Logic and Medvedev Degrees

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History

- Heyting (1930)
- Kolmogorov (1931)
- Kleene (1945)
- Medvedev (1955, 1962)

The Medvedev lattice

A **mass problem** is a subset $\mathcal{A} \subseteq \omega^\omega$.

Medvedev reducibility: $\mathcal{A} \leq_M \mathcal{B}$ if there is a computable functional Ψ such that $\Psi(\mathcal{B}) \subseteq \mathcal{A}$.

There is also a nonuniform version of this:

Muchnik reducibility: $\mathcal{A} \leq_w \mathcal{B}$ if
$(\forall X \in \mathcal{B})(\exists Y \in \mathcal{A})[Y \leq_T X]$.

$\mathfrak{M}$ is the structure of all mass problems modulo $\leq_M$–equivalence. $\mathfrak{M}$ is a *lattice* in a natural way, with lattice operations

$$\mathcal{A} + \mathcal{B} = \left\{ f \oplus g : f \in \mathcal{A} \wedge g \in B \right\}$$

and

$$\mathcal{A} \times \mathcal{B} = 0\widehat{\ }\mathcal{A} \cup 1\widehat{\ }\mathcal{B}.$$

Idem for the Muchnik degrees $\mathfrak{M}_w$.

The Medvedev lattice (continued)

$\mathfrak{M}$ has a zero element $\mathbf{0}$, the degree of any $\mathcal{A}$ containing a recursive set. $\mathbf{1}$ is the degree of $\emptyset$.

It is possible to endow $\mathfrak{M}$ with an operation $\rightarrow$ by

$$\mathcal{A} \rightarrow \mathcal{B} = \text{least}\left\{ \mathcal{C} : \mathcal{A} + \mathcal{C} \geq_M \mathcal{B} \right\}.$$

Negation is defined by $\neg\mathcal{A} = \mathcal{A} \rightarrow \mathbf{1}$.

A propositional formula is **true** in $\mathfrak{M}$ if its interpretation in $\mathfrak{M}$ always yields $\mathbf{0}$. Here $\vee$ is interpreted by $\times$ and $\wedge$ by $+$. $\mathbf{0}$ plays the role of true and $\mathbf{1}$ of false. **(!)**

Example: $\mathcal{A} \rightarrow \mathcal{A}$ is true, $\mathcal{A} \vee \neg\mathcal{A}$ is false.

The **theory** $\mathrm{Th}(\mathfrak{M})$ of $\mathfrak{M}$ consists of all formulas that are true in $\mathfrak{M}$.

The logic of $\mathfrak{M}$

Theorem (Medvedev, Jankov, Sorbi)
$\mathrm{Th}(\mathfrak{M}) = \mathrm{IPC} + \neg\mathcal{A} \vee \neg\neg\mathcal{A}$.

For any principal filter $\mathcal{G} \subseteq \mathfrak{M}$, $\mathfrak{M}/\mathcal{G}$ is again a Brouwer algebra. Negation in $\mathfrak{M}/\mathcal{G}$ is defined by $\neg\mathcal{A} = \mathcal{A} \to \mathcal{G}$.

We can now talk about $\mathrm{Th}(\mathfrak{M}/\mathcal{G})$ for various $\mathcal{G}$. We have the following beautiful result:

Theorem (Skvortsova) There exists $\mathcal{G} \subseteq \mathfrak{M}$ such that $\mathrm{Th}(\mathfrak{M}/\mathcal{G}) = \mathrm{IPC}$.

Join-reducible elements

An element $\mathcal{D}$ is **join-reducible** if there are $\mathcal{A}$ and $\mathcal{B}$ incomparable such that $\mathcal{A} + \mathcal{B} = \mathcal{D}$.

Note that $\mathfrak{M}$ satisfies $\neg\mathcal{A} \vee \neg\neg\mathcal{A}$ because $\mathbf{1}$ is join-irreducible:

Proposition If $\mathcal{D}$ is join-reducible and $\mathcal{G}$ is the principal filter generated by $\mathcal{D}$, then the weak law of the excluded middle does not hold in $\mathrm{Th}(\mathfrak{M}/\mathcal{G})$.

