Realizing Σ_1 Formulas in $\mathcal{R}$

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The Most Trivial Talk

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Another motivation is to investigate incapabilities of simple formulas in term of definability.

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Given a language $\mathcal{L}$ and a model $\mathcal{M}$ of $\mathcal{L}$.

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We define Σ_n (complete) type with the formulas restricted to Σ_n formulas.

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- *all $\leq$-ascending finite tuples in I realize same (Σ_n) types given they are of same length.*

Partial orders

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We say $x, y \in P$ form a minimal pair iff $\mathcal{A}(x) \cap \mathcal{A}(y) = \emptyset$, and call a subset $\overrightarrow{x} \subseteq P$ an antichain iff its elements are pairwise $\leq_P$-incomparable.

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In this talk we will fix $\mathcal{L} = \{\leq\}$ and the structure $\mathcal{R} = (\mathbf{R}, \leq)$.

Slaman-Soare's Theorem

Let Q be partial orders with 0_Q and 1_Q, P a sub partial order of Q, and $S \subset Q$. Define

1. $U(S) = \{x \in P : \forall y \in S(x \geq y)\}$, and $U(y) = U(\{y\})$;
2. $V(S) = \{x \in P : \forall y \in S(x \leq y)\}$, and $V(y) = V(\{y\})$;
3. $Z(y) = \{z \in Q \setminus P : z < y$ and $V(y) \not\subseteq V(U(z))\}$.

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Theorem 4 (Slaman and Soare)**.** *Every embedding of P into $\mathcal{R}$ can be extended to an embedding of Q into $\mathcal{R}$ iff both conditions below fail*

(i) $\exists x, y \in Q(x \not\geq y$ *and* $VU(y) \subseteq VUV(x))$;

(ii) $\exists x \in Q \backslash P(Z(x) \neq \emptyset$ *and* $VU(Z(x) \cup V(x)) \not\subseteq V(x))$.

1-order-indiscernibles ...

Lempp observed that the following follows from Slaman-Soare's Theorem by simple calculations.

Proposition 5. *Every chain in $\mathcal{R}$ not containing $\mathbf{0}$ and $\mathbf{0}'$ is an infinite class of order 1-indiscernibles. Hence no degree besides $\mathbf{0}$ and $\mathbf{0}'$ is Π_1 definable in $\mathcal{R}$.*

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If $x \not\geq y$ then $v_y \notin VUV(x)$. Thus Theorem 4(i) fails:

$$\neg\exists x, y \in Q(x \not\geq y \text{ and } VU(y) \subseteq VUV(x)).$$

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If $Z(x) \neq \emptyset$, then $v_x > u_z$ since $V(x) \nsubseteq VU(z)$ for any $z \in Z(x)$. Thus $VU(Z(x) \cup V(x)) = VUV(x) = V(x)$ and Theorem 4(ii) fails too.

$$\neg \exists x \in Q \backslash P(Z(x) \neq \emptyset \text{ and } VU(Z(x) \cup V(x)) \nsubseteq V(x)).$$

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$$\neg\exists x \in Q \backslash P(Z(x) \neq \emptyset \text{ and } VU(Z(x) \cup V(x)) \not\subseteq V(x)).$$

Hence there is always an extension $g : Q \to \mathcal{R}$.

1-indiscernibles: non-order version

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Given a partial order P and an embedding

$$f : \langle \overrightarrow{x}, \leq_P \rangle \longrightarrow \langle \{\mathbf{d}_n : n < \omega\}, \leq \rangle,$$

when does there exist an extension $g : P \longrightarrow \mathcal{R}$ of f?

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Our strategy is to make the conditions altogether sufficient. In fact we need one more sequence for aid.

1-indiscernibles: a technical lemma

Lemma 7. *There are* $\langle \mathbf{d}_n : n < \omega \rangle$ *and* $\langle \mathbf{a}_{n,k} : n, k < \omega \rangle$ *such that*

(i) $\mathbf{d}_m \wedge \mathbf{d}_n = 0$ *whenever* $m \neq n$*;*

(ii) $\mathbf{a}_{n,k} \leq \mathbf{d}_n$ *for arbitrary* n *and* k*;*

(iii) for any finite subsets $F \subset \omega, G \subset \omega \times \omega$ *and* $\langle n, k \rangle \in \omega \times \omega$*, if* $n \notin F$ *and* $\langle n, k \rangle \notin G$ *then* $\mathbf{a}_{n,k} \not\leq \bigvee_{m \in F} \mathbf{d}_m \cup \bigvee_{\langle i,j \rangle \in G} \mathbf{a}_{i,j}$.

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Note that:

- (ii) and (iii) $\Rightarrow \langle \mathbf{d}_n : n < \omega \rangle$ is an independent sequence;
- (iii) $\Rightarrow$ for F an arbitrary finite subset of ω the class of degrees

$$\{\mathbf{a}_{n,k} : \langle n, k \rangle \in (\omega \backslash F) \times \omega\} \cup \{\mathbf{d}_n : n \in F\}$$

is also independent.

