Derived models associated to mice

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Abstract

This is a set of notes meant to accompany a series of lectures at the workshop *Computational Prospects of Infinity*, to be held at the Institute for Mathematical Sciences of the National University of Singapore, in June 2005. These notes are very much in progress, and likely to be revised and substantially extended in the coming weeks. This version was completed June 26, 2005.

1 Lecture Plan

We shall present some results on the the properties of derived models of mice, and on the existence of mice with large derived models. The basic plan of the lectures is:

Lecture 1. Introduction. Preliminary definitions and background material. The iteration independence of the derived model associated to a mouse. The mouse set conjecture in derived models associated to mice.

Lecture 2. The Solovay sequence in derived models associated to mice. A method of "translating away" extenders overlapping Woodin cardinals in mice.

Lectures 3,4. Some partial results on capturing Σ_1^2 truths in AD^+ models by mice. (That is, partial results on the mouse set conjecture.)

The notes below follow this general plan, although they contain much more background and detail than the author can hope to convey in the lectures themselves. At the same time, these notes are far from self-contained. We have gathered together a fair amount of background material for the reader's convenience, but in order to keep this document to a reasonable size, we have also simply pointed to the literature at many points.

2 Introduction

Here are some well-known Holy Grails of inner model theory:

Conjecture 2.0.1 *Each of the following statements implies the existence of an $\omega_1 + 1$-iterable mouse with a superstrong cardinal:*

(1) There is a supercompact cardinal,

(2) There is a strongly compact cardinal,

(3) PFA*,*

(4) CH *holds, and there is a homogeneous pre-saturated ideal on* ω_1*,*

(5) $\mathsf{AD}_{\mathbb{R}}$ *holds, and* θ *is regular.*

This list could be lengthened substantially. In all cases, we can prove the statement in question implies that there is an iterable mouse with infinitely many Woodin cardinals. In cases (2)-(5), we cannot yet produce a mouse with a Woodin limit of Woodin cardinals. (Case (1) is special; here the best partial results are those of [7], which reach a bit past a Woodin limit of Woodins.)

The partial results we have in cases (2)-(5) come from the *core model induction* method. In this method (due to Woodin), one uses core model theory to construct new mice, while using descriptive set theory to keep track of the degree of correctness and the complexity of the iteration strategies of the mice one has constructed. See [2], [23], and [18] for examples of core model inductions. We suspect that even in case (1), one will not be able to go significantly beyond where we are now without bringing core model induction ideas.

Core model inductions reaching infinitely many Woodins, or further, rely on a key step in which one passes from some kind of *hybrid mouse* to an ordinary $L[\vec{E}]$ mouse. The hybrid mouse $\mathcal{H}$ will have a small number of Woodin cardinals, but these cardinals will be Woodin with respect to predicates coding up a lot of information. The $L[\vec{E}]$ mouse $\mathcal{M}$ is built by a full background extender construction inside $\mathcal{H}$, and may have all sorts of Woodin cardinals, even cardinals strong past a Woodin, and presumably even superstrong cardinals. We sometimes call the ordinary $L[\vec{E}]$ mice *ms-mice*; they are mice in the sense of [11], except that we allow them to be relativised by putting some arbitrary transitive set y at the bottom, in which case we speak of an ms-mouse *over* y). (See e.g. [18].) The following conjecture is the basic test problem for our ability to translate hybrid mice into ordinary ms-mice.

Conjecture 2.0.2 ((Mouse Set Conjectures (MSC))) *Assume* AD^+*, and that there is no* ω_1*-iteration strategy for a mouse with a superstrong cardinal; then*

(I) If $x, y \in HC$, and $x \subseteq y$, and x is ordinal definable from parameters in $y \cup \{y\}$, then there is an ω_1-iterable ms-mouse $\mathcal{M}$ over y such that $x \in \mathcal{M}$, and

(II) if $y \in HC$ and $\exists A \subseteq \mathbb{R}(HC, \in, A) \models \varphi[y]$, then there is an ω_1-iterable ms-mouse $\mathcal{M}$ over y, and a λ such that

$$\mathcal{M} \models \lambda \textit{ is a limit of Woodin cardinals},$$

and

$$\mathcal{M} \models \exists A \in \mathrm{Hom}_{<\lambda}((HC, \in, A) \models \varphi[y]).$$

MSC2 is equivalent to asserting that the Σ^2_1 fact $\exists A \subseteq \mathbb{R}(\mathrm{HC}, \in, A) \models \varphi[y]$ holds in the derived model of $\mathcal{M}$ below λ. It is easy to show that MSC2 implies MSC1, and not too hard to show that MSC1 implies MSC2. We shall therefore use MSC to stand for either MSC1 or MSC2, in situations where we don't need to make a distinction.

Clearly, the conclusion of MSC1 is an equivalence: if x is in an iterable mouse over y, then $x \in \mathrm{OD}(y \cup \{y\})$. In MSC2, one seems to need to add something about the iteration strategy for $\mathcal{M}$ in order to conclude that any Σ^2_1 statement true in the derived model of $\mathcal{M}$ is actually true. This can be done in a natural way. (It is enough that the strategy be "good" in the sense described below.)

MSC is known for a certain initial segment of the Wadge hierarchy. Woodin has shown:

Theorem 2.1 (Woodin, late 90's) MSC *holds if the hypothesis is strengthened to* AD^+ *plus there is no iteration strategy for an ms-mouse $\mathcal{M}$ such that for some λ,*

$$\mathcal{M} \models \lambda \textit{ is a limit of Woodin cardinals },$$

and

$$\{\kappa \mid \kappa \textit{ is } < \lambda\textit{-strong in } \mathcal{M}\} \textit{ has order type } \lambda.$$

The proof builds on Woodin's proof of MSC under the stronger hypothesis that there is no iteration strategy for a nontame mouse. That proof is written up in [15]. There are many additional difficulties in dealing with ms-mice having extenders overlapping Woodin cardinals, however.

More recently, Neeman and the author have shown

Theorem 2.2 (Neeman-Steel, 2004) MSC *holds if the hypothesis is strengthened to* AD^+ *plus there is no iteration strategy for an ms-mouse $\mathcal{M}$ such that for some λ,*

$$\mathcal{M} \models \lambda \textit{ is a limit of Woodin cardinals},$$

and for some $\kappa < \lambda$,

$$\mathcal{M} \models \kappa \textit{ is } < \lambda \textit{ strong, and a limit of } < \lambda\textit{-strongs.}$$

Although the large cardinal reached here is only slightly beyond the large cardinals reached in 2.1, the proof has some new ideas, and seems somewhat simpler, and closer to Woodin's orginal argument in the tame mouse case. Theorem 2.2 is the main goal of lectures 3 and 4.

The mouse set conjectures ask us to construct ms-mice with a given derived models. This leads naturally to the question: *what can one say about the derived model of a mouse?* (See [17] for an exposition of the basic facts about the derived model construction.) This will be the focus of lectures 1 and 2. We shall show that for certain "tractable" $\mathcal{M}, \lambda$ such that $\mathcal{M}$ is a mouse and λ is a limit of Woodins in $\mathcal{M}$, the derived model $\mathcal{D}(\mathcal{M}, \lambda)$ of $\mathcal{M}$ at λ satisfies MSC. We shall show that for these tractable pairs $(\mathcal{M}, \lambda)$, there is a canonical derived model of an iterate of $\mathcal{M}$ whose reals are precisely the reals in V. Finally, we shall investigate the *Solovay sequence* $\langle\theta_\alpha\rangle$ of various $\mathcal{D}(\mathcal{M}, \lambda)$. For example, letting $M^\sharp_{\text{wdn.lim}}$ be the minimal active mouse with a Woodin limit of Woodin cardinals, we shall show

Theorem 2.3 (Closson, Steel) *Let λ be the Woodin limit of Woodins in $M^\sharp_{wdn.lim}$; then*

$$\mathcal{D}(M^\sharp_{wdn.lim}, \lambda) \models \theta = \theta_\theta.$$

We do not know what the cofinality of θ is in the model of 2.3. Possibly θ is regular here, in which case $\mathsf{AD}_\mathbb{R}$ plus "θ is regular" is much weaker than conjecture 2.0.1(5) would have it.

3 Some background and preliminaries

3.1 Homogeneously Suslin sets

The good sets of reals, from the point of view of descriptive set theory, are the homogeneously Suslin sets.

Definition 3.1 *A homogeneity system with support Z is a function $\bar{\mu}$ such that for all $s, t \in Y^{<\omega}$,*

1. *μ_t is a countably complete ultrafilter concentrating on $Z^{\mathrm{dom}(t)}$, and*
2. *$s \subseteq t \Rightarrow \mu_t$ projects to μ_s.*

If all μ_t are κ-complete, then we say $\bar{\mu}$ is κ-complete.

Definition 3.2 *If $\bar{\mu}$ is a homogeneity system, then for $x \in \omega^{<\omega}$,*

$$x \in S_{\bar{\mu}} \iff \text{the tower of measures } \langle \mu_{x \restriction n} \mid n < \omega\rangle \text{ is countably complete.}$$

Let

$$\mathrm{Hom}_\kappa = \{S_{\bar\mu} \mid \bar\mu \text{ is } \kappa\text{-complete } \},$$

$$\mathrm{Hom}_{<\lambda} = \bigcap_{\kappa<\lambda} \mathrm{Hom}_\kappa,$$

$$\mathrm{Hom}_\infty = \bigcap_{\kappa\in\mathrm{OR}} \mathrm{Hom}_\kappa\,.$$

A set of reals is *homogeneously Suslin* just in case it is in Hom_κ, for some κ. Although not literally stated in the paper, one of the main results of Martin [4] is that every homogeneously Suslin set is determined. There are no interesting homogeneously Suslin sets unless there are measurable cardinals. For the most part, we shall be working under the assumption that there are infinitely many Woodin cardinals, in which case one has:

Theorem 3.3 *Let λ be a limit of Woodin cardinals; then*

(a) $\mathrm{Hom}_{<\lambda}$ *is closed under complements and real quantification ([6]),*

(b) every $\mathrm{Hom}_{<\lambda}$ *set has a* $\mathrm{Hom}_{<\lambda}$ *scale ([17]).*

Feng, Magidor, and Woodin ([1]) introduced an important notion which turns out to be equivalent to homogeneity in the presence of Woodin cardinals.

Definition 3.4 *Let T and T^* be trees on $\omega \times X$ and $\omega \times Y$ respectively; then T and T^* are* κ-absolute complements *iff whenever G is V-generic over a poset of size $\leq \kappa$, then $V[G] \models p[T] = \mathbb{R} \setminus p[T^*]$. We say that $A \subseteq \mathbb{R}$ is* κ-universallly Baire *(or κ-UB) iff $A = p[T]$ for some tree T for which there is a κ-absolute complement.*

The results of [5], [21], and [6] show that if λ is a limit of Woodin cardinals, then the $\mathrm{Hom}_{<\lambda}$ sets are precisely the $<\lambda$-universally Baire sets. (See [17].) The theorem and this equivalence also hold for $\lambda = \infty = \mathrm{OR}$.

Stronger large cardinal hypotheses imply stronger closure properties of $\mathrm{Hom}_{<\lambda}$. For example, Woodin has shown that if λ is a limit of Woodin cardinals, and there is a measurable cardinal above λ, then for any $A \in \mathrm{Hom}_{<\lambda}$, $A^\sharp \in \mathrm{Hom}_{<\lambda}$. However, no large cardinal hypothesis is known to imply that sets of reals definable using quantification over $\mathrm{Hom}_{<\lambda}$ must be $\mathrm{Hom}_{<\lambda}$, or even good in some weaker sense. The inner model theory for such a large cardinal hypothesis would be significantly different from the one we have now, as we explain in the next subsection.

3.2 Hom_∞ iteration strategies

Let $\mathcal{M}$ be a *ms-premouse*: a model of the form $(J_\alpha[\vec{E}], \in, \vec{E})$, where $\vec{E}$ is a coherent sequence of extenders. In the *iteration game* $G(\mathcal{M}, \theta)$, I and II cooperate to build an iteration tree $\mathcal{T}$ on $\mathcal{M}$:

- I extends $\mathcal{T}$ at successor steps by applying an extender from the current model to some, possibly earlier, model. The resulting ultrapower is the new current model.
- At limit steps II picks a cofinal branch b, and $\lim_{\alpha \in b} \mathcal{M}_\alpha$ becomes the current model.

II wins if, after θ moves, no illlfounded model has been produced.

Definition 3.5 *A* θ-iteration strategy *for* $\mathcal{M}$ *is a winning strategy for* II *in* $G(\mathcal{M}, \theta)$. $\mathcal{M}$ *is* θ-iterable *if there is such an* iteration strategy.

Notice that if $\mathcal{M}$ is countable, then an ω_1-iteration strategy for $\mathcal{M}$ is essentially a set of reals. "Good" $\mathcal{M}$ (the "standard" ones), have ω_1-iteration strategies which are Hom_∞. This implies θ-iterability, for all θ.

For $L[\vec{E}]$-models in the region where we have a theory (can prove iterability),

$$\begin{aligned} x <^{L[\vec{E}]} y \iff & \exists \mathcal{M}(\mathcal{M} \text{ has a } \mathrm{Hom}_\infty \text{ iteration strategy} \\ & \text{and } \mathcal{M} \models x <^{L[\vec{E}]} y. \end{aligned}$$

So the large cardinal hypotheses true in these models are compatible with there being a wellorder of $\mathbb{R}$ in $L(\mathbb{R}, \mathrm{Hom}_\infty)$.

