OUTLINE OF TALK:

1. Representations of GL_n and (GL_n, GL_m) Duality
2. Tensor Algebras of GL_n and Example $(n = 2)$
3. FFT and SFT for GL_n
4. Using Binary Invariants
   (a) SL_4 Tensor Algebra
   (b) SL_5 Tensor Algebra
5. An Inductive Process
   (a) SL_3 Tensor Algebra (revisited)
   (b) $O_n \times GL_3$ Covariants in $\mathcal{P}(\mathbb{C}^{n,3})$
6. Comments
7. A (Final) Cute Example: Quantum Qubits

1 Irreducible Representations of $\mathbf{GL_n}$

U_n : maximal unipotent subgroup of GL_n

A_n : maximal torus of GL_n which normalises U_n

$\rho_n^{(a_1,\dots,a_n)}$: irreducible GL_n representation with highest weight $(a_1, \dots, a_n)$.

Use Young diagram notation for polynomial representations

$r(D)$: **rank** or **depth** of the Young diagram D, i.e., this is the number of non-zero entries in $D = (a_1, \dots, a_n)$

$\#(D) = a_1 + a_2 + \dots + a_n$: **size** of the Young diagram $D = (a_1, \dots, a_n)$

2 $\mathbf{GL_n \times GL_m}$ Duality

$\mathbb{C}^{n,m} = \mathbb{C}^n \otimes \mathbb{C}^m$: space of complex $n \times m$ matrices

$\mathcal{P}(\mathbb{C}^{n,m})$: Polynomial ring over $\mathbb{C}^{n,m}$

Action of $GL_m \times GL_n$ on $\mathcal{P}(\mathbb{C}^m \otimes \mathbb{C}^n)$:

$$(g,h) \cdot F(x) = F(g^t x h)$$

$GL_n \times GL_m$ Duality:

$$\mathcal{P}(\mathbb{C}^{n,m}) = \mathcal{P}(\mathbb{C}^n \otimes \mathbb{C}^m) = \sum_{r(D) \leq \min(n,m)} \rho_n^D \otimes \rho_m^D.$$

Easy to see:

(a) $\det[X_{kk}]$ is a $GL_m \times GL_n$ highest weight of weight $[1^k] \otimes [1^k]$, for $k = 1, \ldots, \min(n,m)$.

(b) $GL_m \times GL_n$ module of highest weight $D = (d_1, d_2, \ldots, d_k)$ has joint highest weight given (up to a scalar) by:

$$\det[X_{11}]^{d_1 - d_2} \det[X_{22}]^{d_2 - d_3} \ldots \det[X_{k-1,k-1}]^{d_{k-1} - d_k} \det[X_{kk}]^{d_k}$$

3 Partial Models

G: reductive linear algebraic group

$U = U_G$: maximal unipotent subgroup of G

A : maximal torus normalising U

V_ϕ : irreducible representation of G of highest weight $\phi \in \widehat{A_n}^+$.

M_S a **partial model** of G : If

$$M_S = \sum_{\phi \in S \subset \widehat{A}^+} V_\phi,$$

is an **algebra** and a G module comprising a collection S of irreducible modules of G, appearing at most once.

Restrict ourselves to GL_n.

For two partial models of GL_n indexed by sets S and T, we define the **type $(\mathbf{S}, \mathbf{T})$ tensor algebra of $\mathbf{GL_n}$** as the algebra of covariants as follows:

$$(M_S \otimes M_T)^{U_n}$$

If M_S and M_T are full models, we would simply call this the **tensor algebra of $\mathbf{GL_n}$**.

The type (S, T) tensor algebra of GL_n has a graded structure determined by $(\widehat{A_n}^+)^3$.

Understanding the decomposition of this subalgebra with respect to the triple grading is tantamont to understanding the tensor product decomposition of an arbitary tensor product of two irreducible modules $V_\phi \otimes V_\psi$ with $V_\phi \in M_S$ and $V_\psi \in M_T$.

A Little History:

Van der Warden (1930's): Up to $n = 4$

Berenstein-Zelevinsky: SL_n using algebraic combinatorics in 1992

Grosshans used invariant-theoretic methods (of Grosshans-Rota-Stein) to compute the case for GL_4 in 1995

Howe: GL_4 in 1995

Howe-Tan: GL_5 in 2000

Example 1: Highest Weight Theory

$\mathfrak{R}(G/U)$: ring of regular functions on G which are invariant under left translations by U

Theory of Highest Weight:

$$\mathfrak{R}(G/U) = \sum_{\phi \in \widehat{A}^+} V_\phi,$$

i.e., $\mathfrak{R}(G/U)$ gives a **(full) model** of G, i.e., every irreducible finite representation of G appears exactly once.

Example 2: From $\mathbf{GL_n} \times \mathbf{GL_m}$ Duality

Recall that

$$\mathcal{P}(\mathbb{C}^{n,m}) = \sum_{r(D) \leq \min(n,m)} \rho_n^D \otimes \rho_m^D.$$

rank m partial model of GL_n :

$$\mathcal{P}(\mathbb{C}^{n,m})^{U_m} = \sum_{r(D) \leq \min(n,m)} \rho_n^D$$

is a partial model of GL_n if $m < n$, and a model of GL_n otherwise.

rank m$*$ model of GL_n :

$$\mathcal{P}(\mathbb{C}^{n*} \otimes \mathbb{C}^m)^{U_m} = \sum_{r(D) \leq \min(n,m)} \rho_n^{D*}$$

Remark: The tensor algebra of GL_n is the tensor algebra of SL_n with **TWO** extra generators representating the "determinant representation" of either factors.

