Reverse Mathematics and Π^1_2 Comprehension

Stephen G. Simpson

Pennsylvania State University

http://www.math.psu.edu/simpson/

simpson@math.psu.edu

Computational Prospects of Infinity

Institute for Mathematical Sciences

National University of Singapore

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Two books on reverse mathematics (a status report):

1.

S. G. Simpson (editor)
Reverse Mathematics 2001
(a volume of papers by various authors)
Lecture Notes in Logic
Association for Symbolic Logic
401 pages, in press
to appear in July 2005

2.

Stephen G. Simpson
Subsystems of Second Order Arithmetic
Second Edition
Perspectives in Logic
Association for Symbolic Logic
approximately 460 pages, in press
to appear in February 2006

Remark:

This talk represents joint work with Carl Mummert, my recent Ph.D. student at the Pennsylvania State University.

Background:

In my book *SOSOA*, a *complete separable metric space* is defined as the completion $X = (\widehat{A}, \widehat{d})$ of a countable pseudometric space (A, d). Here $A \subseteq \mathbb{N}$ and $d : A \times A \to \mathbb{R}$.

Thus complete separable metric spaces are "coded" by countable objects. Using this coding, a great deal of analysis and geometry is developed in RCA_0, with many reverse mathematics results.

A conceptual difficulty:

Before Mummert/Simpson, there was no reverse mathematics study of general topology.

The obstacle was, there was no way to discuss abstract topological spaces in L_2, the language of second order arithmetic. This was the case even for topological spaces which are separable or second countable.

The solution:

We overcome this obstacle by introducing a restricted class of topological spaces, called *countably based MF spaces*.

This class includes all complete separable metric spaces, as well as many nonmetrizable spaces.

Furthermore, this class of spaces can be discussed in L_2.

Let P be a *poset*, i.e., a partially ordered set.

Definition. A *filter* is a set $F \subseteq P$ such that

1. F is *upward closed*, i.e., $(q \geq p \wedge p \in F) \Rightarrow q \in F$.
2. for all $p, q \in F$ there exists $r \in F$ such that $r \leq p$ and $r \leq q$.

Compare the treatment of forcing in Kunen's textbook of axiomatic set theory.

Definition. A *maximal filter* is a filter which is not properly included in any other filter.

By Zorn's Lemma, every filter is included in a maximal filter.

Definition.

$\text{MF}(P) = \{F \mid F \text{ is a maximal filter on } P\}$.

Definition.
MF(P) $= \{F \mid F$ is a maximal filter on $P\}$.

MF(P) is a topological space with basic open sets
$$N_p = \{F \mid p \in F\}$$
for all $p \in P$.

Definition. An *MF space* is a space of the form MF(P) where P is a poset.

Definition. A *countably based MF space* is a space of the form MF(P) where P is a countable poset.

Thus, the second countable topological space MF(P) is "coded" by the countable poset P.

Therefore, countably based MF spaces can be defined and discussed in L_2. Thus we can do reverse mathematics in the usual setting, subsystems of second order arithmetic.

Characterization problems:

1. To characterize those topological spaces which are homeomorphic to MF spaces.

2. To characterize those topological spaces which are homeomorphic to countably based MF spaces.

These problems seem difficult.

Examples of MF spaces.

Many topological spaces are homeomorphic to MF spaces:

- all complete metric spaces.
- all locally compact Hausdorff spaces.
- the weak-star dual of any Banach space.
- any G_δ subset of any MF space.
- the Baire space ω^ω with the topology generated by the Σ^1_1 sets, i.e., the Gandy/Harrington topology.

The latter is a neat example of a countably based MF space which is Hausdorff but not metrizable. However, the dense open subset $\{f \in \omega^\omega \mid \omega_1^f = \omega_1^{\mathsf{CK}}\}$ is completely metrizable.

Theorem. Every complete (separable) metric space is homeomorphic to a (countably based) MF space.