3.3 The derived model

Given $A \in \mathrm{Hom}_\kappa$, as witnessed by both $\bar{\mu}$ and $\bar{\nu}$, and G V-generic for a poset of size $< \kappa$, we have

$$(S_{\bar{\mu}})^{V[G]} = (S_{\bar{\nu}})^{V[G]}.$$

So we write $A^{V[G]}$ for the common value of all $(S_{\bar{\mu}})^{V[G]}$ such that $A = S_{\bar{\mu}}$.

Now let λ be a limit of Woodin cardinals, and G be V-generic for $\mathrm{Col}(\omega, < \lambda)$, and set

$$\begin{aligned} \mathbb{R}^*_G &= \bigcup_{\alpha < \lambda} \mathbb{R} \cap V[G \upharpoonright \alpha], \\ \mathrm{Hom}^*_G &= \{A^* \mid \exists \alpha < \lambda (A \in \mathrm{Hom}_{<\lambda}^{V[G \upharpoonright \alpha]})\} \end{aligned}$$

where $A^* = \bigcup_{\beta < \lambda} A^{V[G \upharpoonright \beta]}$. Note that one has $\mathrm{Hom}^*_G = \{p[T] \cap \mathbb{R}^*_G \mid \exists \alpha < \lambda (V[G \upharpoonright \alpha] \models T$ has an $< \lambda$-absolute complement$)\}$.

Theorem 3.6 (Derived model theorem, Woodin 1987)

(1) $L(\mathbb{R}^*, \mathrm{Hom}^*) \models \mathsf{AD}^+$.

(2) $\mathrm{Hom}^* = \{A \subseteq \mathbb{R}^* \mid A, \mathbb{R} \setminus A$ *have scales in* $L(\mathbb{R}^*, \mathrm{Hom}^*)$.

If $\mathcal{M} \models \lambda$ is a limit of Woodins, and a reasonable fragment of ZFC, then we write

$$D(\mathcal{M}, \lambda)$$

for the associated derived model. This is analogous to the V^P notation, since we have not specified a generic. Since the forcing is homogeneous, the theory of $D(\mathcal{M}, \lambda)$ does not depend on a generic. We shall speak of "realizations" of $D(\mathcal{M}, \lambda)$ if we have specified a generic G. Note that only $\mathbb{R}^*_G$, not all of G, is needed to realize $D(\mathcal{M}, \lambda)$.

Remark 3.7 This is the "old" derived model; the model is always of the form $L(\Gamma, \mathbb{R})$, where Γ is the class of its Suslin-co-Suslin sets. Not all models of AD^+ have this form. Woodin has shown that the Suslin-co-Suslin sets of any AD^+ model can be realized as the Suslin-co-Suslin sets of a derived model, however, and given a variant construction which hits all models of AD^+ in full.

If $\mathcal{M} = M_\omega$, the minimal mouse with ω Woodins, then letting λ be their sup, $D(\mathcal{M}, \lambda) = L(\mathbb{R}^*)$. On the other hand, for stronger $\mathcal{M}$, $D(\mathcal{M}, \lambda)$ may be larger.

Definition 3.8 *(AD^+.) For $A \subseteq \mathbb{R}$, $\theta(A)$ is the least ordinal α such that there is no surjection of $\mathbb{R}$ onto α which is ordinal definable from A and a real. We set*

$$\begin{aligned}
\theta_0 &= \theta(\emptyset), \\
\theta_{\alpha+1} &= \theta(A), \text{ for any (all) } A \text{ of Wadge rank } \theta_\alpha, \\
\theta_\lambda &= \bigcup_{\alpha<\lambda} \theta_\alpha.
\end{aligned}$$

$\theta_{\alpha+1}$ is defined iff $\theta_\alpha < \Theta$. Note $\theta(A) < \Theta$ iff there is some $B \subseteq \mathbb{R}$ such that $B \notin \mathrm{OD}(\mathbb{R} \cup \{A\})$. In this case, $\theta(A)$ is the least Wadge rank of such a B.

Theorem 3.9 (Woodin, mid 80's) *Assume AD^+, and suppose A and $\mathbb{R} \setminus A$ admit scales; then*

(a) All $\Sigma^2_1(A)$ sets of reals admit scales, and

(b) All $\Pi^2_1(A)$ sets admit scales iff $\theta(A) < \Theta$.

Theorem 3.10 (Martin, Woodin, mid 80's) *Assume* AD^+*; then the following are equivalent:*

(1) $\mathsf{AD}_{\mathbb{R}}$,

(2) Every set of reals admits a scale,

(3) $\Theta = \theta_\lambda$*, for some limit* λ*.*

Sadly, the proofs of these theorems have never fully appeared. There is a good deal on consequences of AD^+ in [21]. There is most of a proof that $\mathrm{Scale}(\Sigma^2_1)$ holds in derived models in [17], section 7.

Theorem 3.11 (Woodin, 1988, 2000) *Suppose* λ *is a limit of Woodins, and* $L(\mathbb{R}^*, \mathrm{Hom}^*)$ *is a derived model at* λ*; then*

(a) $\exists\kappa < \lambda(\kappa$ *is* $< \lambda$*-strong* $\Rightarrow L(\mathbb{R}^*, \mathrm{Hom}^*) \models \theta_0 < \Theta$.

(b) λ *is a limit of* κ *which are* $< \lambda$*-strong* $\Rightarrow L(\mathbb{R}^*, \mathrm{Hom}^*) \models \Theta = \theta_\alpha$*, for some limit* α*.*

(c) λ *is an inaccessible limit of* κ *which are* $< \lambda$*-strong* $\Rightarrow L(\mathbb{R}^*, \mathrm{Hom}^*) \models \Theta = \theta_\alpha$*, where* $\mathrm{cof}(\alpha) \geq \omega_1$.

(d) $\exists\kappa < \lambda(\kappa$ *is* λ*-supercompact* $) \Rightarrow L(\mathbb{R}^*, \mathrm{Hom}^*) \models \mathsf{AD}_{\mathbb{R}} + \Theta$ *is regular.*

Remarks.

(i) Parts (a)-(c) are proved in [17]. Part (d) has not been written up.

(ii) The hypotheses of (a)-(c) follow from λ being a Woodin limit of Woodin cardinals.

(iii) The hypothesis of (b), that there is a λ which is a limit of Woodins and of $< \lambda$-strongs, is called the $\mathsf{AD}_{\mathbb{R}}$ *hypothesis.* Woodin has shown the $\mathsf{AD}_{\mathbb{R}}$ hypothesis is equiconsistent with $\mathsf{AD}_{\mathbb{R}}$. One direction is (b) above.

(iv) There is a big jump going from (c) to (d). How strong is $\mathsf{AD}_{\mathbb{R}} + \Theta$ regular ?

(v) There are other ways of calibrating the strength of determinacy models. One can look at the complexity of the games of length ω_1 which are determined in the model. One can look at the complexity of the mice which have iteration strategies in the model.

3.4 Iterations to make $\mathbb{R}^V = \mathbb{R}^*$

Let $\mathcal{M}$ be countable and (ω_1+1)-iterable, with λ a limit of Woodin cardinals of $\mathcal{M}$. Working in $V^{\mathrm{Col}(\omega,\mathbb{R})}$, we can form an $\mathbb{R}$-*genericity iteration* (of $\mathcal{M}$, below λ), that is, a sequence

$$I = \langle \mathcal{T}_n \mid n < \omega \rangle$$

such that the $\mathcal{T}_n$ are iteration trees whose composition

$$\mathcal{T} = \oplus_n \mathcal{T}_n$$

is a normal, nondropping iteration tree on $\mathcal{M}$, with

$$\mathcal{M}^I_\infty = \lim_n \mathcal{M}^I_n,$$

the direct limit along the main branch of $\mathcal{T}$ (where $\mathcal{M}^I_n$ is the base model of $\mathcal{T}_n$, and the last model of $\mathcal{T}_{n-1}$ if $n > 0$), being such that $\mathbb{R}^V$ is the reals of a symmetric collapse over $\mathcal{M}^I_\infty$ below λ^I_∞, the image of λ. We write

$$\mathrm{Hom}^*_I = \bigcup \{ p[T] \cap \mathbb{R}^V \mid \exists x \in \mathbb{R}^V (\mathcal{M}^I_\infty \models T \text{ is} < \lambda \text{ absolutely complemented})\},$$

and

$$D(\mathcal{M}^I_\infty, \lambda^I_\infty) = L(\mathbb{R}^V, \mathrm{Hom}^*_I)$$

for the derived model of $\mathcal{M}^I_\infty$ at λ^I_∞ whose set of reals is $\mathbb{R}^* = \mathbb{R}^V$.

4 Iteration independence for derived models of mice

In this section we consider derived models of ms-mice.

An $\mathbb{R}$-genericity iteration I of a countable $\mathcal{M}$ cannot belong to V, unless λ happens to be measurable in $\mathcal{M}$. Nevertheless, in many interesting cases, the derived model $D(M^I_\infty)$ is in V.

We need a minor technical strengthening of iterability in order to state this result precisely. Let us call an $\omega_1 + 1$iteration strategy Σ for $\mathcal{M}$ *good* iff whenever $\mathcal{T}$ is a countable normal iteration tree played by Σ with last model $\mathcal{P}$, and $\beta < \mathrm{lh}(\mathcal{T})$, then the phalanx obtained from $\Phi(\mathcal{T} \restriction (\beta + 1))$ obtained by replacing its last model $\mathcal{M}^{\mathcal{T}}_\beta$ with $\mathcal{P}$ is such that Ψ is $\omega_1 + 1$-iterable. All our iterability proofs give good strategies, but we don not see how to show that any $\omega_1 + 1$, or even $(\omega, \omega_1 + 1)$, iteration strategy is good in the abstract.

Proposition 4.0.1 *Let $\mathcal{M}$ be ω-sound and project to ω, and let Σ be a good ω_1+1 iteration strategy for $\mathcal{M}$. Let I be an $\mathbb{R}$-genericity iteration of $\mathcal{M}$ below λ such that I is played according to Σ; then every set in Hom^*_I is projective in $\Sigma \restriction HC$. In particular, $\mathrm{Hom}^*_I \subseteq V$.*

Proof. Let $A \in \mathrm{Hom}_I^*$, as witnessed by $T \in \mathcal{M}_\infty^I[x]$, where $x \in \mathbb{R}^V$. Let $S \in \mathcal{M}_\infty^I[x]$ absolutely complement T for forcings of size $< \lambda_\infty^I$. Let $I = \langle \mathcal{T}_n \mid n < \omega \rangle$, and let $n < \omega$ be large enough that, letting

$$\mathcal{N} = \text{ last model of } \mathcal{T}_n \ ,$$

and

$$\pi : \mathcal{N} \to \mathcal{M}_\infty^I$$

be the canonical embedding, we have that for $\kappa = \mathrm{crit}(\pi)$,

$$x \in \mathcal{N}[g],$$

where $g \in V$ is $\mathcal{N}$-generic over $\mathrm{Col}(\omega, \eta)$, for some $\eta < \kappa$. Note that we can lift π to an embedding of $\mathcal{N}[g]$ into $\mathcal{M}_\infty^I[g]$, which we also call π. We assume that n has been chosen large enough that the g-terms for T and S have pre-images in $\mathcal{N}$, and let then

$$\pi((\bar{T}, \bar{S})) = (T, S).$$

Working now in V, we can define A as follows: for $y \in \mathbb{R}$,

$$\begin{aligned} y \in A \quad \Leftrightarrow \quad & \exists \mathcal{U} \in \mathrm{HC}((\oplus_{k \leq n} \mathcal{T}_k) \oplus \mathcal{U} \text{ is a normal iteration tree} \\ & \text{on } \mathcal{M} \text{ played by } \Sigma, \text{ and} y \in p[i^{\mathcal{U}}(\bar{T})]. \end{aligned}$$

On the right hand side of this equivalence, "$i^{\mathcal{U}}$" stands for the lift of the canonical embedding of $\mathcal{U}$, so that $i^{\mathcal{U}} : \mathcal{N}[g] \to \mathcal{P}[g]$, where $\mathcal{P}$ is the last model of $\mathcal{U}$.

The $\Rightarrow$ direction of the equivalence follows by taking $\mathcal{U} = \oplus_{n < k \leq m} \mathcal{T}_k$, for m sufficiently large. For the other direction, suppose $\mathcal{U}$ is as on the right hand side, and toward contradiction, that $y \notin A$. We can then find an iteration tree $\mathcal{W}$ on $\mathcal{N}$ such that $\oplus_{k \leq n} \mathcal{T}_k \oplus \mathcal{W}$ is a normal tree by Σ with $y \in p[i^{\mathcal{W}}(\bar{S})]$, where $i^{\mathcal{W}}$ is the lift of the canonical embedding of $\mathcal{W}$. (Take $\mathcal{W} = \oplus_{n < k \leq m} \mathcal{T}_k$, for m sufficiently large.) Let $\mathcal{P}$ and $\mathcal{Q}$ be the last models of $\mathcal{U}$ and $\mathcal{W}$ respectively, and let Ψ and Γ be the phalanxes obtained from $\Phi((\oplus_{k \leq n} \mathcal{T}_k)$ by replacing its last model $\mathcal{N}$ with $\mathcal{P}$ and $\mathcal{Q}$, respectively. Since Σ is good, Ψ and Γ are $\omega_1 + 1$ iterable, so we can coiterate them.