Example: Clebsch-Gordon Formula

Consider a rank (1,1) tensor algebra of GL_n:

$$\left(\mathcal{P}(\mathbb{C}^{n,1})^{U_1} \otimes \mathcal{P}(\mathbb{C}^{n,1})^{U_1}\right)^{U_n}$$

Generators (easy):

$$\alpha_{10} = x_1, \qquad \alpha_{01} = y_1, \qquad \alpha_{11} = \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}.$$

There are no relations if $n \geq 2$.

(i) α_{10} has grading $(10; 00; 10)$,

(ii) α_{01} has grading $(00; 10; 10)$,

(iii) α_{11} has grading $(10; 10; 11)$,

Tensor Product of GL_n modules $[s] \otimes [t]$**:** Look for triplets of positive integers (a, b, c) such that

$$\alpha_{10}^a \alpha_{01}^b \alpha_{11}^c$$

is a GL_n highest weight vector.

The corresponding $(\widehat{A_n}^+)^3$ grading is $(a+c, 0; b+c, 0; a+b+c, c)$, and so we need to solve for

$$s = a + c \qquad \text{and} \qquad t = b + c.$$

Thus

$a = s - c$ and $b = t - c$ where $0 \leq c \leq \min(s, t)$,

i.e., $\alpha_{10}^{s-c}\alpha_{01}^{t-c}\alpha_{11}^{c}$ is a GL_n type of highest weight $(s+t-c, c)$ appearing in $[s] \otimes [t]$.

This is the well-known **Clebsch-Gordon Formula**:

$$< \alpha_{10}^{s} > \otimes < \alpha_{01}^{t} > = \sum_{c=0}^{\min(s,t)} < \alpha_{10}^{s-c}\alpha_{01}^{t-c}\alpha_{11}^{c} >$$

as GL_n modules.

4 Example: SL_3 Tensor Product Algebra

Consider a rank (2,2) tensor algebra of GL_n:

$$\left(\mathcal{P}(\mathbb{C}^{n,2})^{U_2} \otimes \mathcal{P}(\mathbb{C}^{n,2})^{U_2}\right)^{U_n}$$

Coordinates:

$$\begin{matrix} x_{11} & x_{12} & y_{11} & y_{12} \\ \vdots & \vdots & \vdots & \vdots \\ x_{n1} & x_{n2} & y_{n1} & y_{n2} \end{matrix}$$

Howe: Generators are:

$$\alpha_{10} = x_{11}, \qquad \alpha_{20} = \begin{vmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{vmatrix},$$

$$\alpha_{01} = y_{11}, \qquad \alpha_{02} = \begin{vmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{vmatrix},$$

$$\alpha_{11} = \begin{vmatrix} x_{11} & y_{11} \\ x_{21} & y_{21} \end{vmatrix}, \qquad \alpha_{12} = \begin{vmatrix} x_{11} & y_{11} & y_{12} \\ x_{21} & y_{21} & y_{22} \\ x_{31} & y_{31} & y_{32} \end{vmatrix},$$

$$\alpha_{21} = \begin{vmatrix} x_{11} & x_{12} & y_{11} \\ x_{21} & x_{22} & y_{21} \\ x_{31} & x_{32} & y_{31} \end{vmatrix}, \qquad \Delta = \begin{vmatrix} x_{11} & x_{12} & 0 & 0 \\ 0 & 0 & y_{11} & y_{12} \\ x_{21} & x_{22} & y_{21} & y_{22} \\ x_{31} & x_{32} & y_{31} & y_{32} \end{vmatrix}.$$

with one relation:

$$\Delta\alpha_{11} = \alpha_{01}\alpha_{20}\alpha_{12} + \alpha_{10}\alpha_{02}\alpha_{21}.$$

Will need these gradings later:

Triple gradings of each of the generators:

(i) α_{10} has grading $(100; 000; 100)$,

(ii) α_{20} has grading $(110; 000; 110)$,

(iii) α_{01} has grading $(000; 100; 100)$,

(iv) α_{02} has grading $(000; 110; 110)$,

(v) α_{11} has grading $(100; 100; 110)$,

(vi) α_{12} has grading $(100; 110; 111)$,

(vii) α_{21} has grading $(110; 100; 111)$,

(viii) Δ has grading $(110; 110; 211)$,

5 First Fundamental Theorem for $\mathbf{GL_n}$

Let

$$V_{s,t} = (\mathbb{C}^n \otimes \mathbb{C}^s) \oplus (\mathbb{C}^{n*} \otimes \mathbb{C}^t) = \mathbb{C}^{n,s} \oplus \mathbb{C}^{n,t}.$$

Select a system of coordinates on $V_{s,t}$ as follows:

$$\begin{matrix} x_{11} & \dots & x_{1s} & y_{11} & \dots & y_{1t} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x_{n1} & \dots & x_{ns} & y_{n1} & \dots & y_{nt} \end{matrix}$$

