LECTURES ON HARMONIC ANALYSIS FOR REDUCTIVE p-ADIC GROUPS

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1. INTRODUCTION

In his paper *The characters of reductive p-adic groups* [14] Harish-Chandra outlines his philosophy about harmonic analysis on reductive p-adic groups. According to this philosophy, there are two distinguished classes of distributions on the group: orbital integrals and characters. Similarly, there are two classes of distributions on the Lie algebra which are interesting: orbital integrals and their Fourier transforms. The real meat of his philosophy states that we ought to treat orbital integrals on both the group and its Lie algebra in the "same way" and similarly, we should think of characters and the Fourier transform of orbital integrals in the same way. This philosophy has many manifestations (see, for example, Robert Kottwitz's excellent article [12]). In this series of lectures, we will examine the various distributions discussed above and discuss one of the deepest connections between them: the Harish-Chandra–Howe local character expansion. Because of requests on the part of participants, I will spend much time reviewing the basics of p-adic fields and discussing some of the uses of Moy-Prasad filtrations in the representation theory of reductive p-adic groups. As such, at a small cost in terms of generality, I will concentrate on those techniques which use this theory. These lectures and the notes are meant as an informal and elementary introduction to the material. For complete, rigorous proofs, please see the references.

Very little of the material in this set of lectures is original. I have borrowed heavily from the work and lectures of Harish-Chandra, Roger Howe, Robert Kottwitz, Allen Moy, Gopal Prasad, Paul Sally, Jr., and J.-L. Waldspurger, among others. I thank the organizers of this conference, in particular, Eng-Chye Tan and Chen-Bo Zhu, for inviting me and allowing me to present this series of tutorials.

2. BASICS

2.1. **An introduction to the p-adics.** The usual introductory mathematical analysis course proceeds roughly as follows: The class agrees to agree that the set of natural numbers

$$\{1, 2, 3, \ldots\}$$

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is very natural and therefore a good place to begin the course. In order to form a group with respect to addition, the additive identity and additive inverses are tossed in to the mix to give us the integers

$$\mathbb{Z} := \{\ldots, -4, -3, -2, -1, 0, 1, 2, 3, \ldots\}.$$

This set does not form a group with respect to multiplication; it is therefore enlarged to form $\mathbb{Q}$, the field of rational numbers. Everything so far has been very natural.

At this point, the incompleteness of the rationals is demonstrated by proving that the square root of two is not rational. To compensate for this, the fact that the rationals are ordered (that is, there is a notion of nonpositive and nonnegative) is invoked to define the absolute value, $|\cdot|$, of any rational number q:

$$|q| = \begin{cases} q & \text{if } q \geq 0 \\ -q & \text{if } q < 0. \end{cases}$$

As usual, the absolute value gives you a metric with respect to which you complete the rationals. We recall that two Cauchy sequences $\{q_n\}$ and $\{q'_n\}$ of rational numbers are said to be equivalent with respect to $|\cdot|$ provided that $\{q_n - q'_n\}$ is a Cauchy sequence which converges to zero. The set of real numbers is then defined to be the set of equivalence classes of Cauchy sequences with respect to $|\cdot|$, that is, the completion of $\mathbb{Q}$ with respect to $|\cdot|$.

It turns out, however, that the normal absolute value provides just one of an infinite number of incompatible ways to complete the rational numbers[1], and, from the proper perspective, this completion is a somewhat unnatural object.

Let p be any prime. If q is a nonzero rational number, then there is a unique integer k such that $q = p^k \cdot a/b$ with $p \nmid a$ and $p \nmid b$. We can then define the *p-adic absolute value*, $|\cdot|_p$, on $\mathbb{Q}$ by setting $|q|_p = 0$ if $q = 0$ and $|q|_p = p^{-k}$ otherwise. The p-adic absolute value has the following properties.

Exercise 2.1.1. If r_1 and r_2 are rational numbers, then

(1) $|r_1|_p \geq 0$, and $|r_1|_p = 0$ if and only if $r_1 = 0$,
(2) $|r_1 \cdot r_2|_p = |r_1|_p \cdot |r_2|_p$, and
(3) $|r_1 + r_2|_p \leq \max(|r_1|_p, |r_2|_p)$.

Exercise 2.1.2. In the last item of the previous exercise, show that if $|r_1|_p \neq |r_2|_p$ then the inequality is an equality. Is the converse of this statement true?

From Exercise 2.1.1, it follows that we can define a metric on $\mathbb{Q}$ with respect to the p-adic absolute value. We define $\mathbb{Q}_p$ to be the completion of $\mathbb{Q}$ with respect to $|\cdot|_p$. The p-adic absolute value on $\mathbb{Q}$ extends continuously (and uniquely) to a p-adic absolute value $|\cdot|_p : \mathbb{Q}_p \to \{0, p^k \mid k \in \mathbb{Z}\}$. We define the *valuation* ν_p on $\mathbb{Q}_p$ by $|x|_p = p^{-\nu_p(x)}$ for $x \in \mathbb{Q}_p^\times$ and $\nu_p(0) = \infty$. Thus $\nu_p(p^m) = m$ for $m \in \mathbb{Z}$.

[1] The field of real numbers is, up to isomorphism, the unique totally ordered completion of the rationals.

We remark that Ostrowski's theorem tells us that the only nondiscrete, locally compact fields of characteristic zero are the field of real numbers, the field of complex numbers, and the finite extensions of $\mathbb{Q}_p$. That is, these are the only fields of characteristic zero on which integration naturally makes sense.