Standard arguments using the fact that $\mathcal{M}$ is ω-sound and projects to ω show that the last model on the two sides in the comparison of Ψ and Γ is the same, call it $\mathcal{R}$, and is above $\mathcal{P}$ and $\mathcal{Q}$ respectively, and the branches $\mathcal{P}$-to-$\mathcal{R}$ and $\mathcal{Q}$-to-$\mathcal{R}$ do not drop. Thus we have

$$\mathcal{N} \xrightarrow{i^{\mathcal{U}}} \mathcal{P} \xrightarrow{j} \mathcal{R}$$

and

$$\mathcal{N} \xrightarrow{i^{\mathcal{W}}} \mathcal{Q} \xrightarrow{k} \mathcal{R},$$

where j and k are given by the comparison of Ψ and Γ. Let

$$\sigma \colon \mathcal{M} \to \mathcal{N}$$

be given by $\oplus_{k \leq n} \mathcal{T}_k$, and let ν be the sup of the generators of extenders in this tree used on $\mathcal{M}$-to-$\mathcal{N}$. Every element of $\mathcal{N}$ is definable from points in $\mathrm{ran}(\sigma)$ and ordinals $< \nu$. On the other hand, $j \circ i^{\mathcal{U}}$ and $k \circ i^{\mathcal{W}}$ are the identity on ν, and agree on $\mathrm{ran}(\sigma)$. Thus

$$j \circ i^{\mathcal{U}} = k \circ i^{\mathcal{W}}.$$

But then $y \in p[i^{\mathcal{U}}(\bar{T}]$, so $y \in p[j(i^{\mathcal{U}}(\bar{T}))]$, so $y \in p[k(i^{\mathcal{W}}(\bar{T}))]$. On the other hand, $y \in p[i^{\mathcal{W}}(\bar{S}]$, so $y \in p[k(i^{\mathcal{W}}(\bar{S}))]$. Thus $k(i^{\mathcal{W}}(\bar{T}))$ and $k(i^{\mathcal{W}}(\bar{S}))$ do not have disjoint projections, although it is satisfied by $\mathcal{R}[g]$ that their projections are disjoint. This contradicts the wellfoundedness of $\mathcal{R}[g]$. □

Remark 4.1 (i) Under sufficiently strong hypotheses, one can expect that the iteration strategy Σ of 4.0.1 will be Hom_∞, in which case we get that Hom^*_I is a Wadge initial segment of Hom_∞.

(ii) It is easy to see that the strategy Σ is not itself in Hom^*_I, as otherwise $\mathcal{M}$ would be ordinal definable over $D(\mathcal{M}^I_\infty, \lambda^I_\infty)$, hence in $\mathcal{M}^I_\infty$, hence in $\mathcal{M}$. Given that Σ is Hom_∞, we conjecture that if $\mathcal{M}$ is a *sharp mouse* (see 4.3 below), then Σ is in any scaled pointclass closed under $\exists^{\mathbb{R}}$ and $\neg$ which properly includes Hom^*_I. This amounts to asserting that any new Suslin sets beyond Hom^*_I bring with them new Σ^2_1 facts, and that a local MSC holds just beyond Hom^*_I. One needs to assume that $\mathcal{M}$ is a sharp mouse here; see **??** below.

(iii) Let $\mathcal{T}$ be a non-dropping iteration tree on $\mathcal{M}$ played by Σ, with last model $\mathcal{N}$, and let $\eta < \delta$) be $\mathcal{N}$-cardinals, with $\nu(E_\alpha)^{\mathcal{T}} < \eta$ for all α, so that $\mathcal{N}$ is η-sound. We write $\Sigma|(\mathcal{T}, [\eta, \delta])$ for the restriction of Σ to iteration trees on $\mathcal{N}|\delta$ which use only extenders with critical point $> \eta$, and do not drop anywhere. (Such trees are normal continuations of $\mathcal{T}$.) We call $\Sigma|(\mathcal{T}, [\eta, \delta])$ a *window-based fragment* of Σ. We call the window $[\eta, \delta]$ *short* if there are no Woodin cardinals of $\mathcal{N}$ strictly between η and δ, and in this case call $\Sigma|(\mathcal{T}, [\eta, \delta])$ a *small fragment* of Σ. The proof of 4.0.1 then shows that every set in Hom^*_I is projective in some small fragment of Σ. In some cases, we can show that all the small fragments of Σ are in Hom^*_I, so that they are Wadge cofinal, and use this to compute the cofinality of θ in $D(\mathcal{M}, \lambda)$. See 8.3.

(iv) In general, the *initial segment based* fragments of Σ, i.e., the fragments based on windows of the form (empty tree,$[0, \delta]$), are not all in $D(\mathcal{M}^I_\infty, \lambda^I_\infty)$. For if they were, they would be Wadge cofinal in Hom^*_I, which would imply that the Wadge ordinal of Hom^*_I has V-cofinality ω. In fact, the V-cofinality of this ordinal is often uncountable, as for example in the case of 3.11(c).

Given that for any $\mathbb{R}$-genericity iteration I, Hom_I^* is in V, it is natural to conjecture that in fact Hom_I^* is independent of I. We do not see how to prove that in full generality, but we can get reasonably close.

There is a natural partial order for producing $\mathbb{R}$-genericity iterations of $\mathcal{M}$.

Definition 4.2 *Let $\mathcal{M}$ be a countable mouse, and $\mathcal{M} \models \lambda$ is a limit of Woodin cardinals. Let Σ be an $\omega_1 + 1$ iteration strategy for $\mathcal{M}$. We let $\mathcal{I}(\mathcal{M}, \lambda, \Sigma)$ be the set of all finite sequences*

$$p = \langle \mathcal{T}_0, ..., \mathcal{T}_n \rangle$$

such that

(a) $\mathcal{T}_0$ is an iteration tree on $\mathcal{M}$, and for $0 < k \leq n$, $\mathcal{T}_k$ is an iteration tree on the last model of $\mathcal{T}_{k-1}$, and $\mathcal{T}_n$ has a last model, and

(b) the composition $\oplus_{k \leq n} \mathcal{T}_k$ is a normal tree on $\mathcal{M}$ played according to Σ, with empty drop-set, and

(c) letting $i \colon \mathcal{M} \to \mathcal{N}$ be the canonical embedding of $\mathcal{M}$ into the last model of $\mathcal{T}_n$, and δ the sup of the lengths of the extenders used in $\oplus_{k \leq n} \mathcal{T}_k$, we have $\delta < i(\lambda)$.

We regard $\mathcal{I}$ as a partial order, under reverse inclusion.

Clause (c) of 4.2 guarantees that every $p \in \mathcal{I}$ has a proper extension in $\mathcal{I}$. It is not hard to see that if G is V-generic over $\mathcal{I}(\mathcal{M}, \lambda, \Sigma)$, then $I = \bigcup G$ is an $\mathbb{R}$-genericity iteration of $\mathcal{M}$. We call such an I an $\mathcal{I}(\mathcal{M}, \lambda, \Sigma)$-*generic* ($\mathbb{R}$-genericity) iteration. We are interested in the case $\mathcal{M}$ is ω-sound and projects to ω, so that Σ is determined by $\mathcal{M}$, and we can write $\mathcal{I}(\mathcal{M}, \lambda)$ for $\mathcal{I}(\mathcal{M}, \lambda, \Sigma)$.

If $p = \langle \mathcal{T}_0, ..., \mathcal{T}_n \rangle \in \mathcal{I}(\mathcal{M}, \lambda, \Sigma)$, then we shall write $\mathcal{T}_i(p) = \mathcal{T}_i$, $\mathcal{M}_0(p) = \mathcal{M}$ and $\mathcal{M}_{i+1}(p) =$ the last model of $\mathcal{T}_i$, $\mathcal{M}_p = \mathcal{M}_{n+1}(p)$, $i^p_{k,l}$ for the canonical embedding of $\mathcal{M}_k(p)$ into $\mathcal{M}_l(p)$, λ_p for $i^p_{0,n+1}(\lambda)$, and δ_p for the sup of the lengths of the extenders used in the $\mathcal{T}_k$, for $k \leq n$.

We shall show that for certain natural $(\mathcal{M}, \lambda)$, any two $\mathcal{I}(\mathcal{M}, \lambda)$-generic iterations give rise to the same derived model. The $\mathcal{M}$ we have in mind here are mice like $M_\omega^\sharp$ (the sharp of the minimal model with ω Woodins), $M^\sharp_{\mathrm{adr}}$ (the sharp of the minimal model of the $\mathsf{AD}_\mathbb{R}$ hypothesis), and $M^\sharp_{\mathrm{wdn.lim}}$ (the sharp of the minimal model with a Woodin limit of Woodins). The main feature of these mice which lets our argument go through is that they reconstruct themselves below the relevant λ. There are some other conditions we seem to need as well. We are by no means sure that the definitions we are about to give abstract the most general hypotheses on $\mathcal{M}$ and λ yielding derived-model invariance.

Definition 4.3 *We call a premouse $\mathcal{M}$ a* sharp mouse *iff*

(a) $\mathcal{M}$ *is active, and for some* $\alpha < \mathrm{crit}(\dot{F}^{\mathcal{M}})$, $\dot{E}^{\mathcal{M}}_\eta = \emptyset$ *for all* $\eta \geq \alpha$,

(b) *there is a sentence* φ *such that for* $\kappa = \mathrm{crit}(\dot{F}^{\mathcal{M}})$, $\mathcal{M}|\kappa \models \varphi$, *but whenever* $\mu = \mathrm{crit}(\dot{E}^{\mathcal{M}}_\eta)$ *for some* η, *then* $\mathcal{M}|\mu \not\models \varphi$, *and*

(c) $\mathcal{M}$ *is* $\omega_1 + 1$*-iterable.*

If α is as in part (a), then (a) says that $\mathcal{M}$ is essentially the sharp of $\mathcal{M}|\alpha$. Part (b) is a φ-minimality condition, and it implies that $\mathcal{M}$ projects to ω. If $\mathcal{M}$ is also ω-sound, the iteration strategy asserted to exist in (c) is unique.

The *reduct* of a sharp mouse $\mathcal{M}$ is $\mathcal{M}||o(\mathcal{M})$, that is, $\mathcal{M}$ with its last extender removed.

Definition 4.4 *Let* $\mathcal{M}$ *be an* ω*-sound sharp mouse, and* λ *a limit cardinal in* $\mathcal{M}$. *We say that* $\mathcal{M}$ reconstructs itself below λ *iff whenever* $\mathcal{N}$ *is a countable, non-dropping iterate of* $\mathcal{M}$ *by its unique* $\omega_1 + 1$ *iteration strategy, with* λ^* *the image of* λ, *and* $\kappa < \lambda^*$, *and* $\mathcal{P}$ *is the extender model to which the maximal plus-1 certified ms-array over* κ *of* $\mathcal{N}$ *converges (see [13]), then there is a sharp mouse* $\mathcal{Q}$ *whose reduct is* $\mathcal{P}$, *and a* Σ_1*-elementary* $\pi\colon \mathcal{M} \to \mathcal{Q}$ *such that* $\pi(\lambda) = \lambda^*$.

The maximal plus-1 certified ms-array of $\mathcal{N}$ is the output of a K^c-construction which uses certain reasonably strong extenders from the $\mathcal{N}$-sequence as background extenders; see [13].

Definition 4.5 *A* tractable pair *is a pair* $\langle \mathcal{M}, \lambda \rangle$ *such that*

(a) $\mathcal{M}$ *is an* ω*-sound sharp mouse,*

(b) $\mathcal{M} \models$ *"*λ *is a limit of Woodin cardinals, and* $\mathrm{cof}(\lambda)$ *is not measurable", and*

(c) $\mathcal{M}$ *reconstructs itself below* λ.

Some (probable) examples of tractable pairs are

(1) $M_\omega^\sharp$, with λ the sup of its Woodin cardinals,

(2) $M^\sharp_{\mathrm{adr}}$, with λ the sup of its Woodin cardinals, or with λ the sup of the first ω Woodin cardinals of $M^\sharp_{\mathrm{adr}}$,

(3) $M^\sharp_{\mathrm{dc}}$, the sharp of the minimal model with a cardinal μ which is a limit of Woodins, and such that the set of $\kappa < \mu$ which are $< \mu$-strong has order type μ, with $\lambda = \mu$, or with λ the sup of the first ω Woodins,

(4) $M^\sharp_{\text{in.lim}}$, the sharp of the minimal model with a cardinal which is inaccessible, and a limit of Woodins, and a limit of $<\lambda$-strong cardinals, with λ the sup of all its Woodins, or with λ the sup of its first ω Woodins,

(5) $M^\sharp_{\text{wdn.lim}}$, with λ the Woodin limit, or with λ the sup of the first ω Woodins.

The author believes that tractability can be verified in each case using [13]. In some cases, though, he has not gone through anything like a thorough proof. $M^\sharp_{\text{adr}}$ is the least mouse with a derived model satisfying $\mathsf{AD}_{\mathbb{R}}$, and $M^\sharp_{\text{dc}}$ is the least mouse with a derived model satisfying $\mathsf{AD}_{\mathbb{R}}$ plus DC (or equivalently, $\theta = \theta_{\omega_1}$. The derived model of $M^\sharp_{\text{wdn.lim}}$ at its top Woodin satisfies $\mathsf{AD}_{\mathbb{R}}$ plus "$\theta = \theta_\theta$". (Again, these claims need checking.)

Theorem 4.6 *Let $\langle \mathcal{M}, \lambda\rangle$ be a tractable pair, and let I and J be $\mathcal{I}(\mathcal{M},\lambda)$- generic iterations; then $D(\mathcal{M}^I_\infty, \lambda^I_\infty) = D(\mathcal{M}^J_\infty, \lambda^J_\infty)$.*

Proof. Fix $\mathcal{M}$ and λ, and let Σ be the unique ω_1+1 iteration strategy for $\mathcal{M}$. Let $\mathcal{I} = \mathcal{I}(\mathcal{M},\lambda)$.