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We simultaneously construct two infinite sequences of c.e. sets $\langle D_n : n < \omega \rangle$ and $\langle A_{n,k} : n, k < \omega \rangle$ such that $D_n = \bigoplus_{k<\omega} A_{n,k}$,

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$$\mathcal{P}_{F,G,n,k,e} : A_{n,k} \neq \Psi_e(D_F \oplus A_G)$$

for all finite $F \subset \omega, G \subset \omega \times \omega, n \notin F$ and $\langle n, k \rangle \notin G$ (where $D_F = \bigoplus_{m \in F} D_m$ and $A_G = \bigoplus_{\langle i,j \rangle \in G} A_{i,j}$).

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A $\mathcal{P}$-strategy: at expansionary stage, enumerate a witness into $A_{n,k}$ and properly restraint $D_F \oplus A_G$.

It is straightforward to combine the strategies.

Let $\mathbf{d}_n = \mathbf{deg}(D_n)$ and $\mathbf{a}_{n,k} = \mathbf{deg}(A_{n,k})$. This end the proof of the lemma.

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Assume otherwise, for each $u \in P \backslash \mathcal{A}(P)$ we add an atom z_u such that $z_u \leq_P u$ and

$$\text{if } u \not\leq_P v \in P \text{ then } z_u \not\leq_P v. \tag{1}$$

Denote the resulting partial order by P'. It suffices to extend f to some $g : P' \to \mathcal{R}$.

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- otherwise let $g(z) = \mathbf{a}_{n,k}$ where $\mathbf{d}_n \notin ran(f)$.

Non-atoms. For $w \in P'\setminus(\mathcal{A}(P') \cup \{0_P\})$, let $g(w)$ be the joint of

$$\bigvee\{\mathbf{d}_n : f^{-1}(\mathbf{d}_n) \leq_P w\}$$

and

$$\bigvee\{g(z) : z \in \mathcal{A}(w) \text{ and } \neg\exists x(z \leq_P x \leq_P w)\}.$$

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If $u \not\leq_P v$, then by (1), $z_u \not\leq_P x$ for $x \leq_P v$. Hence by Lemma 7(iii), $g(z_u) \not\leq g(v)$ and thus $g(u) \not\leq g(v)$.

An easier proof

An infinite partial order P has the universal extensibility property or u.e.p. iff for any finite partial orders $Q \subseteq R$, if Q is a sub partial order of R and there exists an embedding $f : Q \to P$ then there exists an embedding $g : R \to P$ extending f.

It is easy to prove that there exists a countable partial order having u.e.p. and such a partial order must have a countable antichain. Then by embedding such a p.o. into $\mathcal{R}$ one can find that any antichain of the embedding form a set of 1-indiscernibles.

This much simpler proof of Theorem 6 is observed by Slaman.

Questions

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Question 8. *Characterizing 1-indiscernibles.*

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Question 8. *Characterizing 1-indiscernibles.*

Lempp suggested a more basic question.

Question 9. *When do two finite sequences realize same Σ_1 formulas?*

Dividing Σ_1 and Σ_2 formulas

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However, $\mathbf{0}$, $\mathbf{0}'$, the predicate $x \wedge y = z$ and the predicate $x \vee y = z$ are all Π_1 definable, hence Σ_2 definable.

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On the contrary, Σ_2 elementary substructures always contain $\mathbf{0}'$ since the fact there exists a greatest element is Σ_2 in $\mathcal{R}$ and the predicate x is not maximal is Σ_1.

Hence no proper ideal could be a Σ_2 elementary substructure.

Overall strategy

Fix $\mathbf{I}$ a nonprincipal ideal of $\mathcal{R}$. It suffices to prove that for any finite partial order P and an embedding $f : P \to \mathcal{R}$, there exists an embedding $g : P \to \mathbf{I}$ extending $f \upharpoonright P^-$ where $P^- = f^{-1} \upharpoonright \mathbf{I}$.

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Since $\mathbf{I}$ is nonprincipal we can find some $\mathbf{c} \in \mathbf{I}$ such that $\mathbf{c} > \bigvee_{i \leq n} \mathbf{a}_i$. The idea is to extend f in $[\mathbf{0}, \mathbf{c})$.