2.2. **The structure of $\mathbb{Q}_p$ and additive characters.** If we define $\mathbb{Z}_p$ to be the set of $x \in \mathbb{Q}_p$ such that $|x|_p \leq 1$, then $\mathbb{Z}_p$ is a maximal compact open subring in $\mathbb{Q}_p$. As in [33], it can be identified with the completion of $\mathbb{Z}$ with respect to $|\cdot|_p$. We call $\mathbb{Z}_p$ the *ring of integers* in $\mathbb{Q}_p$. The subset of $\mathbb{Z}_p$ consisting of those elements of $\mathbb{Z}_p$ with p-adic absolute value less than one forms a maximal ideal $(p) = p\mathbb{Z}_p$ of $\mathbb{Z}_p$. The quotient $\mathbb{Z}_p/(p)$ is isomorphic to $\mathbb{F}_p$, the field with p elements. It follows that we can write

$$\mathbb{Z}_p = \coprod_{0 \leq i \leq (p-1)} (i + p\mathbb{Z}_p).$$

As a slight tangent, we remark that Hensel's lemma (see nearly any book on local fields) tells us that we can lift a generator of $\mathbb{F}_p^\times$ to $\mathbb{Z}_p^\times$, the group of units in $\mathbb{Z}_p$. Thus, the square root of two is an element of, for example, $\mathbb{Q}_7$.

The set of compact open subrings $p^n\mathbb{Z}_p$ for $n \geq 0$ forms a neighborhood basis of zero. The sets $p^n\mathbb{Z}_p$ also make sense for $n < 0$, and we note that

$$\mathbb{Q}_p = \bigcup_{n \in \mathbb{Z}} p^n\mathbb{Z}_p.$$

Since $\mathbb{Z}_p^\times = \mathbb{Z}_p \smallsetminus p\mathbb{Z}_p$, it follows immediately that

$$\mathbb{Q}_p = \{0\} \cup \coprod_{n \in \mathbb{Z}} p^n\mathbb{Z}_p^\times. \tag{1}$$

From this one can show that for each $x \in \mathbb{Q}_p^\times$ if $|x|_p = p^{-m}$, then there exist unique coefficients $a_n \in \{0, 1, 2, \ldots (p-1)\}$ such that

$$x = \sum_{n=m}^{\infty} a_n \cdot p^n.$$

This is called the *p-adic expansion* of x. For example, the 5-adic expansion of $1/3$ is

$$2 + 3 \cdot (5) + 1 \cdot (5^2) + 3 \cdot (5^3) + 1 \cdot (5^4) + 3 \cdot (5^5) + \cdots .$$

If $x \in \mathbb{Q}_p$ and $|x|_p = p^{-m}$, then we define the *tail* of x to be the integer

$$t(x) = \begin{cases} \sum_{n=m}^{0} a_n \cdot p^n & \text{if } |x|_p = p^{-m} \text{ and } m \leq 0, \text{ and} \\ 0 & \text{if } x \in p\mathbb{Z}_p . \end{cases}$$

We then define

$$\Lambda(x) = e^{\frac{2 \cdot \pi \cdot i \cdot t(x)}{p}}.$$

Exercise 2.2.1. Show that Λ defines an additive character of $\mathbb{Q}_p$ and that the restriction to $p\mathbb{Z}_p$ of Λ is trivial, yet the restriction of Λ to $\mathbb{Z}_p$ is not trivial.

It is a fact that $\mathbb{Q}_p$ is isomorphic to the Pontrjagin dual $\widehat{\mathbb{Q}_p}$ of $\mathbb{Q}_p$ via the map $a \mapsto \big(b \mapsto \Lambda(ba)\big)$.

2.3. **General notation.** From now on, let k denote a finite extension of $\mathbb{Q}_p$, let R denote the ring of integers of k, and let ϖ denote a *uniformizer* so that $\wp = \varpi R$ where $\wp$ is the prime ideal. We define $\wp^m := \varpi^m \cdot R$ for $m \in \mathbb{Z}$. We let $\mathfrak{f} := R/\wp$ denote the *residue field*, and we let $q = |\mathfrak{f}|$. As before, we fix a valuation ν such that $\nu(k) = \mathbb{Z}$. We fix an additive character Λ of k that is trivial on $\wp$ and not trivial on R.

We let G denote the general linear group realized as the set of $n \times n$ matrices with entries in k having nonzero determinant. We let $\mathfrak{g}$ denote the Lie algebra of G. We realize $\mathfrak{g}$ as $\mathrm{M}_n(k)$, the set of $n \times n$ matrices with entries in the field k, having the usual bracket operation. Let dX denote a Haar measure on $\mathfrak{g}$ and let dg denote a Haar measure on G.

We let B denote the Borel subgroup of G consisting of upper triangular matrices, and we let T denote the maximal split torus consisting of diagonal matrices.

The map $(X, Y) \mapsto \mathrm{tr}(X \cdot Y)$ from $\mathfrak{g} \times \mathfrak{g}$ to k defines a G-invariant, nondegenerate, symmetric, bilinear form on $\mathfrak{g}$. As such, it allows us to identify $\mathfrak{g}$ with the Pontrjagin dual $\widehat{\mathfrak{g}}$ of $\mathfrak{g}$ via the map $X \mapsto \big(Y \mapsto \Lambda(\mathrm{tr}(X \cdot Y))\big)$.