$(GL_n, GL_s \times GL_t)$ acts on $\mathcal{P}(V_{s,t})$ as follows:

$$(g, a, b)f(x, y) = f(g^t xa, g^{-1}yb)$$

Theorem 5.1 FFT for GL_n: *The GL_n invariants in $\mathcal{P}(V_{s,t})$ are generated by the inner products*

$$r_{ij} = x_{1i}y_{1j} + \ldots + x_{ni}y_{nj}.$$

Observe that

$$X_{ns}^t Y_{nt} = \begin{bmatrix} x_{11} & \dots & x_{1s} \\ \vdots & \vdots & \vdots \\ x_{n1} & \dots & x_{ns} \end{bmatrix}^t \begin{bmatrix} y_{11} & \dots & y_{1t} \\ \vdots & \vdots & \vdots \\ y_{n1} & \dots & y_{nt} \end{bmatrix}$$

$$= \begin{bmatrix} r_{11} & \dots & r_{1t} \\ \vdots & \vdots & \vdots \\ r_{s1} & \dots & r_{st} \end{bmatrix} = R_{st}.$$

6 Kostant-Rallis Decomposition for $\mathbf{GL_n}$

Define the second order differential operators dual to r_{ij}:

$$\Delta_{ij} = \frac{\partial^2}{\partial x_{1i}\partial y_{1j}} + \ldots + \frac{\partial^2}{\partial x_{ni}\partial y_{nj}}.$$

Define the space of harmonic polynomials:

$$\mathcal{H}(V_{s,t}) = \{f \in \mathcal{P}(V_{s,t}) \mid \Delta_{ij} f = 0 \ \forall \ \Delta_{ij}\}$$

Let

$$\delta_1 = x_{11}, \quad \delta_2 = \det[X_{22}], \quad \ldots, \quad \delta_k = \det[X_{kk}],$$

$$\eta_1 = y_{n1}, \quad \eta_2 = \det[\overline{Y}_{22}] = \begin{vmatrix} y_{n1} & y_{n-1,1} \\ y_{n2} & y_{n-1,2} \end{vmatrix}, \quad \ldots,$$

$$\eta_l = \det[\overline{Y}_{ll}] = \begin{vmatrix} y_{n1} & y_{n-1,1} & \cdots & y_{n-l+1,1} \\ y_{n2} & y_{n-1,2} & \cdots & y_{n-l+1,2} \\ \vdots & \vdots & \vdots & \vdots \\ y_{nl} & y_{n-1,l} & \cdots & y_{n-l+1,l} \end{vmatrix}.$$

Note: δ_k and η_l are of $GL_n \times GL_s \times GL_t$ highest weights:

$$(\underbrace{1, ..., 1}_{k\ copies}, 0, ..., 0) \otimes (\underbrace{1, ..., 1}_{k\ copies}, 0, ..., 0) \otimes (0, ..., 0),$$

and

$$(0, ..., 0, \overbrace{-1, ..., -1}^{l\ copies}) \otimes (0, ..., 0) \otimes (\overbrace{1, ..., 1}^{l\ copies}, 0, ..., 0)$$

Theorem 6.1 *The space $\mathcal{H}(V_{s,t})$ is multiplicity free as a $GL_n \times GL_s \times GL_t$ module, and generated by joint highest weights δ_k and η_l:*

$$\mathcal{H}(V_{s,t}) = \sum \rho_n^{(A;B)} \otimes \rho_s^A \otimes \rho_t^B$$

where the sum above runs over Young's diagrams A and B with $r(A) \leq s$, $r(B) \leq t$ and $r(A) + r(B) \leq n$.

If $A = (a_1, \ldots, a_s)$, $B = (b_1, \ldots, b_t)$ and $r(A) + r(B) \leq n$, then the highest weight of $\rho_n^{(A;B)} \otimes \rho_s^A \otimes \rho_t^B$ is (up to a constant)

$$\delta^A \eta^B = \delta_1^{a_1 - a_2} \delta_2^{a_2 - a_3} \ldots \delta_{s-1}^{a_{s-1} - a_s} \delta_s^{a_s} \eta_1^{b_1 - b_2} \eta_2^{b_2 - b_3} \ldots \eta_{t-1}^{b_{t-1} - b_t} \eta_t^{b_t}.$$

If $A = (a_1, \ldots, a_s)$ and $B = (b_1, \ldots, b_t)$, then

$$(A; B) = (a_1, \ldots, a_s, 0, \ldots, 0, -b_t, \ldots, -b_1).$$

Theorem 6.2 1. *In general, $\mathcal{P}(V_{s,t}) = \mathcal{H}(V_{s,t}) \cdot \mathcal{P}(V_{s,t})^{GL_n}$.*

2. *(**Kostant-Rallis Decomposition**) If $n \geq s + t$, then this is a tensor product:*

$$\mathcal{P}(V_{s,t}) = \mathcal{H}(V_{s,t}) \otimes \mathcal{P}(V_{s,t})^{GL_n}.$$

7 Using Dual Variables

Partial model of GL_n when we extract the $U_s \times U_t$ invariants:

$$\mathcal{H}(V_{s,t})^{U_s \times U_t} \simeq \sum \rho_n^{(A;B)}$$

Consider:

$$\mathcal{H}(V_{s,t})^{U_s \times U_t} \otimes \mathcal{H}(V_{\tilde{s},\tilde{t}})^{U_{\tilde{s}} \times U_{\tilde{t}}}$$

for appropriate (small) values of $s, t, \tilde{s}, \tilde{t}$

Extracting GL_n covariants:

$$\left(\mathcal{H}(V_{s,t})^{U_s \times U_t} \otimes \mathcal{H}(V_{\tilde{s},\tilde{t}})^{U_{\tilde{s}} \times U_{\tilde{t}}}\right)^{U_n}.$$

Note: $\mathcal{H}(V_{s,t})$ is not an algebra but it is possible to treat it as the quotient algebra $\mathcal{H}(V_{s,t}) = \mathcal{P}(V_{s,t})/\mathcal{I}(V_{s,t})$.