In fact we have

Theorem (Sorbi) For every principal filter $\mathcal{G}$ generated by a join-irreducible element greater than

$$\mathbf{0}' = \left\{ f : f \text{ noncomputable} \right\}$$

it holds that $\mathrm{Th}(\mathfrak{M}/\mathcal{G}) = \mathrm{IPC} + \neg\mathcal{A} \vee \neg\neg\mathcal{A}$.

Join-irreducible elements

$\mathbf{0}$ and $\mathbf{1}$ are trivial examples of join-irreducible elements.

Other examples: Let f be noncomputable and define

$$\mathcal{B}_f = \left\{ g : g \not\leq_T f \right\}$$

Then the degree of $\mathcal{B}_f$ is join-irreducible.

A condition for join-reducibility

$C(\mathcal{A}) = \left\{ f : (\exists e)[\Phi_e(f) \in \mathcal{A}] \right\}$.

Splittings in the Turing degrees give many examples of join-reducible elements of $\mathfrak{M}$:

Lemma (Sorbi) Suppose $\mathcal{A}$ is a mass problem such that:

$$\text{There exist functions } g, h \notin C(\mathcal{A}) \text{ such that } g \mid_T h \text{ and } g \oplus h \in C(\mathcal{A}). \quad (1)$$

Then $\mathcal{A}$ is join-reducible.

Proof. If condition (1) holds then

$$\mathcal{A} = (\mathcal{A} \times \{g\}) + (\mathcal{A} \times \{h\}).$$

By incomparability of g and h and the fact that they cannot compute anything in $\mathcal{A}$, it follows that $\mathcal{A} \times \{g\}$ and $\mathcal{A} \times \{h\}$ are incomparable. $\square$

Join-reducibility in $\mathfrak{M}_w$

Proposition Sorbi's condition (1) characterizes the join-reducible Muchnik degrees.

Proof. Sorbi's Lemma also holds for the Muchnik degrees, so we only have to show that if (1) does not hold for $\mathcal{A}$ then $\mathcal{A}$ is join-irreducible. So suppose (1) does not hold, and suppose that $\mathcal{A} \equiv \mathcal{B} + \mathcal{C}$ and $\mathcal{A} \not\leq \mathcal{B}$. We show that $\mathcal{A} \leq \mathcal{C}$. Since $\mathcal{A} \equiv C(\mathcal{A})$ (since $\mathcal{A}$ is Muchnik), $\mathcal{A} \not\leq \mathcal{B}$ implies that there is $g \in \mathcal{B} \setminus C(\mathcal{A})$. Now $\mathcal{A} \leq \{g \oplus h : h \in \mathcal{C}\}$, via Ψ say. But then, since $\Psi(g \oplus h) \leq_T g \oplus h$, all $h \in \mathcal{C}$ must be in $C(\mathcal{A})$ by the failure of (1). Hence $\mathcal{A} \equiv C(\mathcal{A}) \leq \mathcal{C}$ via the identity. □

N.B. Dyment showed that *every* Muchnik degree is meet-reducible.

Join-reducibility in $\mathfrak{M}$

Theorem Condition (1) does not characterize the join-reducible elements of $\mathfrak{M}$: There is a join-reducible $\mathcal{A}$ such that (1) does not hold.

Proof. Construct $\mathcal{A}$ by brute force, using results about lattice embeddings into the Turing degrees. □

Medvedev degrees of Π^0_1 classes

Simpson introduced the structure $\mathfrak{P}$ of Medvedev degrees of nonempty Π^0_1 subsets of 2^ω. $\mathfrak{P}$ is a lattice under $\leq_M$ in the same way as $\mathfrak{M}$, with meet $\times$ and join $+$ defined as before. $\mathfrak{P}$ has smallest element $\mathbf{0}$, the degree of 2^ω, and largest degree $\mathbf{1}$, the degree of the class of all *PA-complete* sets.

A sample of results:

- (Binns and Simpson) Every finite distributive lattice is embeddable into $\mathfrak{P}$.
- (Cenzer and Hinman) $\mathfrak{P}$ is dense.
- (Binns) Every degree in $\mathfrak{P}$ splits in two lesser ones.

Medvedev reducibility and sets of positive measure

$\mathfrak{P}$ posesses many natural elements that are strictly between $\mathbf{0}$ and $\mathbf{1}$. A Π^0_1 class is **special** if has no computable elements.

Theorem (Simpson) Let P be any special Π^0_1 class of positive measure. Then the Medvedev degree of P is strictly between $\mathbf{0}$ and $\mathbf{1}$.