A compact, open, R-submodule of $\mathfrak{g}$ is called a *lattice*. For example, for each integer i, we can define the *standard filtration lattice*

$$\mathfrak{k}_i := \varpi^i \cdot \mathrm{M}_n(R)$$

of $\mathfrak{g}$. As another example, consider the *Iwahori filtration lattices* $\mathfrak{b}_{i/n}$ defined by

$$\mathfrak{b}_{i/n} := \{Y \in \mathfrak{g} \,|\, Y_{jk} \in \varpi^{\lceil \frac{j-k+i}{n} \rceil} \cdot R\}.$$

Note that for all integers i, j, we have $\varpi^j \cdot \mathfrak{k}_i = \mathfrak{k}_{i+j}$ and $\varpi^j \cdot \mathfrak{b}_{i/n} = \mathfrak{b}_{j+\frac{i}{n}}$.

More concretely, for $n = 2$ we have

$$\mathfrak{k}_1 = \begin{pmatrix} \wp & \wp \\ \wp & \wp \end{pmatrix}$$

and

$$\mathfrak{b}_0 = \begin{pmatrix} R & R \\ \wp & R \end{pmatrix} \supset \mathfrak{b}_{1/2} = \begin{pmatrix} \wp & R \\ \wp & \wp \end{pmatrix} \supset \mathfrak{b}_1 = \varpi \cdot \mathfrak{b}_0.$$

For $n = 3$ we have

$$\mathfrak{b}_0 = \begin{pmatrix} R & R & R \\ \wp & R & R \\ \wp & \wp & R \end{pmatrix} \supset \mathfrak{b}_{1/3} = \begin{pmatrix} \wp & R & R \\ \wp & \wp & R \\ \wp & \wp & \wp \end{pmatrix} \supset \mathfrak{b}_{2/3} = \begin{pmatrix} \wp & \wp & R \\ \wp & \wp & \wp \\ \varpi^2 \cdot R & \wp & \wp \end{pmatrix} \supset \mathfrak{b}_1 = \varpi \cdot \mathfrak{b}_0.$$

From these examples, it is easy to see that the image of $\mathfrak{b}_0$ in $\mathfrak{k}_0/\mathfrak{k}_1 = \mathrm{M}_n(\mathfrak{f})$ is a Borel subalgebra with nilradical equal to the image of $\mathfrak{b}_{1/n}$ in $\mathfrak{k}_0/\mathfrak{k}_1 = \mathrm{M}_n(\mathfrak{f})$. (We remark that in all of these examples, the lattice in $\mathfrak{g}$ is a direct sum of copies of lattices of k.)

We have similar filtrations of G by compact open subgroups. The *standard filtration subgroups* are defined by $K_0 = \mathrm{GL}_n(R)$ and $K_i = 1 + \mathfrak{k}_i$ for positive integers i. We note that, up to conjugacy, K_0 is the unique maximal compact open subgroup of G. We also define the Iwahori subgroup $B_0 = \mathfrak{b}_0^\times$ and, for positive integers i, we define the *Iwahori filtration subgroups* $B_{i/n} = 1 + \mathfrak{b}_{i/n}$. It is easy to check that if $0 \leq k \leq j$, then K_j is a normal subgroup of K_k and the lattices $\mathfrak{k}_m$ ($m \in \mathbb{Z}$) are invariant under the action of K_j. Similar statements apply to the Iwahori filtration lattices and subgroups.

If $0 < j \leq k \leq 2j$, then $\mathfrak{k}_j/\mathfrak{k}_k$ is an abelian group which is isomorphic via the map induced by $X \mapsto (1 + X)$ to K_j/K_k. Similarly, $\mathfrak{b}_{j/n}/\mathfrak{b}_{k/n}$ is isomorphic to $B_{j/n}/B_{k/n}$.

Given a lattice L in $\mathfrak{g}$ we define the dual lattice L^* of L by

$$L^* = \{Y \in \mathfrak{g} \mid \operatorname{tr}(X \cdot Y) \in \wp \text{ for all } X \in L\}.$$

For example, for any integer i, the dual lattice of $\mathfrak{k}_i$ is $\mathfrak{k}_{(1-i)}$, and the dual lattice of $\mathfrak{b}_{i/n}$ is $\mathfrak{b}_{(1-i)/n}$. We let $C_c^\infty(\mathfrak{g})$ denote the set of functions on $\mathfrak{g}$ which are compactly supported, complex valued, and locally constant. We recall that a function f on $\mathfrak{g}$ is said to be *locally constant* if for each $X \in \mathfrak{g}$ there exists a lattice L_X such that $f(X+\ell) = f(X)$ for all ℓ in L_X. If we assume that our f is compactly supported, then this lattice can be chosen uniformly. We let $C_c^\infty(G)$ denote the functions on G which are compactly supported, complex-valued, and locally constant (that is, translation invariant with respect to some compact open subgroup of G).

For $f \in C_c^\infty(\mathfrak{g})$ we define $\hat{f}$, the *Fourier transform* of f, by the formula

$$\hat{f}(X) = \int_{\mathfrak{g}} f(Y) \cdot \Lambda(\operatorname{tr}(X \cdot Y))\, dY$$

for X in $\mathfrak{g}$. For example, if $[L]$ denotes the characteristic function of a lattice L in $\mathfrak{g}$, then $\widehat{[L]}$ is $[L^*]$ up to a constant. We normalize the measure dX on $\mathfrak{g}$ so that $\hat{\hat{f}}(X) = f(-X)$ for all $f \in C_c^\infty(\mathfrak{g})$ and $X \in \mathfrak{g}$.

Exercise 2.3.1. Show that this normalization of measures implies that for L sufficiently small, the measure of the lattice L with respect to dX is given by one over the square root of the index of L in L^*, that is,

$$\operatorname{meas}_{dX}(L) = [L^* : L]^{-1/2}.$$

Exercise 2.3.2. Show that the map $f \mapsto \hat{f}$ from $C_c^\infty(\mathfrak{g})$ to itself is a bijection.