Proof (sketch). Let $\widehat{A}$ be a complete metric space with dense subset A. Let $P = A \times \mathbb{Q}^+$ ordered by $(a, r) < (b, s)$ if and only if $d(a, b) + r < s$. We argue that MF(P) is homeomorphic to X. Given a maximal filter F on P, we claim that $\inf\{r \mid (a, r) \in F\} = 0$. Suppose the inf is $h > 0$. Let $(a, r) \in F$ be such that $h \leq r < 4h/3$. We show that $(a, r/3) < (b, s)$ for all $(b, s) \in F$, contradicting maximality. Given $(b, s) \in F$, let $(c, t) \in F$ be such that $(c, t) < (a, r)$ and $(c, t) < (b, s)$. We have $h \leq t < r < 4h/3$ and $d(a, c) + h \leq d(a, c) + t < r < 4h/3$, hence $d(a, c) < h/3$, hence $d(a, c) + r/3 < d(a, c) + 4h/9 \leq h/3 + 4h/9 = 7h/9 < t$ so $(a, r/3) < (c, t) < (b, s)$, proving the claim. Hence F is generated by $(a_0, r_0) > (a_1, r_1) > \cdots > (a_n, r_n) > \cdots$ with $\lim_n r_n = 0$, giving a point of $\widehat{A}$.

Metrization theorems:

Urysohn Metrization Theorem. A second countable topological space is metrizable if and only if it is regular. (A topological space is said to be *regular* if, for every open set U and point $x \in U$, there exists an open set V such that $x \in V$ and the closure of V is included in U. See Kelley, *General Topology*.)

Choquet Metrization Theorem. A topological space is completely metrizable if and only if it is metrizable and has the *strong Choquet property*. (This is a game-theoretic property which is similar to, but stronger than, the property of Baire. See Kechris, *Classical Descriptive Set Theory*.)

Theorem. All MF spaces have the strong Choquet property. (See Mummert's Ph.D. thesis, 2005.)

Metrization theorems, continued.

Combining the above results, we have the following metrization theorem for countably based MF spaces.

> MFMT: A countably based MF space is completely metrizable if and only if it is regular.

Note that the statement MFMT can be formalized as a sentence in the language of second order arithmetic.

We study the reverse mathematics of MFMT.

We consider the following subsystems of second order arithmetic.

ACA_0 = arithmetical comprehension.

Π^1_1-CA_0 = Π^1_1 comprehension.

Π^1_2-CA_0 = Π^1_2 comprehension.

Remark. The basic theory of MF spaces can be formalized in ACA_0. In particular, it is provable in ACA_0 that every complete separable metric space is homeomorphic to a countably based MF space.

Theorem. MFMT is equivalent to Π^1_2-CA_0. The equivalence is provable in Π^1_1-CA_0.

We outline the proof of this theorem.

Lemma 1. MFMT is provable in Π^1_2-CA$_0$.

Proof. **Part 1**. Assume MF(P) is regular. Use Π^1_2 comprehension to form the set $\{(p,q) \in P \times P \mid N_p \supseteq$ closure of $N_q\}$. Use this set as a parameter. Follow Matthias Schröder's effective adaptation of the original Urysohn argument, to find a metric d_1 for MF(P). Thus MF(P) is metrizable.

Part 2. Fix a countable dense set $A \subseteq$ MF(P). Use Π^1_2 comprehension to form the sets $\{(a,r,p) \in A \times \mathbb{Q}^+ \times P \mid B(a,r) \subseteq N_p\}$ and $\{(a,r,p) \in A \times \mathbb{Q}^+ \times P \mid N_p \subseteq B(a,r)\}$, where $B(a,r) = \{x \in$ MF(P) $\mid d_1(a,x) < r\}$. Using these sets as parameters, construct a G_δ set in $(\widehat{A}, \widehat{d}_1)$ which has the same points as MF(P) and is homeomorphic to MF(P). It follows that MF(P) is homeomorphic to a complete separable metric space $(\widehat{A}, \widehat{d}_2)$.

Note: Choquet's game-theoretic argument is not formalizable in second order arithmetic. Instead, we argue directly within Π^1_2-CA$_0$.

Lemma 2. Over Π^1_1-CA_0, MFMT implies Π^1_2 comprehension.