A technical lemma

Lemma 12. *There exist an independent sequence of degrees in* $[\mathbf{0}, \mathbf{c}]$, *say* $\langle \mathbf{b}_{i,k} : i \leq n, k \in \omega \rangle$, *such that* $\mathbf{a}_i \leq \mathbf{b}_{i,k}$, *and for any finite* $H \subset \{0, 1, \ldots, n\} \times \omega$,

$$\mathbf{a}_i \not\leq \bigvee\!\!\!\bigvee_{j \in H_0} \mathbf{a}_j \Rightarrow \mathbf{a}_i \not\leq \bigvee\!\!\!\bigvee_{\langle j,k \rangle \in H} \mathbf{b}_{j,k} \qquad (2)$$

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Let $A_i \subseteq \omega^{[2i]}$ represent $\mathbf{a}_i$ for $i \leq n$ and C represent $\mathbf{c}$. We will construct c.e. sets $\langle B_{i,k} : i \leq n, k \in \omega \rangle$ such that

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- $B_{i,k} \leq_{\mathrm{T}} C$;
- $\langle B_{i,k} : i \leq n, k \in \omega \rangle$ are pairwise disjoin;
- $A \cap B = \emptyset$ where $A = \bigcup_{i \leq n} A_i$ and $B = \bigcup_{i \leq n, k \in \omega} B_{i,k}$.

Proving the lemma: the requirements

To make the sequence independent, for $m \in \omega$, $\langle i, k\rangle \in (n+1) \times \omega$ and finite $H \subset (n+1) \times \omega$, we make

$$\mathcal{P}_e : B_{i,k} = \Phi_m(\bigcup_{j \in H_0} A_j \cup \bigcup_{\langle j,l\rangle \in H} B_{j,l}) \Rightarrow C \leq_{\mathrm{T}} A.$$

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Note that there are only finitely many such pairs (i, H_0).

We will use the trick of true stages computations and the true stages will be uniformly defined as $A \cup B$-true stages.

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At stage $s+1$, we enumerate $\langle x, t, 2e+1\rangle$ in $B_{i,k}$ iff $x \in C_{s+1}$, $\langle x, t, 2e+1\rangle$ is not restrained and $(\forall v)(t \leq v \leq s \rightarrow x < l^{\Phi}(e, v))$.

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Since $C \not\leq_{\mathrm{T}} A$, this restraint will drop at true stages.

An easier proof of the lemma

Slaman observed that Lemma 12 follows from Slaman-Soare Theorem.

Suppose P is a finite p.o with greatest element 1_P. Let $a_0 = 0_P, a_1, \ldots, a_{n-1}, a_n = 1_P$ enumerate the elements of P. For each $m > 0$ we define Q a finite super p.o. of P as below.

1. for $i < n$ and $j < m$ we will have $b_{ij} \in Q - P$ such that $a_k \leq b_{ij} \Leftrightarrow a_k \leq a_i$ and $a_k \geq b_{ij} \Leftrightarrow a_k = 1_P$.
2. for $H \subseteq n \times m$ and $|H| > 1$ we will have $c_H \in Q - P$ such that $a_k \leq c_H \Leftrightarrow a_k \leq \bigvee\!\!\!\bigvee_{i \in H_0} a_i$ (where $H_0 = \{j : \exists k(\langle k, j \rangle \in H)\}$) and $b_{ij} \leq c_H \Leftrightarrow \langle i, j \rangle \in H$.
3. for H', H'', $c_{H'} \leq c_{H''} \Leftrightarrow H' \subseteq H''$.

Recall the definitions before Slaman-Soare Theorem, for $z \in Q - P$, $U(z) = \{1_P\}$ and $VU(z) = P$. Hence for $x \in Q - P$, $Z(x) = \emptyset$ and the second condition of Slaman-Soare Theorem fails. By easy caculations, the first condition also fails.

Thus embeddings of P into $\mathcal{R}$ can always be extended to embeddings of Q into $\mathcal{R}$. It is easy to see that the b_{ij}'s introduced satisfy the requirements of Lemma 12.

Extending the embedding

Let $y_1, \ldots, y_m$ enumerate $P \backslash P^-$, define

$$g(y_k) = \bigvee \{b_{i,l} : x_i <_P y_k \text{ and } y_l \leq_P y_k\}.$$

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Hence $f \restriction P^- \subseteq g : P \to [\mathbf{0}, \mathbf{c}) \subset \mathbf{I}$.

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The differences above might suggest that finding 2-indiscernibles is a much harder job.

However we still raise the question.

Question 14. *For $n > 1$, does there exist n-indiscernibles?*

1-indiscernibles generating $\mathbb{R}$?

Since meet-inaccessible degrees are downward dense (by Ambos-Spies; in fact, dense in $\mathcal{R}$ by Ding and Zhang independently), 1-indiscernibles cannot generate any downward closed subset of $\mathcal{R}$ (by joins and meets).

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- By 1-indiscernibility, the meets of n-tuples of $\mathbf{I}$ are always $\mathbf{0}$. While by the choice of n, all element of $\mathbf{I}$ are cappable.

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Questions

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Question 16. Σ_1 *Skölem hulls of 1-indiscernibles?*

What happen if we turn to $(\mathbf{R}, \vee, \leq)$?

Question 17. *Does there exist n-indiscernibles in* $(\mathbf{R}, \vee, \leq)$ *for* $n \geq 1$*?*

Thanks for your attentions.