Finally, we normalize dg so that the map $X \mapsto (1+X)$ takes dX into dg. So, for example, we have that for all $j \in \mathbb{Z}_{>0}$,

$$\operatorname{meas}_{dX}(\mathfrak{k}_j) = \operatorname{meas}_{dg}(K_j).$$

3. MOY-PRASAD FILTRATIONS

In this section we describe the Moy-Prasad filtration lattices of $\mathfrak{g}$ and subgroups of G.

3.1. **The apartment of T.** Recall that T is the group consisting of diagonal matrices in G. We write $t \in T$ as $t = (t_1, t_2, \ldots, t_n)$ with $t_j \in k^\times$ for $1 \leq j \leq n$. Let $\Phi \subset \mathbf{X}^*(T) = \operatorname{Hom}(T, k^\times)$ denote the set of roots of G with respect to T (that is, the nontrivial eigencharacters for the action of T on $\mathfrak{g}$). More explicitly

$$\Phi = \{\alpha_{ij} \mid 1 \leq i \neq j \leq n \text{ and } \alpha_{ij}(t) = t_i/t_j\}.$$

With respect to our Borel subgroup B we let Δ denote the set of simple roots; that is, $\Delta := \{\alpha_{i(i+1)} \mid 1 \leq i \leq (n-1)\}$. We let Φ^+ denote the set of positive roots in Φ. We fix a Chevalley basis

$$\{Z, H_\beta, X_\gamma \mid \beta \in \Delta \text{ and } \gamma \in \Phi\}.$$

Here

$$(Z)_{kl} = \delta_{kl}$$

and

$$(H_{\alpha_{ij}})_{kl} = \begin{cases} 1 & \text{if } k = l = i \\ -1 & \text{if } k = l = (i+1) \\ 0 & \text{otherwise,} \end{cases}$$

and

$$(X_{\alpha_{ij}})_{kl} = \delta_{ik}\delta_{jl}.$$

We see that the center of $\mathfrak{g}$ is $k \cdot Z$ and the H_β together with Z form a basis for $\mathfrak{t}$, the Lie algebra of T.

For example, for $\mathrm{GL}_2(k)$ we have

$$Z = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right),\ H_{\alpha_{12}} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right),\ X_{\alpha_{12}} = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right),\ \text{ and } X_{\alpha_{21}} = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right).$$

Let $Z(G) \leq T$ denote the center of G. For $1 \leq k \leq n$ define $\lambda_k \in \mathbf{X}_*(T) = \mathrm{Hom}(k^\times, T)$ by setting

$$(\lambda_k(s))_{ij} = \begin{cases} s & \text{if } i = j = k \\ 1 & \text{if } i = j \neq k \\ 0 & \text{otherwise,} \end{cases}$$

for $s \in k^\times$.

With respect to our choice of a Chevalley basis, we can identify $\mathcal{A} = \mathcal{A}(T)$, the *apartment*[2] corresponding to T, with the real vector space

$$(\mathbf{X}_*(T) \otimes \mathbb{R})/(\mathbf{X}_*(Z(G)) \otimes \mathbb{R}).$$

Let x_0 denote the origin in $\mathcal{A}$. The apartment of T is an $(n-1)$-dimensional Euclidean space spanned by the set $\{x_0 + \bar{\lambda}_k \mid 1 \leq k \leq n\}$. Note that these spanning vectors satisfy the relation

$$0 = \sum_k \bar{\lambda}_k.$$

3.2. **A simplicial structure for the apartment of T.** We let

$$\Psi := \{\delta + n \mid \delta \in \Phi \text{ and } n \in \mathbb{Z}\}$$

denote the set of affine roots of G with respect to T and ν. For $\psi = \delta + n \in \Psi$ we let $\dot{\psi} = \delta \in \Phi$.

If $\psi = \delta + n \in \Psi$ and $\sum \lambda_i \otimes r_i \in \mathbf{X}_*(T) \otimes \mathbb{R}$, then we define

$$\psi\left(\sum \lambda_i \otimes r_i\right) = n + \sum r_i \langle \delta, \lambda_i \rangle.$$

Here $\langle\, ,\, \rangle$ denotes the usual pairing between characters and cocharacters. Note that if $v_T \in \mathbf{X}_*(T)$ and $v_Z \in \mathbf{X}_*(Z(G))$, then $\psi(v_T + v_Z) = \psi(v_T)$. Consequently, we can and shall think of an affine root as a function on $\mathcal{A}(T)$. For $\psi \in \Psi$, we define

$$H_\psi = \{x \in \mathcal{A}(T) \mid \psi(x) = 0\}.$$

[2] In these notes, we are always looking at the reduced building and apartments.

The H_ψ are hyperplanes in $\mathcal{A}(T)$ and they provide us with a simplicial decomposition of $\mathcal{A}(T)$.

For example, for $\mathrm{GL}_2(k)$ the apartment $\mathcal{A}(T)$ is one-dimensional and it is spanned by the vectors $x_0 + \bar{\lambda}_1 = x_0 - \bar{\lambda}_2$. See Figure 1.

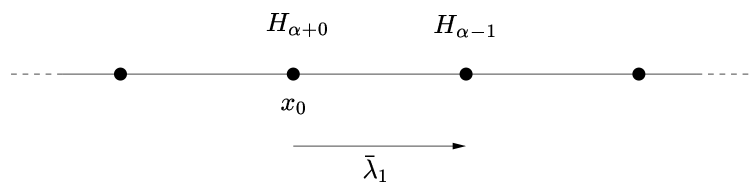

FIGURE 1. The standard apartment for $\mathrm{GL}_2(k)$.