Thus

$$
\begin{aligned}
&\left(\mathcal{H}(V_{s,t})^{U_s\times U_t}\otimes\mathcal{H}(V_{\tilde{s},\tilde{t}})^{U_{\tilde{s}}\times U_{\tilde{t}}}\right)^{U_n}\\
&=\left(\left(\frac{\mathcal{P}(V_{s,t})}{\mathcal{I}(V_{s,t})}\right)^{U_s\times U_t}\otimes\left(\frac{\mathcal{P}(V_{\tilde{s},\tilde{t}})}{\mathcal{I}(V_{\tilde{s},\tilde{t}})}\right)^{U_{\tilde{s}}\times U_{\tilde{t}}}\right)^{U_n}\\
&=\left(\frac{\mathcal{P}(V_{s+\tilde{s},t+\tilde{t}})}{\mathcal{I}(V_{s,t})\cdot\mathcal{I}(V_{\tilde{s},\tilde{t}})}\right)^{U_s\times U_t\times U_{\tilde{s}}\times U_{\tilde{t}}\times U_n}\\
&=\left(\mathcal{H}(V_{s+\tilde{s},t+\tilde{t}})\otimes\frac{\mathcal{I}(V_{s+\tilde{s},t+\tilde{t}})}{\mathcal{I}(V_{s,t})\cdot\mathcal{I}(V_{\tilde{s},\tilde{t}})}\right)^{U_s\times U_t\times U_{\tilde{s}}\times U_{\tilde{t}}\times U_n}\\
&=\left(\mathcal{H}(V_{s+\tilde{s},t+\tilde{t}})\otimes\mathcal{P}(\mathbb{C}^{s,\tilde{t}}\oplus\mathbb{C}^{\tilde{s},t})\right)^{U_s\times U_t\times U_{\tilde{s}}\times U_{\tilde{t}}\times U_n}\\
&=\left(\left(\sum_{r(E)\le s+\tilde{s},\ r(F)\le t+\tilde{t}}\rho^E_{s+\tilde{s}}\otimes\rho^F_{t+\tilde{t}}\right)\otimes\mathcal{P}(\mathbb{C}^{s,\tilde{t}}\oplus\mathbb{C}^{\tilde{s},t})\right)^{U_s\times U_t\times U_{\tilde{s}}\times U_{\tilde{t}}}
\end{aligned}
$$

As a consequence, it is essential to understand the restriction problems:

$$
\left(\sum_{r(E)\le s+\tilde{s}}\rho^E_{s+\tilde{s}}\right)^{U_s\times U_{\tilde{s}}}
$$

and

$$
\left(\sum_{r(F)\le t+\tilde{t}}\rho^F_{t+\tilde{t}}\right)^{U_t\times U_{\tilde{t}}}.
$$

Some examples ...

8 Example: Restriction from $\mathbf{GL_{n+2}}$ to $\mathbf{GL_n} \times \mathbf{GL_2}$

Select coordinates for $\mathbb{C}^n \otimes \mathbb{C}^{n+2}$ as follows:

$$\begin{array}{ccccc} x_{11} & \dots & x_{1n} & y_{11} & y_{12} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x_{n1} & \dots & x_{nn} & y_{1n} & y_{2n} \end{array}$$

Generators of $\mathcal{P}(\mathbb{C}^n \otimes \mathbb{C}^{n+2})^{U_n \times (U_n \times U_2)}$:

$$\gamma_k^{(k,0)} = \det[X_{kk}], \qquad k = 1, ..., n,$$

$$\gamma_k^{(k-1,1)} = \det[X_{k,k-1} Y_{k1}], \qquad k = 1, ..., n,$$

$$\gamma_k^{(k-2,2)} = \det[X_{k,k-2} Y_{k2}], \qquad k = 1, ..., n,$$

$$\delta_{(k,l)} = \det \begin{bmatrix} X_{kk} & 0 & Y_{k2} \\ 0 & X_{l,l-1} & Y_{l2} \end{bmatrix}, \qquad k, l = 1, ..., n.$$

9 Example: SL_4 Tensor Product Algebra

Select a coordinate system $V_{4,2} = V_{2,1} \oplus V_{2,1}$ as follows:

$$\begin{matrix} x_{11} & x_{12} & y_{11} & \quad & x_{13} & x_{14} & y_{12} \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ x_{n1} & x_{n2} & y_{n1} & & x_{n3} & x_{n4} & y_{n2} \end{matrix}$$