Given $A \subset \mathbb{R}$, let us say (p,g) *captures* A iff ($p \in \mathcal{I}$, and g is $\mathcal{M}_p$-generic over $\mathrm{Col}(\omega, \delta_p)$, and there are trees $S, T \in \mathcal{M}_p[g]$ such that

$$\mathcal{M}_p[g] \models S, T \text{ are } < \lambda_p\text{-absolute complements,}$$

and

$$A = \bigcup\{p[i^{\mathcal{U}}(S)] \mid p^\frown\langle \mathcal{U}\rangle \in \mathcal{I}\}.$$

Here $i^{\mathcal{U}}$ is the embedding along the main branch of $\mathcal{U}$. By the proof of 4.0.1, it is enough to prove

Lemma 4.7 *Suppose (p,g) captures A; then there are densely many $r \in \mathcal{I}$ such that for some h, (r,h) captures A.*

Proof. Let (p,g) capture A, and let $q \in \mathcal{I}$. We seek an $r \leq q$ and h such that (r,h) captures A. The idea is just to reconstruct inside $\mathcal{M}_q$, above δ_q, a model $\mathcal{R}$ into which $\mathcal{M}$ embeds. We can then iterate $\mathcal{M}_q$ above δ_q so that we get an $r \leq q$, with $\mathcal{M}_p$ embedding into the image $\mathcal{S}$ of $\mathcal{R}$ under this iteration. Our universally Baire representation of A over $\mathcal{M}_p$ then lifts to $\mathcal{S}$, and then to the background universe $\mathcal{M}_r$ for $\mathcal{S}$, as desired.

In order to lift the universally Baire representations, we must work with their associated extender normal forms. Recall that an extender normal form is a map

$$\mathcal{E} = (s \mapsto \mathcal{E}_s), \text{ for } s \in \omega^{<\omega},$$

where each $\mathcal{E}_s$ is an alternating chain with $2\cdot\mathrm{dom}(s)$ many models, and $\mathcal{E}_s$ extends $\mathcal{E}_t$ whenever s extends t. For $x \in \omega^\omega$, we write $\mathcal{E}^e_x$ and $\mathcal{E}^o_x$ for the direct limits along the even and odd branches of the infinite alternating chain associateed to x. If $\mathcal{E}$ is an extender normal form which uses only extenders with critical point $> \kappa$, and whenever G is V-generic for a poset of size $< \kappa$,

$$V[G] \models \forall x(\text{ exactly one of } \mathcal{E}^e_x \text{ and } \mathcal{E}^o_x \text{ is wellfounded},$$

then we call $\mathcal{E}$ a *κ-ENF*. We say that $\mathcal{E}$ *represents* C iff $C \subseteq \mathbb{R}$, and for all $x \in \mathbb{R}$,

$$x \in C \Leftrightarrow \mathcal{E}^e_x \text{ is wellfounded }.$$

It is worth noting the following absoluteness property of such representations: if S, T are κ-absolute complements, and $\mathcal{E}$ is a κ-ENF which represents $p[S]$ in V, then whenever G is V-generic for a poset of size $\leq \kappa$, we have that $\mathcal{E}$ represents $p[S]$ in $V[G]$. (This uses the uniqueness of wellfounded branches in $V[G]$ which we have built into our definition of κ-ENF.)

We say that $\mathcal{E}$ is an *$\mathcal{N}$-based κ-ENF in $\mathcal{N}[g]$* iff $\mathcal{N}$ is a premouse, and g is $\mathcal{N}$-generic over some poset of size $< \kappa$ in $\mathcal{N}$, and $\mathcal{N}[g] \models \mathcal{E}$ is a κ-ENF, and $\mathcal{E}$ uses only extenders from the sequence of $\mathcal{N}$ and its images whose sup-of-generators is a cardinal in the model in which they appear. The following carries over a basic fact about ENF's to this fine-structural setting.

Sublemma 4.7.1 *[[17],[6]] Let $\mathcal{N}$ be a premouse, and $\mathcal{N} \models \mathsf{ZFC}+$ "λ is a limit of cardinals which are Woodin via extenders on my sequence". Let g be $\mathcal{N}$-generic over some poset of size $< \lambda$; then working inside $\mathcal{N}[g]$, the following are equivalent, for any set C of reals:*

(a) C is $< \lambda$-universally Baire,

(b) for any $\kappa < \lambda$ there is an $\mathcal{N}$-based κ-ENF $\mathcal{E}$ which represents C.

We continue with the proof of 4.7. By the sublemma, $\mathcal{M}_p[g]$ satisfies the statement that there is a sequence $\langle \mathcal{E}(\kappa) \mid \kappa < \lambda_p \rangle$ such that for each κ, $\mathcal{E}(\kappa)$ is an $\mathcal{M}_p$-based κ-ENF representing $p[S]$. Fix such an $\langle \mathcal{E}(\kappa) \mid \kappa < \lambda_p \rangle$. Using a g-name for this sequence, we can fix a sequence $\langle X_\kappa \mid \kappa < \lambda_p \rangle \in \mathcal{M}_p$ such that for all $\kappa < \lambda_p$,

(a) $\mathcal{M}_p \models |X_\kappa| \leq \delta_p$, and

(b) for all $s \in \omega^{<\omega}$, $\mathcal{E}_s(\kappa) \in X_\kappa$.

Now let $\mathcal{N}_q$ be the version of $\mathcal{M}$ reconstructed inside $\mathcal{M}_q$, with all critical points above δ_q. Thus we have a Σ_1 elementary

$$\pi\colon \mathcal{M} \to \mathcal{N}_q \subseteq \mathcal{M}_q$$

such that $\pi(\lambda_p) = \lambda_q$. Using the fact that λ does not have measurable cofinality, and a simple comparison argument, one can show

$$\text{ran}(\pi) \text{ is cofinal in } \lambda_q.$$

This was the reason for our restriction on the cofinality of λ. We leave the details to the reader.

We can use π to lift the iteration from $\mathcal{M}$ to $\mathcal{M}_p$ to an iteration tree on $\mathcal{N}_q$, and thence via the background-extender iterability proof to an iteration tree on $\mathcal{M}_q$ which is above δ_q. This gives us a condition $t \leq q$ in $\mathcal{I}$, and a Σ_1 elementary

$$\sigma \colon \mathcal{M}_p \to \mathcal{N}_t \subseteq \mathcal{M}_t,$$

where $\mathcal{N}_t$ is a sharp mouse whose reduct is the limit model of a plus-1 certified ms-array in the sense of $\mathcal{M}_t$.

Let h_0 be $\mathcal{M}_t$-generic over $\text{Col}(\omega, \sigma(\delta_p))$. Letting

$$Y = \sigma(\langle X_\kappa \mid \kappa < \lambda_p \rangle),$$

we have that Y_μ is countable in $\mathcal{M}_t[h_0]$ for all $\mu < \lambda_t$, and $Y_{\sigma(\mu)}$ covers $\sigma``X_\mu$, for all $\mu < \lambda_p$. Fix a sequence $\langle f_\mu \mid \mu < \lambda_t \rangle$ n $\mathcal{M}_t[h_0]$ such that f_μ maps ω onto Y_μ, for all μ. Finally, fix a real z such that for all $\mu < \lambda_p$, the map

$$s \mapsto f^{-1}_{\sigma(\mu)}(\sigma(\mathcal{E}_s(\mu)),$$

which is essentially a real, is recursive in z.

Now let δ be a Woodin cardinal of $\mathcal{M}_t[h_0]$ such that $\sup(\sigma(\delta_p), \delta_t) < \delta < \lambda_t$. We do a genericity iteration of $\mathcal{M}_t[h_0]$ via an $\mathcal{M}_t$-based tree which is above $\sup(\sigma(\delta_p), \delta_t)$, and below δ, and obtain thereby a condition $r \leq t$ in $\mathcal{I}$, and a generic h on $\text{Col}(\omega, \delta_r)$, so that $h_0, z \in \mathcal{M}_r[h]$. Let

$$\tau \colon \mathcal{M}_t \to \mathcal{M}_r$$

be the natural map, and let

$$\psi = \tau \circ \sigma \colon \mathcal{M}_p \to \mathcal{N}_r,$$

where $\mathcal{N}_r = \tau(\mathcal{N}_t)$.

Claim. (r, h) captures A.

Proof. Recall that there is a $\nu < \lambda_r$ such that in $\mathcal{M}_r[h_0]$, any ν-homogeneous set of reals is $< \lambda_r$-universally Baire. Fix such a $\nu = \nu_0$. Now pick $\mu < \lambda_p$ such that

(a) $\nu_0 < \psi(\mu)$,

(b) $\gamma_0 < \mu$, where γ_0 is the least Woodin cardinal γ of $\mathcal{M}_p$ such that $\delta_r < \psi(\gamma)$, and

(c) there is a Woodin cardinal κ_0 of $\mathcal{M}_r$ such that $\psi(\gamma_0) < \kappa_0 < \psi(\mu)$.

Since λ_r is a limit of Woodin cardinals in $\mathcal{M}_r$, and $\mathrm{ran}(\psi)$ is cofinal in λ_r, we can easily find such a μ, γ_0, and κ_0.

Let

$$\mathcal{E}_s = \mathcal{E}_s(\mu),$$

and

$$\mathcal{F}_s = \psi(\mathcal{E}_s),$$

and

$$\mathcal{F}^*_s = \text{ alternating chain on } \mathcal{M}_r[h] \text{ induced by } \mathcal{F}_s,$$

for all $s \in \omega^{<\omega}$. Here we mean that $\mathcal{F}^*_s$ arises from $\mathcal{F}_s$ as in the iterability proof, using the background extenders provided by the K^c-construction in $\mathcal{M}_r$ whose output is $\mathcal{N}_r$.

We need the following small extension of the theorem 3.3 of [13] on UBH in extender models. We omit the proof.

Lemma 4.8 *Let $\mathcal{P} \models \mathsf{ZFC}$ be fully iterable mouse, and let G be generic over $\mathcal{P}$ for a poset of size $< \kappa$ in $\mathcal{P}$; then there are no $\mathcal{T}, b, c \in \mathcal{P}[G]$ such that*

$$\mathcal{P}[G] \models \mathcal{T} \textit{ is a } \mathcal{P}\textit{-based plus-2 iteration tree on } \mathcal{P}[G] \textit{ of limit length,}$$

with all critical points of extenders in $\mathcal{T}$ above κ, and $D^{\mathcal{T}} = \emptyset$, and

$$\mathcal{P}[G] \models \textit{ b and c are distinct, cofinal, wellfounded branches of } \mathcal{T}.$$

It follows immediately from lemma 4.8 that $\mathcal{M}_r[h] \models \mathcal{F}$ is a $\psi(\mu)$-ENF. We therefore have trees $U, W \in \mathcal{M}_r[h]$ such that

$$\mathcal{M}_r[h] \models U, W \text{ are } < \lambda_r\text{-absolute complements},$$

and

$$\mathcal{M}_r[h] \models p[U] = \{x \mid (\mathcal{F}^*)^e_x \text{ is wellfounded}\}.$$

We claim that U, W witness that (r, h) captures A. For that, fix $x \in \mathbb{R}$. It is enough to show that if $x \in A$, then there is a condition $u \le r$ such that $x \in p[i_{r,u}(U)]$, and if $x \notin A$, then there is a condition $u \le r$ such that $x \in p[i_{r,u}(W)]$. So assume $x \in A$; the proof when $x \notin A$ is completely parallel.

Let $a \le p$ come from iterating $\mathcal{M}_p$ above δ_p, and below γ_0, so as to make x generic over $\mathcal{M}_a[g]$ for the image of the extender algebra at γ_0. We have then that $x \in p[i_{p,a}(S)]$, and thus working in $\mathcal{M}_a[g][x]$, we can conclude that $i_{p,a}(\mathcal{E})^o_x$ is illfounded. Let $b \le r$ come from lifting the genericity iteration from $\mathcal{M}_p$ to $\mathcal{N}_r$ using ψ, then further lifting it to $\mathcal{M}_r$ using the iterability proof for background-certified models. It is important here to note that the iteration strategy Σ on normal extensions of $\mathcal{T}_p$ is unique, so that it agrees with the strategy of lifting to $\mathcal{M}_r$ as above, and using Σ to pick branches of the evolving tree on $\mathcal{M}_r$. This shows that the lifting process succeeds, and does give us a condition in $b \in \mathcal{I}$.

Let $\phi\colon \mathcal{M}_a \to \mathcal{N}_b$ be the natural lifting map, where $\mathcal{N}_b = i_{r,b}(\mathcal{N}_r)$. We have from the commutativity of the copy maps that

$$\phi \circ i_{p,a} = i_{r,b} \circ \psi,$$

and from this we get that

$$\phi(i_{p,a}(\mathcal{E})) = i_{r,b}(\mathcal{F}).$$

Here we understand the identity just displayed by letting the embeddings act on $\mathcal{E}$ "pointwise", for ϕ is not actually defined on $\mathcal{M}_a[g]$. (This was the reason we moved to extender normal forms in the first place.) We have then that

$$(i_{r,b}(\mathcal{F}^*))^o_x \text{ is illfounded.}$$

Finally, let $u \leq b$ come from a genericity iteration of $\mathcal{M}_b[h]$ above δ_b and below $i_{r,b}(\kappa_0)$ which makes x generic over $\mathcal{M}_u[h]$ for the extender algebra at $i_{r,u}(\kappa_0)$. Clearly, we still have

$$(i_{r,u}(\mathcal{F}^*))^0_x \text{ is illfounded.}$$

But note that x is generic over $\mathcal{M}_u[h]$ for a poset of size $< i_{r,u}(\psi(\mu))$, and that the universally Baire representation $i_{r,u}(U)$ is satisfied to be equivalent to $i_{r,u}(\mathcal{F})$ in $\mathcal{M}_u[h]$, and that this equivalence is absolute for forcing of size $< i_{r,u}(\psi(\mu))$ over $\mathcal{M}_u[h]$. It follows that $x \in p[i_{r,u}(U)]$, as desired.