For $\mathrm{GL}_3(k)$ the apartment $\mathcal{A}(T)$ is two dimensional, and it is spanned by the vectors $x_0 + \bar{\lambda}_1$, $x_0 + \bar{\lambda}_2$, and $x_0 + \bar{\lambda}_3 = x_0 - (\bar{\lambda}_1 + \bar{\lambda}_2)$. See, for example Figure 2.

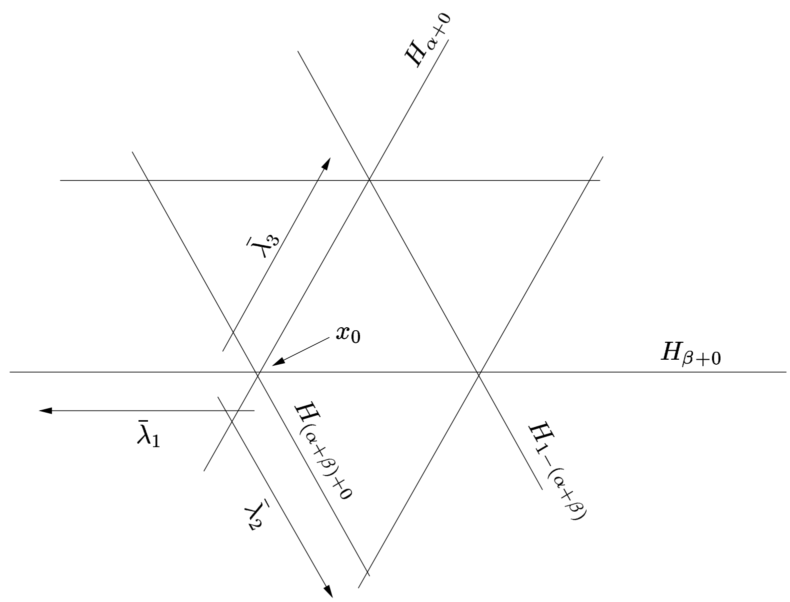

FIGURE 2. The standard apartment for $\mathrm{GL}_3(k)$

The maximal facets in this simplicial decomposition of $\mathcal{A}(T)$ are called *alcoves* or *chambers* of $\mathcal{A}(T)$. For $\mathrm{GL}_2(k)$, the alcoves are open line segments. For $\mathrm{GL}_3(k)$, an alcove is the interior of an equilateral triangle. For $\mathrm{GL}_4(k)$, an alcove is the interior of a regular tetrahedron.

3.3. **Some subgroups of G.** Note that for all $\alpha \in \Phi$, the root group

$$U_\alpha := 1 + k \cdot X_\alpha$$

is naturally isomorphic to k as an additive group. From (1), k, regarded as an additive group, has a natural filtration indexed by $\mathbb{Z} \cup \infty$. Namely,

$$\varpi^\infty \cdot R := \{0\} \subset \cdots \subset \varpi^2 \cdot R \subset \varpi \cdot R \subset R = \varpi^0 \cdot R \subset \varpi^{-1} \cdot R \subset \cdots \subset k.$$

Similarly, U_α has a natural filtration indexed by $\mathbb{Z} \cup \infty \cong \{\psi \in \Psi \,|\, \dot{\psi} = \alpha\} \cup \infty$. The problem is: how do we decide which $\psi \in \{\psi \in \Psi \,|\, \dot{\psi} = \alpha\}$ corresponds to which subgroup of U_α? The solution to our problem lies with our choice of a Chevalley basis. The choice of this basis determines the subgroup $K_0 = \mathrm{GL}_n(R)$. We define

$$U_{\alpha+0} := U_\alpha \cap K_0.$$

This completely determines a natural indexing of subgroups in U_α by the set $\{\psi \in \Psi \,|\, \dot{\psi} = \alpha\}$.

For example, for $\mathrm{GL}_3(k)$, we have

$$U_{\alpha_{12}+j} = \begin{pmatrix} 1 & \varpi^j \cdot R & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Similarly, we have a filtration of $\mathfrak{g}_\alpha$, the α-eigenspace in $\mathfrak{g}$, by setting

$$\mathfrak{g}_{\alpha+0} := \mathrm{M}_n(R) \cap \mathfrak{g}_\alpha.$$

Note that for all $\psi \in \Psi$ we have an isomorphism of the corresponding additive subgroups $\mathfrak{g}_\psi$ and U_ψ via the map $X \mapsto (1 + X)$.

The maximal compact open subgroup

$$T_0 = \{(t_1, t_2, \ldots t_n) \in T \,|\, t_i \in R^\times\}$$

of T also has a filtration indexed by the integers. Because of future considerations, we define, for $r \in \mathbb{R}_{\geq 0}$,

$$T_r := \{t \in T_0 \,|\, \nu\big(1 - \chi(t)\big) \geq r \text{ for all } \chi \in \mathbf{X}^*(T)\}.$$

We also define

$$T_{r^+} := \{t \in T_0 \,|\, \nu\big(1 - \chi(t)\big) > r \text{ for all } \chi \in \mathbf{X}^*(T)\}.$$

It is a matter of definition to see that $T_r = T_{\lceil r \rceil}$, and, for $i \in \mathbb{Z}_{\geq 0}$, if $i < r \leq (i+1)$, then

$$T_r = \{(t_1, t_2, \ldots, t_n) \in T_0 \,|\, t_j \in 1 + \varpi^{(i+1)} \cdot R \text{ for } 1 \leq j \leq n\}.$$