GL_2^L, GL_2^R: two GL_2 acting on the two sets of (left and right) x-columns

U_L and U_R: unipotent subgroups of GL_2 acting on each of the copies of $\mathcal{H}(V_{2,1})$

Consider: For $n \geq 6$,

$$\begin{aligned} &\mathcal{H}(V_{2,1})^{U_L} \otimes \mathcal{H}(V_{2,1})^{U_R} \\ &= (\mathcal{P}(V_{2,1})/\mathcal{I}(r_{11}, r_{21}))^{U_L} \otimes (\mathcal{P}(V_{2,1})/\mathcal{I}(r_{32}, r_{42}))^{U_R} \\ &= (\mathcal{P}(V_{4,2})/\mathcal{I}(r_{11}, r_{21}, r_{32}, r_{42}))^{U_L \times U_R} \\ &= (\mathcal{H}(V_{4,2}) \otimes < r_{12}, r_{22}, r_{31}, r_{41} >)^{U_L \times U_R} . \end{aligned}$$

Consider GL_n covariants of $\mathcal{H}(V_{4,2})$:

$$\mathcal{H}(V_{4,2})^{U_n} = \sum \left(\rho_n^{(D,E)} \right)^{U_n} \otimes \rho_4^D \otimes \rho_2^E \simeq \sum \rho_4^D \otimes \rho_2^E$$

where the sum runs through all Young diagrams D, E with $r(D) \leq 4$ and $r(E) \leq 2$. This is basically the tensor of a (partial) model for GL_4 with a model of GL_2.

Need the $GL_2^L \times GL_2^R$ decomposition of $\sum_{r(D) \leq 4} \rho_4^D$:

This is a quotient of

$$\mathcal{P}((\sigma_L \otimes \mathbb{C}^2) \oplus (\sigma_R \otimes \mathbb{C}^2) \oplus (\sigma_L \otimes \sigma_R) \oplus \tilde{Z}).$$

Here:

$\sigma_L \simeq \mathbb{C}^2$: standard representation of GL_2^L.

$\sigma_R \simeq \mathbb{C}^2$: standard representation of GL_2^R.

Our objective is then to compute

$$\begin{aligned} &\left(\mathcal{H}(V_{2,1})^{U_L} \otimes \mathcal{H}(V_{2,1})^{U_R}\right)^{U_n} \\ &\simeq \left(\mathcal{H}(V_{4,2})^{U_n} \otimes < r_{12}, r_{22}, r_{31}, r_{41} >\right)^{U_L \times U_R} \\ &\simeq \left(\mathcal{H}(V_{4,2})^{U_n} \otimes \mathcal{P}(\sigma_L \oplus \sigma_R)\right)^{U_L \times U_R}, \end{aligned}$$

which is a subquotient of the following algebra:

$$\mathcal{P}((\sigma_L \otimes \mathbb{C}^3) \oplus (\sigma_R \otimes \mathbb{C}^3) \oplus (\sigma_L \otimes \sigma_R) \oplus Z)^{U_L \times U_R}$$

Result:

1. $n \geq 6$: 21 generators and 12 relations
2. $n = 4, 5$: 18 generators and 12 relations

SL$_5$ Tensor Product Algebra:

Need to compute the following binary covariants:

$$\begin{aligned} \mathcal{P}((\sigma_1 \otimes \mathbb{C}^2) \oplus (\sigma_2 \otimes \mathbb{C}^2) \oplus (\sigma_3 \otimes \mathbb{C}^2) \oplus (\sigma_4 \otimes \mathbb{C}^2) \oplus (\sigma_1 \otimes \sigma_3) \\ \oplus (\sigma_1 \otimes \sigma_4) \oplus (\sigma_2 \otimes \sigma_3) \oplus (\sigma_2 \otimes \sigma_4) \oplus Z)^{N_1 \times N_2 \times N_3 \times N_4}. \end{aligned}$$

Here N_i are unipotent subgroups of SL_2.

10 An Inductive Process

Scheme: Exploit two ways to look at

$$\mathcal{P}(V_{s,t})^{U_n \times U_s \times U_t}$$

For $n \geq s+t$,

$$\mathcal{P}(V_{s,t})^{U_n \times U_s \times U_t}$$

$$= \big(\mathcal{H}(V_{s,t}) \otimes \mathcal{P}(V_{s,t})^{GL_n}\big)^{U_n \times U_s \times U_t}$$

$$= \Big(\Big(\sum \mathcal{H}^{(A;B)}(V_{s,t})^{U_n}\Big) \otimes \mathcal{P}(\mathbb{C}^{s,t})\Big)^{U_s \times U_t}$$

$$= \left(\left(\sum_{r(A)\leq s,\ r(B)\leq t} \rho_s^A \otimes \rho_t^B\right) \otimes \left(\sum_{r(C)\leq s,t} \rho_s^C \otimes \rho_t^C\right)\right)^{U_s \times U_t}$$

$$= \Big(\Big(\sum \rho_s^A\Big) \otimes \Big(\sum \rho_s^C\Big)\Big)^{U_s} \otimes \Big(\Big(\sum \rho_t^B\Big) \otimes \Big(\sum \rho_t^C\Big)\Big)^{U_t}$$

This is the tensor product of a rank $(s, \min(s,t))$ tensor algebra of GL_s with another rank $(t, \min(s,t))$ tensor algebra of GL_t, with an **identification**.