This completes the proof of the claim, and hence the proof of lemma 4.7, and hence the proof of theorem 4.6. $\square$

5 Mouse operators

The sharp mice we introduced earlier all relativise: e.g., we can form $M^\sharp_{\mathrm{adr}}(y)$ for any transitive set y. Similarly, the λ's which were part of our tractable pairs relativise. Here is an abstraction of some properties of the resulting operators.

Definition 5.1 *A* -mouse operator *is a function $\mathcal{M} = y \mapsto \mathcal{M}(y)$ defined on all $y \in HC$, and such that there is a sentence φ in the language of relativised premice such that for all y*

(a) $\mathcal{M}(y) \models \varphi$,

(b) for all $\xi < o(\mathcal{M}(y))$, $\mathcal{M}(y)|\xi \not\models \varphi$, and

(c) $\mathcal{M}(y)$ is y-sound and ω_1+1-iterable.

Definition 5.2 *A* sharp-mouse operator *is a function $\mathcal{M} = y \mapsto \mathcal{M}(y)$ defined on all $y \in HC$, and such that there is a sentence φ in the language of relativised premice such that for all y*

(a) $\mathcal{M}(y)$ is active, and for some $\alpha < \mathrm{crit}(\dot{F}^{\mathcal{M}(y)})$, $\dot{E}_\eta^{\mathcal{M}(y)} = \emptyset$ for all $\eta \geq \alpha$,

(b) for $\kappa = \mathrm{crit}(\dot{F}^{\mathcal{M}(y)})$, $\mathcal{M}(y)|\kappa \models \varphi$, but whenever $\mu = \mathrm{crit}(\dot{E}_\eta^{\mathcal{M}(y)})$ for some η, then $\mathcal{M}(y)|\mu \not\models \varphi$, and

(c) $\mathcal{M}(y)$ is y-sound and ω_1+1-iterable.

Definition 5.3 *Let $\mathcal{M}$ be an sharp mouse operator, and $\lambda = (y \mapsto \lambda(y))$ be such that for all $y \in HC$, $\lambda(y)$ is a limit cardinal in $\mathcal{M}(y)$. We say that $\mathcal{M}$* reconstructs itself below λ *iff whenever $y \in HC$ and $\mathcal{N}(y)$ is a countable, non-dropping iterate of $\mathcal{M}(y)$ by its unique ω_1+1 iteration strategy, with λ^* the image of $\lambda(y)$, and $\kappa < \lambda^*$, and z is $\mathcal{N}(y)$-generic over a poset of size $< \kappa$ in $\mathcal{N}(y)$ and $\mathcal{P}(z)$ is the extender model to which the maximal plus-1 certified ms-array over κ of $\mathcal{N}(y)[z]$ converges (see [13]), then there is a sharp mouse $\mathcal{Q}(z)$ whose reduct is $\mathcal{P}(z)$, and a Σ_1-elementary $\pi\colon \mathcal{M}(z) \to \mathcal{Q}(z)$ such that $\pi(\lambda(z)) \leq \lambda^*$.*

Definition 5.4 *A* tractable operator *is a pair $(\mathcal{M}, \lambda)$ such that*

(a) $\mathcal{M}$ is a sharp-mouse operator,

(b) $\lambda = (y \mapsto \lambda(y))$ is such that $\lambda(y)$ is definable from y over $\mathcal{M}(y)$, uniformly in y,

(c) for all y, $\mathcal{M}(y) \models$ "$\lambda(y)$ is a limit of Woodin cardinals, and $\mathrm{cof}(\lambda(y))$ is not measurable",

(d) $\mathcal{M}$ reconstructs itself below λ.

With these definitions, we can extend 4.6 to:

Theorem 5.5 *Let $\langle \mathcal{M}, \lambda \rangle$ be a tractable operator, let $x, y \in HC$, and let I and J be $\mathcal{I}(\mathcal{M}(x), \lambda(x))$ and $\mathcal{I}(\mathcal{M}(y), \lambda(y))$ generic iterations; then $D(\mathcal{M}(x)^I_\infty, \lambda^I_\infty) = D(\mathcal{M}(y)^J_\infty, \lambda^J_\infty)$.*

Later we shall need a further strengthening of 4.6, in which the relativised mice are *hybrid mice* of the form $\mathcal{M}^\Sigma(y)$, where Σ is an appropriate set of reals in the derived model of $\mathcal{M}(y)$.

6 The mouse set conjecture in $D(\mathcal{M}, \lambda)$

If $\mathcal{M}$ is a mouse, then $\mathcal{M}$ itself "captures" all reals which are OD in its derived model. So the following result is not surprising:

Theorem 6.1 *Let $(\mathcal{M}, \lambda)$ be a tractable operator, and suppose D is a realization of $D(\mathcal{M}(z), \lambda)$, and $x, y \in HC^D$, with $x \subseteq y$. Then the following are equivalent:*

(a) $D \models x \in OD(y \cup \{y\})$,

(b) $x \in \mathcal{M}(y)$,

(c) $D \models x$ is in an ω_1-iterable mouse over y.

Proof. $(a) \Rightarrow (b)$: We easily get that $D^* \models x \in \mathrm{OD}(y \cup \{y\})$, where $D^* = D(\mathcal{M}(z)^I_\infty)$ for an $\mathcal{I}(\mathcal{M}(z), \lambda(z))$-generic I. But then $D^* = D(\mathcal{M}(y)^J_\infty)$ for an $\mathcal{I}(\mathcal{M}(y), \lambda(y))$-generic J. Since the forcing is homogeneous, $x \in \mathcal{M}(y)^J_\infty$. Therefore $x \in \mathcal{M}(y)$.

$(b) \Rightarrow (c)$ (*Sketch*): It is enough to show that if $\mathcal{P}$ is a proper initial segment of $\mathcal{M}(y)$, and $\mathcal{P}$ projects to y, then $\mathcal{P}$ has an ω_1-iteration strategy in $D(\mathcal{M}(y), \lambda)$. (One can then pass to $D = D(\mathcal{M}(z), \lambda(z))$ by 5.5.) For that, we show that the unique strategy Σ for $\mathcal{P}$ has a μ-UB code in $\mathcal{M}(y)$, for any $\mu < \lambda(y)$. Fix $\mu < \lambda(y)$. Because $\mathcal{M}(y)$ reconstructs itself below $\lambda(y)$, the full background extender construction $\langle \mathcal{C}_k(\mathcal{N}_\eta) \rangle$ of $\mathcal{M}(y)$, done with all critical points $> \mu$ (and using μ-closed extenders from the sequence of $\mathcal{M}(y)$ as backgrounds), reaches $\mathcal{P} = \mathcal{C}_k(\mathcal{N}_\eta)$. Let $\mathcal{M}(y)|\xi \models \mathcal{P} = \mathcal{C}_k(\mathcal{N}_\eta) \wedge \mathsf{ZFC}^-$. By the results of section 12 of [8], Σ is induced by lifting trees on $\mathcal{P}$ to trees on $\mathcal{M}(y)$, then following the unique strategy for $\mathcal{M}(y)$. The lifting process is uniformly definable over $\mathcal{M}(y)[g]$, for any size $< \mu$ generic g over $\mathcal{M}(y)$, and this definition has the generic absoluteness required to obtain a μ-UB code. The key to the generic absoluteness is the fact that UBH holds in ms-mice, or more precisely, the sharper version of this fact stated as Theorem 3.4 in [13]. The iteration trees on $\mathcal{P}$ being lifted have size less than the closure of the background extenders, so by UBH they are continuously illfounded off the branches they choose, which guarantees both the existence-in-$\mathcal{M}(y)[g]$ and the absolute definabilty of the branches chosen by Σ.

[Here is a bit more detail. Let φ be the formula which defines Σ over $< \mu$-generic extensions of $\mathcal{M}(y)|\xi$ via this lifting process. It is enough to show that club many hulls of $\mathcal{M}(y)|\xi$ are generically correct. So let $\pi N \to \mathcal{M}(y)$ be elementary, where N is transitive, with $\pi \restriction (\mathcal{P} \cup \{\mathcal{P}\}) =$ identity. Let g be $< \pi^{-1}(\mu)$-generic over N. Let $\mathcal{T}$ be an iteration tree satisfying φ over $N[g]$. We need to see $\mathcal{T}$ is by Σ. Since Σ is the unique strategy for $\mathcal{P}$, it suffices to see that the phalanx $\Phi(\mathcal{T})$ is iterable. Let $\mathcal{U}$ be the lift of $\mathcal{T}$ to N, as in [8]. It suffices to see $\Phi(\mathcal{U})$ is iterable. But $\Phi(\mathcal{U})$ is continuously illfounded off the branches it chooses, and therefore must be according to any iteration strategy for N.]

$(c) \Rightarrow (b)$: This is one of the elementary corollaries of the comparison lemma. (Note ω_1-iterability implies $(\omega_1 + 1)$-iterability, granted AD.) $\square$

The independence of the derived model $D(\mathcal{M}^I_\infty)$ for *arbitrary* $\mathbb{R}$-genericity iterations (as opposed to $\mathcal{I}$-generic ones) is perhaps not so important. However, 6.1 does give us something there.

Corollary 6.2 *Let $(\mathcal{M}, \lambda)$ be a tractable operator, and let D_0 and D_1 be derived models associated to $\mathbb{R}$-genericity iterations of $\mathcal{M}(x)$ and $\mathcal{M}(y)$. Then $(\Sigma^2_1)^{D_0} = (\Sigma^2_1)^{D_1}$. Thus if $D_0 \models \theta = \theta_0$, then $D_0 = D_1$.*

The proof is an easy consequence of the fact that, assuming AD^+, $x \in \mathrm{OD}(y)$ is a Σ^2_1-complete relation on reals. (Note also $D_0 \equiv D_1$ by 4.6, and since we are talking about "old" derived models, if $\theta_0 = \theta$ in such a model, then it is of the form $L(\mathbb{R} \cup \{U\})$, for U and Σ^2_1-complete set.)

We also get a sort of converse to 3.11(a), in the case the ground model is a mouse.

Corollary 6.3 *Let $(\mathcal{M}, \lambda)$ be a tractable operator, and suppose that for some x, $\mathcal{M}(x) \models \lambda(x)$ is a limit of cutpoints. Then for all x, $\mathcal{M}(x) \models \lambda(x)$ is a limit of cutpoints, moreover and derived model $D(\mathcal{M}(x), \lambda(x))$ satisfies $\theta_0 = \theta$.*

Proof. Let D be a realization of $D(\mathcal{M}(x), \lambda(x))$, ans suppose toward contradiction that $D \models \theta_0 < \theta$. AD^+ and $\theta_0 < \theta$ together imply that the relation on reals $z \notin \mathrm{OD}(y)$ can be uniformized by a Suslin-co-Suslin function. Let $f(y) = z$ be such a function in the sense of D. Because the cutpoints of $\mathcal{M}(x)$ are cofinal below $\lambda(x)$, we can find such a cutpoint η and a g which is $\mathcal{M}(x)$-generic for $\mathrm{Col}(\omega, \eta)$ such that $f = p[T]$, for some tree T in $\mathcal{M}(x)[g]$. Let y be a real in $\mathcal{M}(x)[g]$ which codes $\langle \mathcal{M}(x)|\eta, g\rangle$ is some simple way, and let $z = f(y)$. Because $T \in \mathcal{M}(x)[g]$, we get $z \in \mathcal{M}(x)[g]$. But η is a cutpoint, so $\mathcal{M}[x][g]$ can be re-arranged as a mouse over y. It is not hard to see that this mouse must be $\trianglelefteq \mathcal{M}(y)$, so $z \in \mathcal{M}(y)$, so $z \in \mathrm{OD}(y)$ in D, a contradiction. □

7 The Solovay sequence in $D(\mathcal{M}, \lambda)$

We are not sure how to properly state or prove the results of this section in the case of arbitrary tractable operators. We can handle the specific operators introduced above: $M^\sharp_{\mathrm{adr}}, M^\sharp_{\mathrm{dc}}, M^\sharp_{\mathrm{in.lim}}, M^\sharp_{\mathrm{wdn.lim}}$, with λ the sup of the Woodin cardinals of the mouse in each case. We can handle other tractable operators like these, but have not abstracted a good definition. So for now, let us call those 4 operators *paradigmatic*. Woodin's 3.11 implies that for our paradigmatic $(\mathcal{M}, \lambda)$, the derived model $D(\mathcal{M}, \lambda)$ satisfies $\mathsf{AD}_\mathbb{R}$, or equivalently, $\theta = \theta_\xi$ for some limit ordinal ξ. In fact, one can give a somewhat simpler proof of 3.11 in this case, one which gives more information as to what sets of reals sit at the θ_α's in the Wadge hierarchy of the derived model. To begin with

Theorem 7.1 *Let* $(\mathcal{M}, \lambda)$ *be paradigmatic, and let* I *be* $\mathcal{I}(\mathcal{M}(x), \lambda(x))$*-generic, and* $D = D(\mathcal{M}^I_\infty, \lambda^I_\infty)$. *Then the function*

$$z \mapsto \mathcal{M}(z), \ \textit{for } z \in \mathbb{R}$$

is in D*, and has Wadge rank* θ_0 *in* D.

Proof. We may as well take $x = \emptyset$. Let us write $\mathcal{M} = \mathcal{M}(\emptyset), \lambda = \lambda(\emptyset)$. Let κ be the least $< \lambda$-strong cardinal, and δ the least Woodin cardinal $> \kappa$, in $\mathcal{M}$. By 4.6, we may assume that the first normal tree in I is a genericity iteration below δ, with all critical points above κ, giving rise to

$$i_{0,1} \colon M^I_0 \to M^I_1,$$

in such a way that

$$\mathcal{M} = M^I_0 \in M^I_1[g],$$

with g being $\mathrm{Col}(\omega, i_{0,1}(\delta))$-generic over M^I_1. We can also assume that $\mathrm{crit}(i_{1,\infty}) > i_{0,1}(\delta)$.