We can define filtrations of $\mathfrak{t} = \mathrm{Lie}(T)$ in a similar fashion. For example, for $\mathrm{GL}_3(k)$, we have

$$\mathfrak{t}_{5.2} = \begin{pmatrix} \varpi^6 \cdot R & 0 & 0 \\ 0 & \varpi^6 \cdot R & 0 \\ 0 & 0 & \varpi^6 \cdot R \end{pmatrix}.$$

3.4. **The Moy-Prasad filtrations.** Suppose $r \in \mathbb{R}$ and $x \in \mathcal{A}(T)$. In [28, 27], Moy and Prasad define filtration lattices of $\mathfrak{g}$ according to the formulae

$$\mathfrak{g}_{x,r} := \mathfrak{t}_r \oplus \sum_{\{\psi \in \Psi \mid \psi(x) \geq r\}} \mathfrak{g}_\psi$$

and

$$\mathfrak{g}_{x,r^+} := \mathfrak{t}_{r^+} \oplus \sum_{\{\psi \in \Psi \mid \psi(x) > r\}} \mathfrak{g}_\psi.$$

For some people, it is easier to process these definitions when they are presented in the following form.

$$\mathfrak{g}_{x,r} := \mathfrak{t}_r \oplus \sum_{\alpha \in \Phi} \sum_{\substack{m \in \mathbb{Z} \\ (\alpha+m)(x) \geq r}} \mathfrak{g}_{\alpha+m}$$

and

$$\mathfrak{g}_{x,r^+} := \mathfrak{t}_{r^+} \oplus \sum_{\alpha \in \Phi} \sum_{\substack{m \in \mathbb{Z} \\ (\alpha+m)(x) > r}} \mathfrak{g}_{\alpha+m}.$$

Similarly, if $r \geq 0$, then they define

$$G_{x,r} := \langle T_r, U_\psi \rangle_{\{\psi \in \Psi \mid \psi(x) \geq r\}}$$

and

$$G_{x,r^+} := \langle T_{r^+}, U_\psi \rangle_{\{\psi \in \Psi \mid \psi(x) > r\}}.$$

These lattices (resp., subgroups) are referred to as the Moy-Prasad filtration lattices of $\mathfrak{g}$ (resp., subgroups of G).

It follows immediately from the definitions that if $X \in \mathfrak{g}_{x,r}$ and $Y \in \mathfrak{g}_{x,s}$, then $X \cdot Y \in \mathfrak{g}_{x,(r+s)}$. The following exercise is highly recommended; we have previously considered these statements for the congruence and Iwahori filtrations.

Exercise 3.4.1. Suppose $s \in \mathbb{R}$ and $r \in \mathbb{R}_{\geq 0}$.

(1) Up to a constant $\widehat{[\mathfrak{g}_{x,s}]} = [\mathfrak{g}_{x,(-s)^+}]$.
(2) If $g \in G_{x,r}$, then $g\mathfrak{g}_{x,s}g^{-1} = \mathfrak{g}_{x,s}$ and $g\mathfrak{g}_{x,s^+}g^{-1} = \mathfrak{g}_{x,s^+}$.
(3) If $0 < r \leq s \leq 2r$, then $X \mapsto (1+X)$ induces an abelian group isomorphism of $\mathfrak{g}_{x,r}/\mathfrak{g}_{x,s}$ and $G_{x,r}/G_{x,s}$.

3.4.1. *The special case* $r = 0$. We first attempt to understand these definitions when $r = 0$. In this case, if x is a point in $\mathcal{A}(T)$, then $G_{x,0}$ is called the *parahoric subgroup* attached to x and $G_{x,0^+}$ is called the *pro-unipotent radical* of $G_{x,0}$. Let F be a facet in $\mathcal{A}$ and let $x, y \in F$. It follows from the definitions of both the simplicial structure of $\mathcal{A}$ and the Moy-Prasad filtrations that $G_{x,0} = G_{y,0}$, $\mathfrak{g}_{x,0} = \mathfrak{g}_{y,0}$, $G_{x,0^+} = G_{y,0^+}$, and $\mathfrak{g}_{x,0^+} = \mathfrak{g}_{y,0^+}$. Thus, it is enough to understand these filtrations on a facet by facet basis, and it is natural to label the filtrations using facets rather than points. For $\mathrm{GL}_2(k)$, Figures 3 and 4 describe the subgroups $G_{F,0}$ and $G_{F,0^+}$ for F a facet in the apartment of T.

Note that as we move to the right, that is, in the positive direction as defined by the spherical Weyl chamber corresponding to B, the positive root spaces "expand" and the

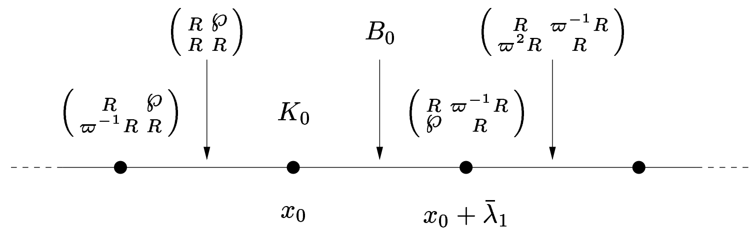

FIGURE 3. Parahoric subgroups $G_{F,0}$ for $\mathrm{GL}_2(k)$

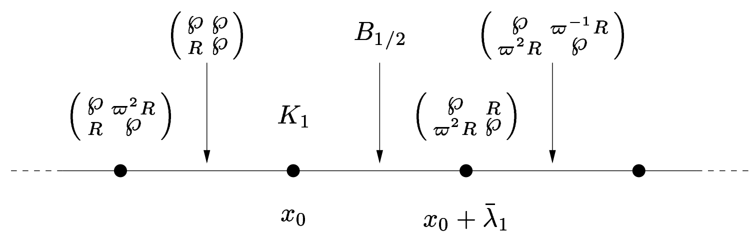

FIGURE 4. The subgroups $G_{F,0^+}$ for $\mathrm{GL}_2(k)$

negative root spaces "shrink". This simple observation will play a significant role in what is to come.