On the other hand,

$$\mathcal{P}(V_{s,t})^{U_n\times U_s\times U_t}$$

$$= \big(\mathcal{P}(\mathbb{C}^n\otimes\mathbb{C}^s)\otimes\mathcal{P}(\mathbb{C}^{n*}\otimes\mathbb{C}^t)\big)^{U_n\times U_s\times U_t}$$

$$= \left(\left(\sum_{r(E)\leq s}\rho_n^E\otimes\rho_s^E\right)^{U_s}\otimes\left(\sum_{r(F)\leq t}\rho_n^{F*}\otimes\rho_t^F\right)^{U_t}\right)^{U_n}$$

$$= \left(\left(\sum_{r(E)\leq s}\rho_n^E\right)\otimes\left(\sum_{r(F)\leq t}\rho_n^{F*}\right)\right)^{U_n}.$$

Thus this is a rank $(s,t*)$ tensor algebra of GL_n.

By appropriate choice of of s, t and n, we have an inductive process of computing the tensor algebra of GL_n.

11 Example: Re-deriving The SL_3 Tensor Product Algebra

Now set we set $s = t = 2$ and $n \geq 4$ in the above, we have:

$$\left(\left(\sum_{r(E)\leq 2} \rho_n^E\right) \otimes \left(\sum_{r(F)\leq 2} \rho_n^{F*}\right)\right)^{U_n}$$
$$= \left(\left(\sum \rho_2^A\right) \otimes \left(\sum \rho_2^C\right)\right)^{U_2} \otimes \left(\left(\sum \rho_2^B\right) \otimes \left(\sum \rho_2^C\right)\right)^{U_2}.$$

RHS is the tensor product of two GL_2 tensor product algebras, identified so the seconding grading is the same.

Recall GL_2 tensor product algebra:

$$\alpha_{10} = x_{11}, \qquad \alpha_{20} = \begin{vmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{vmatrix},$$

$$\alpha_{01} = y_{11}, \qquad \alpha_{02} = \begin{vmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{vmatrix},$$

$$\alpha_{11} = \begin{vmatrix} x_{11} & y_{11} \\ x_{21} & y_{21} \end{vmatrix}.$$

Observe that they have the following grading:

(i) α_{10} has grading $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$,

(ii) α_{20} has grading $\begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$,

(iii) α_{01} has grading $\begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$,

(iv) α_{02} has grading $\begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$,

(v) α_{11} has grading $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$,

Let the generators of the second GL_2 tensor algebra be

$$\tilde{\alpha}_{10}, \quad \tilde{\alpha}_{20}, \quad \tilde{\alpha}_{01}, \quad \tilde{\alpha}_{02}, \quad \tilde{\alpha}_{11}.$$

Thus, the rank (2,2*) GL_3 tensor algebra is generated by the following 9 generators:

(i) $\alpha_{10} \otimes \mathbb{1}$, $\alpha_{20} \otimes \mathbb{1}$, $\mathbb{1} \otimes \tilde{\alpha}_{10}$, $\mathbb{1} \otimes \tilde{\alpha}_{20}$,

(ii) $\alpha_{01} \otimes \tilde{\alpha}_{01}$, $\alpha_{01} \otimes \tilde{\alpha}_{11}$, $\alpha_{11} \otimes \tilde{\alpha}_{01}$, $\alpha_{11} \otimes \tilde{\alpha}_{11}$,

(iii) $\alpha_{02} \otimes \tilde{\alpha}_{02}$,

with one relation:

$$[\alpha_{01} \otimes \tilde{\alpha}_{11}][\alpha_{11} \otimes \tilde{\alpha}_{01}] = [\alpha_{01} \otimes \tilde{\alpha}_{01}][\alpha_{11} \otimes \tilde{\alpha}_{11}].$$

12 Example: $\mathbf{O_n \times GL_3}$ Highest Weights in $\mathcal{P}(\mathbb{C}^{\mathbf{n,3}})$

Similar scheme of looking at

$$\mathcal{P}(\mathbb{C}^{n,3})^{U_{O_n} \times U_{GL_3}} = \left(\mathcal{H}(\mathbb{C}^{n,3})^{U_{O_n}} \otimes \mathcal{P}(\mathbb{C}^{n,3})^{O_n}\right)^{U_3}$$

......

Recall the SL_3 triple gradings of the 8 generators in the tensor algebra of SL_3:

(1) α_{10} has grading $(100; 000; 100)$,

(2) α_{20} has grading $(110; 000; 110)$,

(3) α_{01} has grading $(000; 100; 100)$,

(4) α_{02} has grading $(000; 110; 110)$,

(5) α_{11} has grading $(100; 100; 110)$,

(6) α_{12} has grading $(100; 110; 111)$,

(7) α_{21} has grading $(110; 100; 111)$,

(8) Δ has grading $(110; 110; 211)$.