The proof uses considerations about 1-random sets. Define the **left shift** $T : 2^\omega \to 2^\omega$ by $T(X)(n) = X(n+1)$. Let T^k denote the k-iteration of T.

Theorem (Kučera) For any Π^0_1 class $\mathcal{P}$ of positive measure we have the following: For every 1-random X there exists k such that $T^k(X) \in \mathcal{P}$.

Sets of positive measure (continued)

It follows from the theorem that in particular

$$(\forall X \text{ 1-random })(\exists Y \in \mathcal{P})[X \equiv_T Y]. \qquad (2)$$

Next we show that Kučera's theorem and its corollary (2) do not hold *uniformly*:

Proposition Let $\mathcal{P}$ be any special Π^0_1 class of positive measure. Then $\mathcal{P} \not\leq_M \mathcal{R}$, where $\mathcal{R}$ is the class of 1-random sets.

Proof. Suppose that $\mathcal{P} \leq_M \mathcal{R}$ via Φ. Then $\mathrm{dom}(\Phi)$ contains a computable set: Since $\mathcal{R}$ is dense in 2^ω we can recursively find for any $\sigma \in 2^{<\omega}$ an extension $\tau \sqsupset \sigma$ such that $\Phi(\tau)\downarrow$. Let $X \in \mathrm{dom}(\Phi)$ be computable. Then $\Phi(X)$ is also computable, hence since $\mathcal{P}$ is special there is n such that $\Phi(X{\upharpoonright}n) \in \mathcal{U}$, where $\mathcal{U}$ is the open complement of $\mathcal{P}$. Now let Z be a 1-random set extending $X{\upharpoonright}n$. Then $\Phi(Z) \in \mathcal{U}$, but also $\Phi(Z) \in \mathcal{P}$ by assumption on Φ, contradiction. □

Sets of positive measure (continued)

Simpson, using Kučera's theorem, proved that the *Muchnik* degrees of positive measure have a largest element. The next theorem shows that this does not hold for the Medvedev degrees, answering a question posed by Simpson.

Theorem (Simpson and Slaman, Terwijn) The set of elements of $\mathfrak{P}$ of positive measure does not have a maximal element.

Connection with constructive logic

Theorem (Terwijn) $\mathfrak{P}$ is not a Heyting algebra.

Proof builds on results by Jockusch and Soare, Sorbi, and Binns and Simpson.

Conjecture $\mathfrak{P}$ is not a Brouwer algebra.

If $\mathfrak{P}$ does not allow for suitable implication operators, we can go back to the larger structure $\mathfrak{M}$, and look at natural quotients there.

Conjecture $\mathrm{Th}(\mathfrak{M}/PA) = \mathrm{IPC}$.

The conjecture $\mathrm{Th}(\mathfrak{M}/PA) = \mathrm{IPC}$

- $\mathrm{IPC} \subseteq \mathrm{Th}(\mathfrak{M}/PA)$
- $\mathrm{Th}(\mathfrak{M}/PA) \subseteq \mathrm{IPC} + \neg\alpha \vee \neg\neg\alpha$ because PA is closed and by a general result of Sorbi and Terwijn.
- $\mathrm{Th}(\mathfrak{M}/PA) \neq \mathrm{IPC} + \neg\alpha \vee \neg\neg\alpha$ because PA is join-reducible (Binns) and by a general result by Sorbi.

The conjecture $\text{Th}(\mathfrak{M}/PA) = \text{IPC}$

- Skvortsova: If $\mathbf{A} \in \mathfrak{M}$ is Muchnik then $\text{Th}(\mathfrak{M}/\mathbf{A})$ satisfies the Kreisel-Putnam formula

  $$(\neg p \to q \vee r) \to (\neg p \to q) \vee (\neg p \to r)$$

  and hence $\text{Th}(\mathfrak{M}/\mathbf{A}) \supsetneq \text{IPC}$.

- The Turing degrees of PA are upwards closed (Solovay), but the Medvedev degree of PA is not Muchnik.

- PA is *effectively homogeneous*. Hence by Skvortsova it satisfies

  $$(p \to q \vee r) \to (p \to q) \vee (p \to r)$$

  where p is interpreted by PA and q and r by arbitrary Medvedev degrees.