It is highly recommended that the reader complete the following exercise.

Exercise 3.4.2. Make diagrams similar to those above, but for the filtration lattices $\mathfrak{g}_{F,0}$ and $\mathfrak{g}_{F,0^+}$. Now, do the same for $\mathrm{GL}_3(k)$.

The origin x_0 is the facet in $\mathcal{A}$ defined by the intersection of the hyperplanes $H_{\alpha+0}$ for $\alpha \in \Phi$. We see from the above examples (and definitions) that $G_{x_0,0} = K_0$ and $G_{x_0,0^+} = K_1$.

More generally, up to conjugation, the facets of $\mathrm{GL}_n(k)$ are indexed by

$$\mathcal{P}(n) := \{(\mu_1, \mu_2, \ldots, \mu_\ell) \mid \mu_1 \geq \mu_2 \geq \cdots \geq \mu_\ell \geq 1 \text{ and } \sum_{i=1}^{\ell} \mu_i = n\},$$

the set of ordered partitions of n. We briefly describe how this works. We let C_0 be the alcove in $\mathcal{A}$ with vertices $\{v_1 = x_0 = x_0 + \sum_{i=1}^n \bar{\lambda}_i, v_2 = v_1 - \bar{\lambda}_1, v_3 = v_2 - \bar{\lambda}_2, \ldots, v_n = \bar{\lambda}_n\}$. To $\mu \in \mathcal{P}(n)$ we attach the facet F_μ of dimension ℓ with vertices $v_1, v_{(\mu_1+1)}, v_{(\mu_1+\mu_2+1)}, \ldots, v_{(\mu_1+\mu_2+\cdots\mu_\ell+1)}$. The parahoric subgroup $G_{F_\mu,0}$ can be described as $K_1 \cdot Q_\mu(R)$ where Q_μ is the "standard" parabolic subgroup of G containing B which corresponds to the partition $\mu = (\mu_1, \mu_2, \ldots, \mu_\ell)$. The pro-unipotent radical $G_{F_\mu,0^+}$ is then the inverse image in $G_{F_\mu,0}$ of the unipotent radical of the image of $G_{F_\mu,0}$ in $\mathrm{GL}_n(\mathfrak{f}) \cong K_0/K_1$. Note that if $\mu, \nu \in \mathcal{P}(n)$ with $\mu < \nu$ in the usual partial ordering of ordered partitions, then $Q_\mu \supset Q_\nu$ and similarly for the associated parahoric subgroups.

More generally, note that if F_1 and F_2 are facets in $\mathcal{A}(T)$ with $F_1 \subset \overline{F_2}$ where $\overline{F_2}$ denotes the closure of F_2, then it follows that

$$G_{F_1,0^+} \subset G_{F_2,0^+} \subset G_{F_2,0} \subset G_{F_1,0}.$$

Moreover, as the examples and exercises above show, $G_{F_2,0}/G_{F_1,0^+}$ is a parabolic subgroup of the connected reductive group $G_{F_1,0}/G_{F_1,0^+}$. The unipotent radical of $G_{F_2,0}/G_{F_1,0^+}$ is $G_{F_2,0^+}/G_{F_1,0^+}$ and $G_{F_2,0}/G_{F_1,0^+}$ has Levi component isomorphic to $G_{F_2,0}/G_{F_2,0^+}$.

As a tangent, we also observe that if $W = N_G(T)/T \cong (N_G(T) \cap K_0)/T_0$ denotes the Weyl group, then there exist $|W|$ alcoves in $\mathcal{A}(T)$ which contain x_0 in their closure. This is because these alcoves correspond to the Borel subgroups in $\mathrm{GL}_n(\mathfrak{f}) = K_0/K_1 = G_{x_0,0}/G_{x_0,0^+}$ which contain "$T(\mathfrak{f})$".

3.4.2. *The Moy-Prasad filtrations for arbitrary r.* For arbitrary r, there does not exist such a nice description of what is happening. We first consider $\mathrm{GL}_2(k)$. In Figure 5 we have