Claim 1. The function $z \mapsto \mathcal{M}(z)$, for $z \in \mathrm{HC}^V$, is in the symmetric model $M^I_\infty(\mathbb{R}^V)$.

Proof.

Note that we can extend $i_{1,\infty}$ to act on $M_1[g]$. Fix $z \in \mathrm{HC}^V$; we show informally how to compute $\mathcal{M}(z)$ inside $M_\infty(\mathbb{R}^V)$. First, let $\alpha < \lambda_\infty$ be an inaccessible cardinal of M_∞ large enough that z is generic over $M_\infty[g]|\alpha$, and $i_{0,1}(\delta) < \alpha$. Let E be an extender on the M_∞ sequence such that $\kappa = \mathrm{crit}(E)$ and $\alpha \leq \mathrm{lh}(E)$. In $M_\infty[g]$ we have $\mathcal{M}$ *as a set*, and hence we can form $\mathrm{Ult}(\mathcal{M}, E)$ *as a set.* Since $\mathcal{M}|\kappa^{+\mathcal{M}} = M_\infty|\kappa^{+M_\infty}$, we have that

$$\mathrm{Ult}(\mathcal{M}, E)|i^{\mathcal{M}}_E(\kappa) = \mathrm{Ult}(M_\infty, E)|i^{M_\infty}_E(\kappa),$$

and thus

$$\mathrm{Ult}(\mathcal{M}, E)|\alpha = M_\infty|\alpha$$

by coherence. Thus z is generic over $\mathrm{Ult}(\mathcal{M}, E)$ for a partial order of size $< i_E(\lambda)$. Since the $(\mathcal{M}, \lambda)$ operator is tractable, we can now use the modified background-extender construction of [13] to re-build $\mathcal{M}(z)$ from $\mathrm{Ult}(\mathcal{M}, E)[z]$. Once again, the key is that we get $\mathcal{M}(z)$ *as a set* in $M_\infty[g][z]$. □

Claim 2. The function $F = z \mapsto \mathcal{M}(z)$, for $z \in \mathbb{R}^V$, is in $D(M_\infty, \lambda_\infty)$; in fact, it has a universally Baire code in $M_\infty[g]$.

Proof. Working in $M_\infty[g]$, where we have $\mathcal{M}$, let α be any inaccessible cardinal as in claim 1. Let E be an extender on the M_∞-sequence such that $\kappa = \mathrm{crit}(E)$ and $\alpha < \mathrm{lh}(E)$. Let φ be the formula defining F on size $< \alpha$ generic extensions of $M_\infty[g]$ from $\mathcal{M}$ and E which is implicit in the proof of claim 1.

Let ξ be sufficiently large. Since any tractable operator condenses to itself, we get that in $\mathcal{M}_\infty[g]$, there are club many countable $X \prec M_\infty|\xi$ such that if N is the transitive collapse of X, and H is N-generic for a poset of size $<$ the collapse of α, then φ defines $F \restriction N[H]$ over $N[H]$ from $\mathcal{M}$ and the collapse of E. (I.e., there are club many generically correct X.) As is well known, this gives us an α-universally Baire code for $F \restriction M_\infty[g]$ in $M_\infty[g]$. It is easy to see that the code (T, T^*) has the property that $p[T] \cap M_\infty[g][H] = F \cap M_\infty[g][H]$ for all size $< \alpha$ generic H. □

*Claim 3.*The function $F = z \mapsto \mathcal{M}(z)$, for $z \in \mathbb{R}^V$, has Wadge rank θ_0 in $D(M_\infty, \lambda_\infty)$.

Proof. Suppose F were OD(z) in $D(M_\infty, \lambda_\infty)$, where $z \in \mathbb{R}$. By 6.1, we then get $F(z) \in \mathcal{M}(z)$, a contradiction. Thus F has Wadge rank at least θ_0.

Let $\mathcal{M}(z)^-$ be the proper class model obtained by iterating away the last extender of $\mathcal{M}(z)$, and put

$$F_n(z) = \text{ type of 1st } n \text{ indiscernibles over } \mathcal{M}(z)^-,$$

for $n < \omega$ and $z \in \mathbb{R}$. It is clear that F is Wadge equivalent to the join of the F_n's. (This is actually just a matter of definition; strictly speaking, $F(z)$ isn't even a real. It is coded by its 1st order theory, however, and this theory is easily intercomputable with the join of the $F_n(z)$.) It is enough then to show that each F_n is OD($\mathbb{R}$) in $D(M_\infty, \lambda_\infty)$.

For this, note first that F_n is uniformly Turing invariant, that is, it is a jump operator. Also, $F_n(z) \in \mathcal{M}(z)$ for all z, so we can set

$$g(z) = \text{ least } \alpha \text{ such that } F_n(z) \in \mathcal{M}(z)|\alpha,$$

and

$$G(z) = \Sigma_1 \text{ theory of } p \text{ in } \mathcal{M}(z)|g(z),$$

where p is the first standard parameter of $\mathcal{M}(z)|g(z)$. Clearly $z \equiv_T y \Rightarrow g(z) = g(y)$, so g induces a function from the Turing degrees $\mathcal{D}$ to OR, which has an ordinal rank γ in $\mathrm{OR}^{\mathcal{D}}$ mod the Martin measure. Also, G is a jump operator, so by the prewellordering of jump operators (see [19]) we have a recursively pointed perfect set P and an e such that $\forall x \in P$, $F_n(x) = \{e\}^{G(x)}$. Clearly γ and P determine the value of $F_n(x)$ on a pointed perfect set's worth of x. However, it is then easy to see that this determines the value of $F_n(x)$ at all x. Thus F_n is definable over $D(M_\infty, \lambda_\infty)$ from γ and P. □

The claims complete our proof. □

Remark 7.2 Something close to 7.1 was first proved by Woodin, by another method.

We can produce mouse operators sitting at the higher θ_α in $D(M^I_\infty, \lambda^I_\infty)$ by *nesting* the $\mathcal{M}$-operator at various depths. For the mouse operators $\mathcal{M}^\alpha$ we define this way, $\mathcal{M}^\alpha(x)$ will only be defined for a "cone" of x, namely, those α which are countable in the first admissible set over x. This is because we want $\mathcal{M}^\alpha(x)$ to project to x. So we need to extend definition 5.1 in order to call them mouse operators.

Definition 7.3 *Let $\mathcal{M}$ be a mouse operator, and let $\mathcal{P}$ be a premouse, with $\lambda \leq o(\mathcal{P})$. We say that $\mathcal{P}$ is* $\mathcal{M}$-closed below λ *iff whenever $\xi < \lambda$ and ξ is a cutpoint of $\mathcal{P}$, then $\mathcal{M}(\mathcal{P}|\xi) \trianglelefteq \mathcal{P}$.*

Notice that for each of our paradigmatic operators $(\mathcal{M}, \lambda)$ we have a large cardinal hypothesis φ such that λ is the first η such that the truth of φ in $\mathcal{M}$ is witnessed by the *total* extenders from the $\mathcal{M}|\eta$-sequence. Let us call φ the *hypothesis associated to* $(\mathcal{M}, \lambda)$.

Definition 7.4 *Let $(\mathcal{M}, \lambda)$ be paradigmatic, with associated hypothesis φ. For $\alpha < \omega_1$, we define the mouse operator $\mathcal{M}^\alpha$ by setting $\mathcal{M}^0 = \mathcal{M}$, and for x such that α is countable in the least admissible set over x:*

$$\begin{aligned} \mathcal{M}^{\alpha+1}(x) \quad = \quad & \textit{minimal active } x\textit{-mouse } \mathcal{P} \textit{ such that} \\ & \exists \gamma < \mathrm{crit}(\dot{F}^{\mathcal{P}}) (\ \mathcal{P} \textit{ is } \mathcal{M}^\alpha\textit{- closed below } \gamma \\ & \ \textit{and } \mathcal{P} \models \varphi, \textit{ as witnessed by total extenders of } \mathcal{P}|\gamma). \end{aligned}$$

Fot η a limit ordinal which is countable in the least admissible set over x:

$$\mathcal{M}^\eta(x) = \bigcup_{\alpha<\eta} \mathcal{M}^\alpha(x).$$

Theorem 7.5 *Let $(\mathcal{M}, \lambda)$ be paradigmatic, and $D = D(\mathcal{M}^I_\infty, \lambda^I_\infty)$ where I is $\mathcal{I}(\mathcal{M}, \lambda)$-generic. Then*

(a) if $\mathcal{M} = M^\sharp_{adr}$, then for all $n < \omega$, the $\mathcal{M}^n$-operator is in D and has Wadge rank θ_n in D;

(b) if $\mathcal{M} = M^\sharp_{dc}, M^\sharp_{in.lim}$, or $M^\sharp_{wdn.lim}$, then for all $\alpha < \omega_1$, the $\mathcal{M}^\alpha$ operator is in D, and has Wadge rank θ_α in D.

Proof sketch. We show that (a) and (b) hold when $\alpha = n = 1$, and leave the rest to the reader.

First, 4.6 generalizes in the following way.

Lemma 7.6 *Let I and J be generic $\mathbb{R}$-genericity iterations of $\mathcal{M}(x)$ below $\lambda(x)$ and $\mathcal{M}^1(y)$ below $\lambda^1(y)$, respectively. Then $D(\mathcal{M}(x)^I_\infty, \lambda(x)^I_\infty) = D(\mathcal{M}^1(y)^J_\infty, \lambda^1(y)^J_\infty)$.*

The proof is like that of 4.6. The main additional point is that an appropriate generic extenision of $\mathcal{M}(x)$ can reconstruct $\mathcal{M}^1(y)$ arbitrarily high below $\lambda(x)$. This follows from 7.1.

Next, 6.1 generalizes as follows:

Lemma 7.7 *Under the hypotheses of theorem 7.5, we have that for x, y countable transitive with $x \subseteq y$,*

$$x \in OD(z \mapsto \mathcal{M}(z), y)^D \Leftrightarrow x \in \mathcal{M}^1(y).$$

The proof is just like that of 6.1, using 7.6 in place of 4.6. These two lemmas imply that if the $\mathcal{M}^1$ operator is in D, then it has Wadge rank θ_1 there. The proof is the same as that which shows the $\mathcal{M}^0$ operator sits at θ_0.

To see that the $\mathcal{M}^1$ operator is in D, we relativise the argument of 7.1. Let δ be the least Woodin cardinal above the least $< \lambda$-strong cardinal of $\mathcal{M}$, and let

$$i \colon \mathcal{M} \to \mathcal{N}$$

come from a genericity iteration below δ and above that strong cardinal, with $\mathcal{M} \in \mathcal{N}[g]$, where g is $\mathrm{Col}(\omega, i(\delta))$-generic. We will show in the next section that

$$\mathcal{N}[g] = \mathcal{M}^1[\langle \mathcal{N}|i(\delta), g\rangle].$$

We can therefore apply the argument of 7.1, doing a further genericity iteration $j \colon \mathcal{N} \to \mathcal{P}$ in the window between the first $< i(\lambda)$-strong cardinal of $\mathcal{N}[g]$ and its next Woodin, and add a universally Baire code of the $\mathcal{M}^1$ operator to some $\mathcal{P}[g][h]$. By derived model invariance, we may assume the I in our theorem started out going to $\mathcal{N}$ and then $\mathcal{P}$, so we are done. $\square$

One can avoid using the identity $\mathcal{N}[g] = \mathcal{M}^1[\langle \mathcal{N}|i(\delta), g\rangle]$, borrowed from the next section, in the proof above. It is enough that $\mathcal{N}[g]$ can reconstructi an iterate $\mathcal{P}$ of $\mathcal{M}^1[\langle \mathcal{N}|i(\delta), g\rangle]$ via a construction which provides full background extenders for the total extenders on the $\mathcal{P}$-sequence. Note here that because of the background condition, every set in $V_{i(\lambda)} \cap \mathcal{N}[g]$ is generic over $\mathcal{P}$ by a forcing of size $< i(\lambda)$. This means that $\mathcal{P}$ is close enough to $\mathcal{N}[g]$ that the arument of 7.1 still works.

8 The $*$-transform

In this section, we describe a method for translating mice with extenders overlapping some δ into relativised mice having δ as a cutpoint.

Suppose that $\mathcal{M}$ is an $\omega_1 + 1$-iterable, sound x-mouse, and $\mathcal{M}$ projects to x. Let $\mathcal{T}$ be a normal iteration tree on $\mathcal{M}$ played according to its unique iteration strategy, and let $\mathcal{N}$ be the last model of $\mathcal{T}$. Suppose δ is a cardinal of $\mathcal{N}$, and not measurable in $\mathcal{N}$ or any $\mathrm{Ult}(\mathcal{N}, E)$ with $\mathrm{lh}(E) \geq \delta$. Let τ be the order type of

$$\begin{aligned} S_\delta^{\mathcal{N}} \;=\; & \{\kappa < \delta \mid \exists E (E \text{ is on the } \mathcal{N}\text{-sequence} \\ & \text{and } \mathrm{crit}(E) = \kappa \text{ and } \mathrm{lh}(E) \geq \delta\}. \end{aligned}$$

For $\alpha < \tau$, let
$$\kappa_\alpha = \ \alpha^{\text{th}} \text{ member of } S_\delta^{\mathcal{N}},$$
and
$$\gamma_\alpha = \ \text{least } \gamma \text{ such that } \kappa_\alpha < \nu(E_\gamma^{\mathcal{T}}),$$
where we assume that γ_α exists for all $\alpha < \tau$, and
$$\mathcal{P}_\alpha = \mathcal{M}_{\gamma_\alpha}|\xi_\alpha,$$
where ξ_α is the largest ξ such that $\mathcal{M}_{\gamma_\alpha}|\xi$ and $\mathcal{M}_{\gamma_\alpha}|\,\text{lh}(E_{\gamma_\alpha}^{\mathcal{T}}$ have the same subsets of κ_α. Note that $\mathcal{P}_\alpha$ is the model to which an extender with critical point κ_α would be applied in any normal continuation of $\mathcal{T}$. We now define
$$\Phi_\delta(\mathcal{T}) = \langle \mathcal{P}_\alpha \mid \alpha < \tau \rangle.$$

$\Phi_\delta(\mathcal{T})$ is essentially the sub-phalanx of $\Phi(\mathcal{T})$ containing the models we might actually go back to in a normal continuation of $\mathcal{T}$ using extenders of length $\geq \delta$. For example, if $\mathcal{T}$ is the genericity iteration of $\mathcal{M}$ we used in the proof of 7.1, which lived in the window between the least $< \lambda$-strong of $\mathcal{M}$ and the next Woodin δ, and $i \colon \mathcal{M} \to \mathcal{N}$ is the iteration map of $\mathcal{T}$, then $\Phi_{i(\delta)}(\mathcal{T}) = \langle \mathcal{M} \rangle$.