Boils down to: Looking at tensor products where the second copy comes from **even** diagrams ... we can write down the 14 generators of $\mathcal{P}(\mathbb{C}^{n,3})^{U_{O_n} \times U_{GL_3}}$ as follows:

(a) α_{10} and α_{20},

(b) Choose any product of two elements from $\{\alpha_{01}, \alpha_{11}, \alpha_{21}\}$, i.e., these are $[\alpha_{01}^2]$, $[\alpha_{11}^2]$, $[\alpha_{21}^2]$, $[\alpha_{01}\alpha_{11}]$, $[\alpha_{01}\alpha_{21}]$, $[\alpha_{11}\alpha_{21}]$,

(b) Choose any product of two elements from $\{\alpha_{02}, \alpha_{12}, \Delta\}$, i.e., these are $[\alpha_{02}^2]$, $[\alpha_{12}^2]$, $[\Delta^2]$, $[\alpha_{02}\alpha_{12}]$, $[\alpha_{02}\Delta]$, $[\alpha_{12}\Delta]$.

General Procedure to write down relations: Total of 21 relations:

(a) **Three** relations come from multiplying two different elements in $\{[\alpha_{01}^2], [\alpha_{11}^2], [\alpha_{21}^2]\}$:

(a)(1) $[\alpha_{01}^2][\alpha_{11}^2] = [\alpha_{01}\alpha_{11}]^2$,

(a)(2) $[\alpha_{01}^2][\alpha_{21}^2] = [\alpha_{01}\alpha_{21}]^2$,

(a)(3) $[\alpha_{11}^2][\alpha_{21}^2] = [\alpha_{11}\alpha_{21}]^2$,

(b) **Three** relations come from multiplying two different elements in $\{[\alpha_{01}\alpha_{11}], [\alpha_{01}\alpha_{21}], [\alpha_{11}\alpha_{21}]\}$:

(b)(1) $[\alpha_{01}\alpha_{11}][\alpha_{01}\alpha_{21}] = [\alpha_{01}^2][\alpha_{11}\alpha_{21}]$,

(b)(2) $[\alpha_{01}\alpha_{11}][\alpha_{11}\alpha_{21}] = [\alpha_{11}^2][\alpha_{01}\alpha_{21}]$,

(b)(3) $[\alpha_{01}\alpha_{21}][\alpha_{11}\alpha_{21}] = [\alpha_{21}^2][\alpha_{01}\alpha_{11}]$,

(c) **Three** relations come from multiplying two different elements in $\{[\alpha_{02}^2], [\alpha_{12}^2], [\Delta^2]\}$:

(c)(1) $[\alpha_{02}^2][\alpha_{12}^2] = [\alpha_{02}\alpha_{12}]^2$,

(c)(2) $[\alpha_{02}^2][\Delta^2] = [\alpha_{02}\Delta]^2$,

(c)(3) $[\alpha_{12}^2][\Delta^2] = [\alpha_{12}\Delta]^2$,

(d) **Three** relations come from multiplying two different elements in $\{[\alpha_{02}\alpha_{12}], [\alpha_{02}\Delta], [\alpha_{12}\Delta]\}$:

(d)(1) $[\alpha_{02}\alpha_{12}][\alpha_{02}\Delta] = [\alpha_{02}^2][\alpha_{12}\Delta]$,

(d)(2) $[\alpha_{02}\alpha_{12}][\alpha_{12}\Delta] = [\alpha_{12}^2][\alpha_{02}\Delta]$,

(d)(3) $[\alpha_{02}\Delta][\alpha_{12}\Delta] = [\Delta^2][\alpha_{02}\alpha_{12}]$,

(e) **Nine** relations come from multiplying by an element in $\{\alpha_{01}, \alpha_{11}, \alpha_{21}\}$ and another element from $\{\alpha_{02}, \alpha_{12}, \Delta\}$ to the relation

$$\Delta\alpha_{11} = \alpha_{01}\alpha_{20}\alpha_{12} + \alpha_{10}\alpha_{02}\alpha_{21}.$$

For instance,

(e)(1) Multliplying by $\alpha_{01}\alpha_{02}$ to give

$$[\alpha_{01}\alpha_{11}][\alpha_{02}\Delta] = [\alpha_{01}^2]\alpha_{20}[\alpha_{02}\alpha_{12}]+\alpha_{10}[\alpha_{02}^2][\alpha_{01}\alpha_{21}],$$

(e)(2) Multliplying by $\alpha_{01}\alpha_{12}$ to give

$$[\alpha_{01}\alpha_{11}][\alpha_{12}\Delta] = [\alpha_{01}^2]\alpha_{20}[\alpha_{12}^2]+\alpha_{10}[\alpha_{02}\alpha_{12}][\alpha_{01}\alpha_{21}],$$

Compare and contrast with the results from [ATZ]:

Theorem 12.1 *Assume $n \geq 6$. The algebra of $O_n \times GL_3$ highest weight vectors in $\mathcal{P}(\mathbb{C}^{n,3})$ is generated by the following 15 $O_n \times GL_3$ highest weight vectors:*

$$\alpha_i = \det[Z_{ii}], \quad i = 1, 2, 3,$$
$$\gamma_i = \det[R_{ii}], \quad i = 1, 2, 3,$$

$$\beta_1 = \begin{vmatrix} z_{11} & z_{12} \\ r_{11} & r_{12} \end{vmatrix}, \quad \beta_2 = \begin{vmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \\ r_{11} & r_{12} & r_{13} \end{vmatrix},$$