Proof. Let $\psi(n, X)$ be a Π^1_1 formula. Assuming MFMT, we prove the existence of the Σ^1_2 set $S = \{n \mid \exists X\, \psi(n, X)\}$.

We write $\psi(n, X) \equiv \neg\, \exists f\, \forall m\, R(n, X[m], f[m])$ where $X[m] = \langle X(0), \ldots, X(m-1)\rangle$ and $f[m] = \langle f(0), \ldots, f(m-1)\rangle$. Let P be the countable poset consisting of all $(n, X[k], f[k])$ such that $(\forall m \leq k)\, R(n, X[m], f[m])$, plus all $(n, X[k])$, partially ordered by:

1. $(n, X[k], f[k]) < (n', X'[k'], f'[k'])$ iff $n = n'$ and $X[k] \supset X'[k']$ and $f[k] \supset f'[k']$.

2. $(n, X[k]) < (n', X'[k'])$ iff $n = n'$ and $X[k] \supset X'[k']$.

3. $(n, X[k], f[k]) < (n', X'[k'])$ iff $n = n'$ and $X[k] \supset X'[k']$.

4. $(n, X[k]) < (n', X'[k'], f'[k'])$ never.

The maximal filters on P are of three types:

1. $F = \{p \in P \mid q \leq p\}$,
where q is a minimal element of P.

2. $F = \{(n, X[k], f[k]), (n, X[k]) \mid k \in \mathbb{N}\}$,
where n, X, f are such that
$\forall m\, R(n, X[m], f[m])$ holds.

3. $F = \{(n, X[k]) \mid k \in \mathbb{N}\}$,
where n, X are such that $\psi(n, X)$ holds.

Let C be the closed set in MF(P) consisting of all F of type 3. The complement of C is the open set $\bigcup_{n \in \mathbb{N}} N_{(n,\langle\rangle,\langle\rangle)}$.

By Kondo's Π^1_1 Uniformization Theorem (provable in Π^1_1-CA_0, SOSOA §VI.2), we may assume that $\forall n\,(\exists$ at most one $X)\, \psi(n, X)$. Thus, for each n, $C \cap N_{(n,\langle\rangle)}$ contains at most one point.

Under this assumption, it is straightforward to show that MF(P) is regular.

By MFMT, there is a homeomorphism $\Phi : \mathsf{MF}(P) \cong \widehat{A}$, where $\widehat{A}$ is a complete separable metric space. In particular, $\Phi(C) \subseteq \widehat{A}$ is closed, and the open sets $\Phi(N_{(n,\langle\rangle)}) \subseteq \widehat{A}$ are arithmetical uniformly in n, using a code of Φ^{-1} as a parameter. Hence by Π^1_1 comprehension we may form the set

$$\begin{aligned} S &= \{n \mid \Phi(C) \cap \Phi(N_{(n,\langle\rangle)}) \neq \emptyset\} \\ &= \{n \mid C \cap N_{(n,\langle\rangle)} \neq \emptyset\} \\ &= \{n \mid \exists X\, \psi(n, X)\}\,. \end{aligned}$$

This completes the proof.

Remark. This is the first instance of a core mathematical theorem equivalent to Π^1_2 comprehension. Previous reverse mathematics results within second order arithmetic have involved only weaker set existence axioms.

(However, Heinatsch and Möllerfeld have shown that Π^1_2-CA_0 proves the same Π^1_1 sentences as $\mathsf{ACA}_0 + {<}\omega$-$\Sigma^0_2$ determinacy.)

Another result:

Theorem. The following are equivalent over ACA_0.

1. In any countably based MF space, any uncountable closed set contains a perfect set.

2. $\forall X\,(\aleph_1^{L(X)}$ is countable).

References:

Carl Mummert and Stephen G. Simpson, Reverse Mathematics and Π^1_2 Comprehension, Bulletin of Symbolic Logic, to appear.

Carl Mummert, Ph.D. thesis, *On the Reverse Mathematics of General Topology*, 2005, Pennsylvania State University.

Mummert's forthcoming papers.

THE END