Theorem 8.1 (Closson, Neeman, Steel) *If $\mathcal{M}, \mathcal{T}, \mathcal{N}$, and δ are as above, and g is $\text{Col}(\omega, \xi)$ generic over $\mathcal{N}$, for some $\xi \leq \delta$, and $\Phi_\delta(\mathcal{T})$ is in the least admissible set over $\langle \mathcal{N}|\delta, g \rangle$, then*
$$\mathcal{N}[g] =^* \mathcal{R},$$
for some $\langle \mathcal{N}|\delta, g \rangle$-mouse $\mathcal{R}$.

Remark 8.2 The equality $\mathcal{N}[g] = \mathcal{R}$ cannot literally be true, since the two are structures for different languages. What we mean is that the two structures are fine-structurally equivalent, in that they have the same projecta, standard parameters, and Levy hierarchy past some point. (It can happen that their universes are different, although they are the same if $\mathcal{N}[g]$ is admissible.) We prefer not to spell out here the details of this notion of equivalence, which we call *intertranslatability*. We write $=^*$ for it.

Proof. We define the $*$-*transform*, which associates to initial segments $\mathcal{Q}[g]$ of $\mathcal{N}[g]$ mice $\mathcal{Q}[g]^*$ over $\langle \mathcal{N}|\delta, g \rangle$, in such a way that $\mathcal{Q}[g]$ and $\mathcal{Q}[g]^*$ are intertranslatable. Let $\Phi_\delta(\mathcal{T}) = \langle \mathcal{P}_\alpha \mid \alpha < \tau \rangle$.

To begin with, letting $\alpha > \delta$ be least such that $\mathcal{N}|\alpha \models \mathsf{KP}$, there is clearly a unique mouse $\mathcal{R}$ over $\langle \mathcal{N}|\delta, g \rangle$ such that $\mathcal{N}|\alpha[g] =^* \mathcal{P}$, and we let $(\mathcal{N}|\alpha[g])^*$ be this unique $\mathcal{R}$.

Let U be the tree of all finite sequences $\langle E_0, ..., E_n \rangle$ such that each E_i is an extender with $\text{crit}(E_i) < \delta$ and $\text{lh}(E_i) \geq \delta$, and $\mathcal{E}_0$ is on the $\mathcal{N}$-sequence, and E_{i+1} is on the sequence of $\mathrm{Ult}(\mathcal{M}_{\gamma_\alpha}, E_i)$ for α such that $\mathrm{crit}(E_i) = \kappa_\alpha$. For $\langle E_0, ..., E_n\rangle$ in U, we write $\mathcal{P}(\vec{E})$ for $\mathrm{Ult}(\mathcal{M}_{\gamma_\alpha}, E_n)$, where $\kappa_\alpha = \mathrm{crit}(E_n)$. Here we understand that $\mathcal{P}(\emptyset) = \mathcal{N}$. Because our initial $\mathcal{M}$ is iterable, U is wellfounded. We shall define $\mathcal{Q}[g]^*$ for all $\mathcal{Q} \trianglelefteq \mathcal{P}(\vec{E})$ where $\vec{E} \in U$ such that $o(\mathcal{Q})$ is at least the first admissible over $\mathcal{N}|\delta$. The definition is by induction on the U-rank of $\vec{E}$, with a subinduction on $o(\mathcal{Q})$.

The inductive clauses are as follows:

(a) if $o(\mathcal{Q}) = \omega\alpha + \omega$, then $\mathcal{Q}[g]^*$ is obtained from $\mathcal{Q}|\alpha[g]^*$ by taking one step in the J-hierarchy.

(b) if $o(\mathcal{Q})$ is a limit ordinal and $\mathcal{Q}$ is passive, then $\mathcal{Q}[g]^* = \bigcup\{(\mathcal{Q}|\eta)[g]^* \mid \eta < o(\mathcal{Q})\}$.

(c) if $\mathcal{Q}$ is active with last extender E, and $\mathrm{crit}(E) > \delta$, then letting $\mathcal{Q} = (\mathcal{R}, E)$, we set $\mathcal{Q}[g]^* = (\mathcal{R}[g]^*, E)$.

(d) if $\mathcal{Q}$ is active with last extender E, and $\mathrm{crit}(E) = \kappa_\alpha < \delta$, then $\mathcal{Q}[g]^* = \mathrm{Ult}_n(\mathcal{P}_\alpha, E)[g]^*$, where n is least such that $\rho_{n+1}(\mathcal{P}_\alpha) \leq \kappa_\alpha$. .

Our non-measurability assumption on δ guarantees that these cases are exhaustive.

The detailed verification that $\mathcal{Q}[g]$ and $\mathcal{Q}[g]^*$ are intertranslatable takes some work. However, the basic idea is quite simple. Since $\Phi_\delta(\mathcal{T})$ is coded into $\langle \mathcal{Q}|\delta, g\rangle$, $\mathcal{Q}[g]$ can recover $\mathcal{Q}[g]^*$ by employing the inductive definition we just gave. Conversely, if we want to recover $\mathcal{Q}[g]$ from $\mathcal{Q}[g]^*$, all is trivial unless inductive clause (d) applies. Adopting the notation there, we simply note that $\mathcal{Q}^*$ can recover $\mathcal{P}_\alpha$ and E as an appropriate core of itself, and associated core-embedding extender. It is worth noting that in this direction, we do not need to use g.

(For example, let $\mathcal{Q}$ be the first level of $\mathcal{N}$ such that case (d) applies in defining $\mathcal{Q}[g]^*$. It is not hard to see that $\mathrm{crit}(E) = \kappa_0$ then. Let n be least such that $\rho_{n+1} \leq \kappa_0$. We have $\mathrm{Ult}_n(\mathcal{P}_0, E)[g]^* = \mathrm{Ult}_n(\mathcal{P}_0, E)[g]$ because E is the first extender overlapping δ. We can regard $\mathrm{Ult}_n(\mathcal{P}_0, E)[g]$ as a mouse $\mathcal{Q}[g]^*$ over $\langle \mathcal{N}|\delta, g\rangle$. The universes of $\mathcal{Q}[g]$ and $\mathcal{Q}[g]^*$ are different in this case, but one has that the $r\Sigma_{n+1}^{\mathcal{Q}[g]^*}$ subsets of δ are the same as the $r\Sigma_1^{\mathcal{Q}[g]}$ subsets of δ. Moreover, $\rho_{n+1}(\mathcal{Q}[g]^*) \leq \delta$, and $\mathcal{Q}[g]^*$ is $n+1$-sound as a mouse over $\langle \mathcal{N}|\delta, g\rangle$. Similarly, $\rho_1(\mathcal{Q}) \leq \delta$, and $\mathcal{Q}$ is 1-sound. So $\mathcal{Q}[g]$ and $\mathcal{Q}[g]^*$ are intertranslatable.) $\square$

We can use the $*$-transform to compute the length of certain Solovay sequences.

Theorem 8.3 *(a) For $\mathcal{M} = M^\sharp_{adr}$ and λ the sup of its Woodins, $D(\mathcal{M}, \lambda) \models \theta = \theta_\omega$.*

(b) For $\mathcal{M} = M^\sharp_{dc}$ or $\mathcal{M} = M^\sharp_{in.lim}$, and λ the sup of its Woodins in either case, we have $D(\mathcal{M}, \lambda) \models \theta = \theta_{\omega_1}$.

Proof. We prove (b), the proof of (a) being similar. Let I be $\mathcal{I}(\mathcal{M}, \lambda)$-generic, and $A \in \mathrm{Hom}^*_I$. By 7.5, it suffices to show that A is projective in some operator $\mathcal{M}^\tau$, where $\tau < \omega_1$.

Let A have a $< \lambda_\infty$-UB code in $M_\infty[g]$, where g is generic on $\mathrm{Col}(\omega, \mu_0)$. Let n be large enough that $\mu = \mathrm{crit}(i_{n,\infty}) > \mu_0$. Let $\mathcal{T}$ be the iteration tree giving rise to $i_{0,n}$. Let ξ be the least Woodin of M^I_n above μ. We can do a genericity iteration of M^I_n in the window (μ, ξ), producing a normal tree $\mathcal{U}$ and associated embedding $k \colon M^I_n \to \mathcal{P}$ so that $\Phi_\mu(\mathcal{T}) \in \mathcal{P}[g][h]$ for some h which is $\mathcal{P}$- generic over $\mathrm{Col}(\omega, k(\xi))$.

Let γ be the least Woodin of $\mathcal{P}$ strictly above $k(\xi)$, and let Σ be the iteration strategy for $\mathcal{M}$. By the proof of 4.0.1, A is projective in the window-based fragment $\Gamma = \Sigma|(\mathcal{T}^\frown\mathcal{U}, [k(\xi), \gamma])$, and so it suffices to show this fragment is projective in some $\mathcal{M}^\tau$. Let τ be the order type of the cardinals of $\mathcal{P}$ which are strong past γ. Note that because ξ and γ were chosen to be *least*, the cardinals of $\mathcal{P}$ which are strong past γ are all $< \mu$, and moreover

$$\Phi_\gamma(\mathcal{T}^\frown\mathcal{U}) = \Phi_\mu(\mathcal{T}).$$

We can compute Γ from $\mathcal{M}^\tau$ using $\mathcal{Q}$-structures. Letting $\mathcal{W}$ be played according to Γ, and $b = \Gamma(\mathcal{W})$, we have that b is the unique cofinal branch c of $\mathcal{W}$ such that $\mathcal{Q}(c, \mathcal{W})$ is iterable when backed up by the phalanx $\Phi_{\delta(\mathcal{W})}(\mathcal{T}^\frown\mathcal{U}^\frown\mathcal{W}) = \Phi_\mu(\mathcal{T})$. (We use here that $\mathcal{W}$ does not drop.) The branch oracle $\mathcal{Q}(b, \mathcal{W})$ may involve extenders overlapping $\delta(\mathcal{W})$, but these can be transformed away, and we get b is the unique cofinal branch c of $\mathcal{W}$ such that $\mathcal{Q}(\mathcal{W})[g][h]^* \trianglelefteq \mathcal{M}^\tau(\langle \mathcal{M}(\mathcal{W}), (g, h)\rangle)$. The key here is that $\mathcal{Q}(\mathcal{W})[g][h]^*$ reaches no further than this in the hierarchy of mice over $\langle \mathcal{M}(\mathcal{W}), (g, h)\rangle$. That can be proved by looking a little more closely at the definition of the $*$-transform. $\square$

Remark 8.4 The author first discovered a verion of the way $\mathcal{Q}[g]^*$ recovers $\mathcal{Q}[g]$ which works without the g coding $\Phi_\delta(\mathcal{T})$, and used this to prove 8.3. Itay Neeman then observed that the process simplified considerably in the presence of a g coding $\Phi_\delta(\mathcal{T})$, and would likely lead in that case to a level-by-level intertranslation. Erik Closson worked out the intertranslation in full detail.

9 A long Solovay sequence

We shall show in this section that $\theta_{\omega_1} < \theta$ in the derived model associated to $M^\sharp_{\mathrm{wdn.lim}}$, and then give some indication as to how to prove that in fact $\theta = \theta_\theta$ holds there.

It is easy to say what the mouse operator sitting at θ_{ω_1} is. Let us fix $\mathcal{M} = M^\sharp_{\mathrm{wdn.lim}}$ throughout this section, and let the $\mathcal{M}^\alpha$-operator be obtained from $\mathcal{M}$ as in 7.4.

Definition 9.1 *For any countable transitive set x,*

$$\mathcal{M}^{\omega_1}(x) = \mathcal{M}^{\omega_1^x}(x),$$

where ω_1^x is the height of the least admissible set to which x belongs.

Clearly $\mathcal{M}^\alpha$ is projective in $\mathcal{M}^{\omega_1}$, for all $\alpha < \omega_1$. It is enough then to show that, letting λ be the Woodin limit of Woodins in $\mathcal{M}$, and $D = D(\mathcal{M}^I_\infty, \lambda^I_\infty)$ where $I \in \mathcal{I}(\mathcal{M}, \lambda)$-generic, that $\mathcal{M}^{\omega_1} \in D$.

Claim 1. If $\mathcal{M}^{\omega_1}|V^{M_\infty[g]}_{\lambda_\infty} \in M_\infty[g]$, for some g generic over M_∞ for a poset of size $< \lambda_\infty$, then $\mathcal{M}^{\omega_1} \in D$.

