly converges. The goal q is $\alpha 2^{-|\sigma|}$.
- It is necessary to call in parallel procedures based on different inputs σ. So we necessarily have a tree of decanters.
- At the golden run P_{j+1}, we can show that A is low for K, by emulating the cost function construction of a low for K set. When $P_j(q, \beta)$ associated with $U^A(\sigma) = y$ returns, we have the right to put a request $\langle |\sigma| + c, y \rangle$ into a set W (c is some constant). An A change has a cost, since we put a request for a wrong computation. The fact that P_{j+1} does not reach its goal implies that the cost is bounded. Hence W is KC.

Further applications

The golden run method shows:

- The cost function construction is necessary even for K-trivial Δ^0_2 sets A
- This is used to show that there is a c.e. K-trivial set Turing above A
- For each K-trivial set A, the relativized Chaitin probability Ω^A is left-c.e. (a real r is left-c.e. if $\{q \in \mathbb{Q} : q < r\}$ is c.e.)

  (The converse also holds here, in case A is Δ^0_2: if Ω^A is left-c.e. then it is Turing complete, so A is a basis for ML-randomness, hence K-trivial.)

Summary of properties of $\mathcal{K}$

The K-trivial sets form an ideal $\mathcal{K}$ in the Δ^0_2 Turing degrees. It can be characterized in many ways, and has the following nice properties.

- $\mathcal{K}$ is the downward closure of its r.e. members
- $\mathcal{K}$ is Σ^0_3
- $\mathcal{K}$, like any Σ^0_3 ideal, is contained in $[\mathbf{o}, \boldsymbol{b}]$ for some c.e. low_2 $\boldsymbol{b}$
- each $A \in \mathcal{K}$ is super-low.

Also, $X \equiv_T Y$ implies $\mathcal{K}^X = \mathcal{K}^Y$.

Question 16 *Is $\mathcal{K}$ definable in the Δ^0_2, or the c.e. degrees? Does it have an exact pair in the c.e. degrees?*

Part III: Recent developments

Effective descriptive set theory

Π^1_1 sets of numbers are a high-level analog of c.e. sets, where the steps of an effective enumeration are recursive ordinals. Hjorth and Nies (2005) have studied the analogs of K and of ML-randomness based on Π^1_1-sets.

- The KC-theorem and Schnorr's Theorem still hold, but the proofs takes considerable extra effort because of limit stages
- There is a Π^1_1 set of numbers which is K-trivial (in this new sense) and not hyperarithmetic.

The classes are different now

Theorem 17 *If A is low for Π^1_1-ML-random, then A is hyperarithmetic.*

First we show that $\omega_1^A = \omega_1^{CK}$. This is used to prove that A is in fact K-trivial at some $\eta < \omega_1^{CK}$, namely

$$\forall n \; K_\eta(A \upharpoonright n) \leq K_\eta(n) + b.$$

Then A is hyperarithmetic, by the same argument Chaitin used in the c.e. case to show that K-trivial sets are Δ^0_2:

The collection of Z which are K-trivial at η form a hyperarithmetical tree of width $O(2^b)$.

Subclasses of $\mathcal{K}$

Next we look at subclasses of the (c.e.) K-trivial sets which may be proper. We have already seen one: the c.e. sets A such that there an incomplete ML-random Z above A.

Here is a further one. This recent development was initiated by Kučera(2004).

ML-cuppability

A Δ^0_2 set A is ML-cuppable if

$$A \oplus Z \equiv_T \emptyset' \text{ for some ML-random } Z <_T \emptyset'.$$

Many sets are ML-cuppable: If A is not K-trivial, then

- $A \not\leq_T \Omega^A$ by the hungry sets theorem, and
- $A' \equiv_T \Omega^A \oplus A \geq_T \emptyset'$.

If A is also low, then $Z = \Omega^A <_T \emptyset'$, so A is ML-cuppable. This shows for instance that each c.e. non-K-trivial set B is ML-cuppable, since one can split it into low c.e. sets, $B = A_0 \cup A_1$, and one of them is also not K-trivial.

Existence

Theorem 18 *[N, 2005] There is a promptly simple set which is not ML-cuppable.*

The proof combines cost functions with the priority method. See my web page.

Question 19 *Can a K-trivial set be ML-cuppable?*

The cost functions I used in the proof of the existence theorem are much more restricted then the one used to characterize the K-trivial sets. This gives some positive evidence.

The same applies to the Kučera construction of a simple A below a Δ^0_2 ML-random $Z \upharpoonright e$. Recall here that, if $Z \upharpoonright e$ changes another time at s, then $[0, s)$ becomes taboo for S_e.

Is there a characterization of K-trivial independent of randomness and K?

Figueira, N, Stephan have tried the following strengthening of super-lowness:

For each order function h, A' has an approximation that changes at most $h(x)$ times at x.

They build a c.e. noncomputable such set, via a construction that resembles the cost function construction. No relationship to K-trivial is known.

When $\emptyset'$ looks trivial to you

B is near complete if $\emptyset'$ is K-trivial relative to A.

I observed in the 2003 lowness properties paper that Jockusch-Shore inversion applied to the operator given by the cost function construction yields such a set.

- Binns, Kjos-Hanssen, Lerman and Solomon:

  If A is a Δ^0_2 uniformly a.e. dominating set, then ML-random in A implies ML-random in $\emptyset'$. Hence A is near complete.

- Cholak, Greenberg, Miller: There is a c.e. incomplete uniformly a.e. dominating set.

Open questions

The final version of the Miller-Nies open questions paper is available on my web page. Thanks to all for the comments.

It contains lots of questions (including the ones from this talk), with definitions, motivation and background info.

Some have been solved already.

Good luck.