$$\beta_3 = \begin{vmatrix} z_{11} & z_{12} & z_{13} \\ r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \end{vmatrix}, \quad \beta_4 = \begin{vmatrix} 0 & z_{11} & z_{12} \\ z_{11} & r_{11} & r_{12} \\ z_{12} & r_{21} & r_{22} \end{vmatrix},$$

$$\beta_5 = \begin{vmatrix} 0 & z_{11} & z_{12} & z_{13} \\ z_{11} & r_{11} & r_{12} & r_{13} \\ z_{12} & r_{21} & r_{22} & r_{23} \\ z_{13} & r_{31} & r_{32} & r_{33} \end{vmatrix},$$

$$\beta_6 = \begin{vmatrix} 0 & z_{11} & z_{12} & z_{13} \\ 0 & z_{21} & z_{22} & z_{23} \\ z_{11} & r_{11} & r_{12} & r_{13} \\ z_{21} & r_{21} & r_{22} & r_{23} \end{vmatrix}, \quad \beta_7 = \begin{vmatrix} z_{11} & 0 & z_{12} & z_{13} \\ z_{21} & 0 & z_{22} & z_{23} \\ 0 & r_{11} & r_{12} & r_{13} \\ 0 & r_{21} & r_{22} & r_{23} \end{vmatrix},$$

$$\beta_8 = \begin{vmatrix} 0 & 0 & z_{11} & z_{12} & z_{13} \\ 0 & 0 & z_{21} & z_{22} & z_{23} \\ z_{11} & z_{21} & r_{11} & r_{12} & r_{13} \\ z_{12} & z_{22} & r_{21} & r_{22} & r_{23} \\ z_{13} & z_{23} & r_{31} & r_{32} & r_{33} \end{vmatrix}, \quad \beta_9 = \begin{vmatrix} z_{11} & z_{12} & 0 & 0 & 0 \\ 0 & 0 & z_{11} & z_{12} & z_{13} \\ z_{11} & z_{21} & r_{11} & r_{12} & r_{13} \\ z_{12} & z_{22} & r_{21} & r_{22} & r_{23} \\ z_{13} & z_{23} & r_{31} & r_{32} & r_{33} \end{vmatrix}.$$

1. General Setting of Flag Manifolds and Examples
   (a) Covariants of One Subspace
   (b) Spherical Double Cones
   (c) Multiplicity Free Actions on Flag Manifolds
   (d) Covariants of Two Subspaces
   (e) GL_n Covariants of $\mathcal{G}r_k^n \times \mathcal{MF}_{\{1,m\}}$
   (f) Invariants of Four Subspaces
2. Some Applications
   (a) Branching Rules
   (b) Analysis of Degenerate Principal Series
   (c) A Cute Example: Quantum Qubits

13 Quantum Q-bits

Interested in studying the action of $(SU_2)^k$ acting on the k-fold tensor $(\mathbb{C}^2)^{\otimes k}$.

Question: What are the real orbits? In other words, what are the real polynomial invariants?

Complexify:

$$\Delta(SL_2^k) \hookrightarrow SL_2^k \times SL_2^k$$

acting on

$$(\mathbb{C}^2)^{\otimes k} \oplus (\mathbb{C}^2)^{\otimes k}$$

Understand the SL_2^k covariants:

$$\mathcal{P}((\mathbb{C}^2)^{\otimes k})^{U_2^k}$$

and then resolve the invariants:

$$\mathcal{P}((\mathbb{C}^2)^{\otimes k} \oplus (\mathbb{C}^2)^{\otimes k})^{SL_2^k}$$

(a) $k = 1$: $\mathcal{P}(\mathbb{C}^2)^{U_2} \simeq \mathbb{C}[x]$.

(b) $k = 2$: Use spherical harmonics because $Spin_4 \simeq SL_2 \times SL_2$ and

$$\mathcal{P}(\mathbb{C}^2 \otimes \mathbb{C}^2)^{U_2 \times U_2} \simeq \mathcal{P}(\mathbb{C}^4)^{U_{SO_4}}$$

(c) $k = 3$ (Meyer-Wallach): Use the dual pair (SO_4, Sp_4) and

$$\mathcal{P}(\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2)^{U_2 \times U_2 \times U_2} \simeq \mathcal{P}(\mathbb{C}^4 \otimes \mathbb{C}^2)^{U_{SO_4} \times U_2}$$

(d) $k = 4$ (Howe): **314 generators!**

$$\mathcal{P}((\mathbb{C}^2 \otimes \mathbb{C}^2) \otimes (\mathbb{C}^2 \otimes \mathbb{C}^2))^{(U_2 \times U_2) \times (U_2 \times U_2)} \simeq \mathcal{P}(\mathbb{C}^4 \otimes \mathbb{C}^4)^{U_{SO_4} \times U_{SO_4}}$$

General Problem: Describe $\mathcal{P}(\mathbb{C}^n \otimes \mathbb{C}^m)^{U_{SO_n} \times U_{SO_m}}$.

Resolve the $U_{SO_n} \times U_{GL_m}$ invariants

$$\mathcal{P}(\mathbb{C}^{n,m})^{U_{O_n} \times U_{GL_m}} = \left(\mathcal{H}(\mathbb{C}^{n,m})^{U_{O_n}} \otimes \mathcal{P}(\mathbb{C}^{n,m})^{O_n}\right)^{U_{GL_m}}$$

and then compute the restriction algebra .....