Proof. The same proof that worked for $\mathcal{M}^0$ in 7.1 works here. Since $\mathcal{M}^{\omega_1}$ condenses to itself, and determines itself on small generic extensions, we get club many generically correct hulls of $M_\infty[g]$, and hence a UB code in $M_\infty[g]$ for $\mathcal{M}^{\omega_1}$. $\square$

Now let
$$\mathcal{M} \models \kappa \text{ is } \lambda + \omega\text{-reflecting in } \lambda.$$
That there is such a κ follows from the Woodinness of λ, and this is all of the Woodin property we shall need to show that $\mathcal{M}^{\omega_1} \in D$. Let ξ be the least Woodin of $\mathcal{M}$ above κ, and let
$$k\colon \mathcal{M} \to \mathcal{P}$$
come from a genericity iteration $\mathcal{T}$ on $\mathcal{M}|\xi$ with all critical points $> \kappa$, such that
$$\mathcal{M} \in \mathcal{P}[g], \text{ where } g \text{ is } \mathrm{Col}(\omega, k(\xi))\text{-generic over } \mathcal{P}.$$
We may as well assume that $\mathcal{T}$ is the first tree used in I, and therefore it is enough to show

Claim 2. $\mathcal{M}^{\omega_1}|V^{\mathcal{P}[g]}_{k(\lambda)} \in \mathcal{P}[g]$.

Proof. We show that $\mathcal{P}[g]$ can compute $\mathcal{M}^{\omega_1}(x)$ by using the $*$-transform. To this end, let η be a cardinal of $\mathcal{P}[g]$ such that $k(\xi) < \eta < k(\lambda)$. Let F_η be the first extender F on the $\mathcal{P}$-sequence such that $\kappa < \mathrm{crit}(F) \leq \eta$ and $\mathrm{lh}(F) \geq \eta$, if there is one, and let F_η be a principal ultrafilter otherwise. Note that $\mathrm{crit}(F_\eta) > k(\xi)$. Set
$$\mathcal{P}_\eta = \mathrm{Ult}(\mathcal{P}, F_\eta).$$
The choice of F_η guarantees that there are no extenders G on the sequence of $\mathcal{P}_\eta$ such that $\kappa < \mathrm{crit}(G) \leq \eta$ and $\mathrm{lh}(G) \geq \eta$. Thus
$$\Phi_\eta(\mathcal{T}^\frown\langle F_\eta\rangle) = \langle \mathcal{M}\rangle,$$
and since $\mathcal{M}$ is coded into $\langle \mathcal{P}|k(\xi), g\rangle$, we can define the $*$-transform of $\mathcal{P}_\eta[g]$ at η as in the proof of 8.1. We write
$$\mathcal{Q}[g]^{*,\eta}$$
for the $\langle \mathcal{P}|\eta, g\rangle$-mouse we get by applying the transform at η to an appropriate $\mathcal{Q}[g]$.

We now show by induction on $\alpha < k(\lambda)$:

Subclaim. If η is a cardinal of $\mathcal{P}$ such that $k(\xi) < \eta < k(\lambda)$, and $\alpha \leq \omega_1^{\langle \mathcal{P}|\eta, g\rangle}$, then $\mathcal{M}^\alpha(\langle \mathcal{P}|\eta, g\rangle) \trianglelefteq \mathcal{Q}[g]^{*,\eta}$, for some proper initial segment $\mathcal{Q}$ of $\mathcal{P}_\eta$.

Proof. We have done the case $\alpha = 0$ in the proof of 7.1. Let α be a limit ordinal, and $\alpha \leq \omega_1^{\langle \mathcal{P}|\eta, g\rangle}$. By induction, we get that the function $f(\beta) = \mathcal{M}^\beta(\langle \mathcal{P}|\eta, g\rangle)$, defined on all $\beta < \alpha$, is in $\mathcal{P}_\eta[g]$. But then $f \in \mathcal{P}_\eta[g]^{*,\eta}$, so $f \in \mathcal{Q}[g]^{*,\eta}$ for some $\mathcal{Q} \triangleleft \mathcal{P}_\eta$. Clearly then $\mathcal{M}^\alpha(\langle \mathcal{P}|\eta, g\rangle) \trianglelefteq \mathcal{Q}[g]^{*,\eta}$.

Now suppose the subclaim holds for α. Fix η such that $k(\xi) < \eta < \lambda$, η is a cardinal of $\mathcal{P}[g]$, and $\alpha < \omega_1^{\langle \mathcal{P}|\eta, g\rangle}$. We want to show $\mathcal{M}^{\alpha+1}(\langle \mathcal{P}|\eta, g\rangle)$ is a proper initial segment of $\mathcal{P}_\eta[g]^{*,\eta}$. Note that κ is $k(\lambda)+\omega$ reflecting in $k(\lambda)$ in $\mathcal{P}_\eta$, as it is not moved by our embedding from $\mathcal{M}$ to $\mathcal{P}_\eta$. Let A be the theory in $\mathcal{P}_\eta|(k(\lambda)+\omega)$ of parameters in $\mathcal{P}_\eta|k(\lambda)$, and let E be an extender on the $\mathcal{P}_\eta$-sequence so that $\mathrm{crit}(E) = \kappa$, and

$$i_E(A) \cap \eta^+ = A \cap \eta^+$$

holds in $\mathcal{P}_\eta$. Let

$$\mathcal{Q} = \mathcal{P}_\eta|\,\mathrm{lh}(E).$$

It is enough to show $\mathcal{M}^{\alpha+1}(\langle \mathcal{P}|\eta, g\rangle) \trianglelefteq \mathcal{Q}[g]^{*,\eta}$. But $\mathcal{Q}[g]^{*,\eta} = \mathrm{Ult}(M, E)[g]^{*,\eta}$, which reaches the Woodin-limit-of-Woodins hypothesis, so it is enough to show that $\mathrm{Ult}(M, E)[g]^{*,\eta}$ is $\mathcal{M}^\alpha$-closed.

Now by our induction hypothesis, $\mathcal{P}[g] \models$ " for all $\beta \leq \alpha$, for all cardinals ν such that $k(\xi) < \nu < k(\lambda)$ and $\beta \leq \omega_1^{\langle \mathcal{P}|\nu, g\rangle}$, $\mathcal{P}_\nu[g]^{*,\nu}$ has a proper initial segment satisfying "I am $\mathcal{M}^\beta(\langle \mathcal{P}|\nu, g\rangle)$"". Let us call the sentence in quotes $\psi(k(\xi), \mathcal{M}, g, k(\lambda), \alpha)$, where we have displayed the parameters about which it speaks. We then get $\mathcal{P}_\eta[g] \models \psi[k(\xi), \mathcal{M}, g, k(\lambda), i_{F_\eta}(\alpha)]$, using that $k(\xi), k(\lambda), \mathcal{M}$, and g are fixed by i_{F_η}. But $i_{F_\eta}(\alpha) \geq \alpha$, so inspecting ψ, we see $\mathcal{P}_\eta[g] \models \psi[k(\xi), \mathcal{M}, g, k(\lambda), \alpha]$. Letting τ be a term such that $\mathcal{M} = \tau^g$, we can fix $p \in \mathrm{Col}(\omega, k(\xi))$ such that

$$\mathcal{P}_\eta \models p \Vdash \psi(k(\xi), \tau, \dot{g}, k(\lambda), \alpha).$$

Because $i_E(A) \cap \eta^+ = A \cap \eta^+$ holds in $\mathcal{P}_\eta$,

$$\mathrm{Ult}(\mathcal{P}_\eta, E) \models p \Vdash \psi(k(\xi), \tau, \dot{g}, k(\lambda), \alpha).$$

But $\mathcal{M}$ embeds into $\mathcal{P}_\eta$ with critical point $> \kappa$, and hence $\mathrm{Ult}(M, E)$ embeds into $\mathrm{Ult}(\mathcal{P}_\eta, E)$ with critical point $> \eta^{+,\mathcal{P}_\eta}$. Thus

$$\mathrm{Ult}(\mathcal{M}, E) \models p \Vdash \psi(k(\xi), \tau, \dot{g}, k(\lambda), \alpha).$$

It is easy to see that this implies that $\mathrm{Ult}(\mathcal{M}, E)[g]$ is $\mathcal{M}^\alpha$-closed below $i_E(\lambda)$. □

The subclaim completes the proof of Claim 2. □

Now we may assume that the first iteration tree used in I is the tree giving rise to k. In that case, we get the hypothesis of Claim 1 from Claim 2, and so we have $\mathcal{M}^{\omega_1} \in D$.

Now suppose $\gamma < \theta^D$. We want to show $\theta_\gamma < \theta$ holds in D. Assume first that there is a prewellorder $\leq^*$ of $\mathbb{R}$ of order type γ such that $\leq^*$ has a UB code in $\mathcal{M}$, or more precisely, is captured by $(\emptyset, \emptyset)$ in the sense of the proof of 4.6. (This is true if $\leq^*$ is projective, for example.) For $x \in R$ and $n < \omega$, we define a mouse operator $\mathcal{M}^{x,n}$ which, we shall show, has Wadge rank approximately $\omega|x| + n$ in D. (Here and below, $|x|$ is the rank of x in $\leq^*$.

It will suffice to define $\mathcal{M}^{x,n} \restriction \mathbb{R}$, by the following lemma.

Definition 9.2 *A* mouse-jump operator *is a function $x \mapsto \mathcal{N}(x)$ defined for a Turing cone of reals x, such that for some sentence ϕ is the language of relativised ms-mice, we have for all $x \in \mathrm{dom}(\mathcal{N})$*

(a) $\mathcal{N}(x)$ is the least $\omega_1 + 1$-iterable ms-mouse $\mathcal{P}$ over x such that $\mathcal{P} \models \phi$, and

(b) if $y \equiv_T x$, then $\mathcal{N}(y)$ is the canonical re-arrangement of $\mathcal{N}(x)$ as a y-mouse.

Lemma 9.3 *Assume AD^+. Let $\mathcal{N}$ be a mouse-jump operator defined in the cone above x_0; then there is a unique mouse operator $\mathcal{N}^+$ defined on all countable transitive sets Y such that $x_0 \in Y$, and such that whenever g is $\mathcal{N}^+(Y)$-generic over $\mathrm{Col}(\omega, Y)$, and $z(Y, g)$ is a real canonically coding Y and g (so that $x_0 \leq_T z(Y, g)$), then*

$$\mathcal{N}^+(Y)[g] = \mathcal{N}(z(Y, g)).$$

Moreover, the $\mathcal{N}^+$-operator satisfies condensation, in that if $\pi: \mathcal{P} \to \mathcal{N}^+(Y)$ is elementary and $x_0 \in \mathrm{ran}(\pi)$, then $\mathcal{P} = \mathcal{N}^+(\pi^{-1}(Y)$.

Proof. We construct $\mathcal{N}^+(Y)$ as follows. Let g be generic over $\mathrm{Col}(\omega, Y)$ for the collection of all $\mathrm{OD}(Y \cup \{Y\})$ dense sets. (Our only use of AD^+ is to conclude that there is such a g.) Let $z = z(Y, g)$. One can now construct $\mathcal{N}^+(Y)$ by using the extender sequence of $\mathcal{N}(z)$, as in [16]. In the end we have that for our particular g, $\mathcal{N}^+(Y)[g] = \mathcal{N}(z(Y, g))$. But since the $\mathcal{N}$-operator is given by a sentence, the identity we have for g holds for all $\mathcal{N}^+(Y)$-generics, as desired.

It is easy to see that $\mathcal{N}^+$ satisfies condensation. □

We shall define mouse-jump operators $\mathcal{N}^{x,n}$, and then let $\mathcal{M}^{x,n} = (\mathcal{N}^{x,n})^+$. The operator $\mathcal{N}^{x,n}$ will be defined on the Turing cone above x. The definition of $\mathcal{N}^{x,n}$ proceeds by induction on the lexicographic order on $\mathbb{R} \times \omega$ determined by $\leq^*$. If $|x| = 0$, then we set $\mathcal{N}^{x,0} = \mathcal{M} \restriction \mathbb{R}$. If $|x| > 0$, then for $z \in \mathbb{R}$,

$$\mathcal{N}^{x,0}(z) = \bigcup \{\mathcal{N}^{y,n}(z) \mid y \leq_T x \wedge y <^* x \wedge n < \omega\}.$$

It is important here that $\mathcal{N}^{x,0}$ is a mouse-jump operator itself, and for that one needs to use the definability of $\leq^* \restriction \{y \mid y \leq_T z\}$ over $\mathcal{M}(z)$. This is where we use the assumption that $\leq^*$ has a UB code in $\mathcal{M}$.

Finally,

$$\begin{aligned}\mathcal{N}^{x,n+1}(z) \quad = \quad & \text{minimal active } (\mathcal{N}^{x,n})^{+}\text{-closed } z\text{-mouse } \mathcal{R} \\ & \text{such that } \mathcal{R} \models \text{there is a Woodin limit of Woodins.}\end{aligned}$$

One can show that the $\mathcal{M}^{x,n}$ are in D by an argument like that given in the proof of claim 2 above. It is also not hard to show $\mathcal{M}^{x,n+1}$ is not ordinal definable from $\mathcal{M}^{x,n}$ and a real over D, and is essentially Wadge least with this property.

In order to remove the assumption that $\leq^*$ has a UB code in $\mathcal{M}$, we replace our Woodin limit mouse operator $\mathcal{M}$ with a *hybrid mouse operator* $\mathcal{M}^{\Sigma}$, where $\Sigma \in D$ is an iteration strategy with condensation such that $\leq^*$ is Wadge reducible to Σ. Here $\mathcal{M}^{\Sigma}(z)$ is the minimal active Σ-mouse with a Woodin limit of Woodins. The invariance of the derived model proof (see 4.6) extends so as to show that $\mathcal{I}$-generic iterations of $\mathcal{M}^{\Sigma}(z)$ yield the derived model D as well. This can be used as above to show that the nestings of $\mathcal{M}^{\Sigma}$ guided by $\leq^*$ are in D.

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