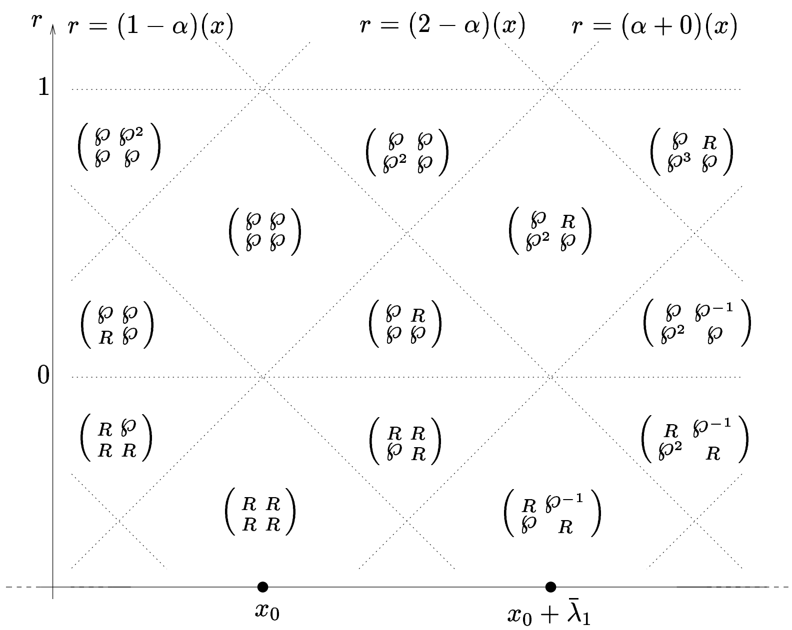

FIGURE 5. $(\mathfrak{gl}_2)_{x,r}$

identified the apartment of T with the horizontal axis. The vertical axis measures r. Given a pair $(x, r) \in \mathcal{A}(T) \times \mathbb{R}$, we wish to describe the lattice $\mathfrak{g}_{x,r}$. Note that the plane has been divided into convex polygons. The diagonal dotted lines are the graphs of the equations $r = \psi(x)$ where $\psi \in \Psi$. These measure where the root subgroup filtrations "change". The horizontal dotted lines are the graphs of the equations $r = n$ where $n \in \mathbb{Z}$. These measure where the toral subgroup filtrations "change". Each of the convex polygons is labeled by a lattice: if (x, r) belongs to the interior of the convex polygon, then $\mathfrak{g}_{x,r}$ is the lattice so identified. For purposes of assigning lattices to every point in the figure, the convex polygons are “closed at the top”. We note that $\mathfrak{g}_{x,r} = \mathfrak{g}_{x,r^+}$ unless the point (x, r) lies on a dotted line. Finally, note that if we fix r and move to the right, then the positive root space “expands” while the negative root space “shrinks”.

Fortunately, even though life is not so nice for arbitrary r, there exists a wonderful result of Moy and Prasad[3]. Define

$$\mathcal{O} := \{x \in \mathcal{A}(T) \mid x \text{ is the barycenter of a facet }\}.$$

An element of $\mathcal{O}$ is called an *optimal point*[4].

Lemma 3.4.3 (Moy and Prasad)**.** *Suppose $z \in \mathcal{A}(T)$ and $r \in \mathbb{R}$. There exist points $x, y \in \mathcal{O}$ such that*

$$\mathfrak{g}_{x,r} \subset \mathfrak{g}_{z,r} \subset \mathfrak{g}_{y,r}$$

and there exist points $x', y' \in \mathcal{O}$ such that

$$\mathfrak{g}_{x',r^+} \subset \mathfrak{g}_{z,r^+} \subset \mathfrak{g}_{y',r^+}.$$

A similar pair of statements can be made for the Moy-Prasad filtration subgroups.

For example, for $\mathrm{GL}_3(k)$, the optimal points, up to the action of G (see below), are the points $\bar\lambda_1 + \bar\lambda_2 + \bar\lambda_3$, $\frac{\bar\lambda_3}{2}$ and $\frac{3\bar\lambda_3+2\bar\lambda_2+\bar\lambda_1}{3}$.

Exercise 3.4.4. Describe (up to conjugacy) the optimal points for $\mathrm{GL}_3(k)$ and their associated filtrations $\mathfrak{g}_{x,r}$ and $\mathfrak{g}_{x,r^+}$.

4. G-DOMAINS

Recall that the ultimate goal of these notes is to relate certain invariant distributions on G with certain invariant distributions on $\mathfrak{g}$. Since the distributions in question are invariant, we will need sets in G and $\mathfrak{g}$ which are invariant. Such a set can never be compact, so the most we can hope for is to have sets which are invariant, open, and closed. A set with these properties is called a *G-domain*.

As we show in this section, it is possible to use the Moy-Prasad filtrations to define G-domains which have very nice properties. In particular, we will define a family of G-domains which form a “neighborhood basis” of $\mathcal{N}$, the set of nilpotent elements.

By using the Cartan decomposition of G (see [33]), in [18, Lemma 2.4] Howe demonstrates that, for all integers i, the $\mathrm{GL}_n(k)$-orbit of $\mathfrak{k}_i$ is contained in $\mathfrak{k}_i + \mathcal{N}$. From, for example, [13, Lemma 12.2], it is also true that for any compact set $\omega \subset \mathfrak{g}$, there exists a lattice $\mathcal{L} \subset \mathfrak{g}$ such that ${}^G\omega \subset \mathcal{L} + \mathcal{N}$. We analyze these statements from the perspective of Moy-Prasad filtration lattices. After doing this, we look at the situation on the group.

4.1. **Some comments on the Bruhat-Tits building of G.** Before we can continue, we must recall some facts about $\mathcal{B}$, the *reduced Bruhat-Tits building of G*. To each maximal split torus T' in G we can attach an apartment $\mathcal{A}(T')$ exactly as we did above. The building of G can then be thought of as the “gluing” together of all these various apartments. In Figure 6 we present a picture of $\mathcal{B}$ for $\mathrm{GL}_2(k)$. An apartment in $\mathcal{B}$ is the image of any

[3]You will not find the result stated exactly as follows, but it is easy to derive this formulation.

[4]Beware, $\mathcal{O}$ plays many roles in these notes. However, there is no possibility of confusion.

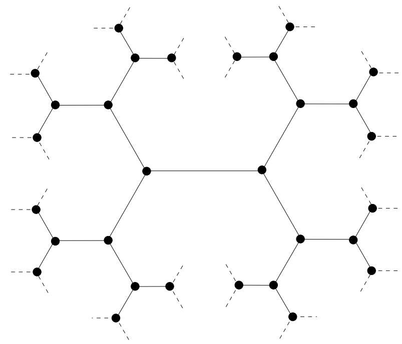

FIGURE 6. A picture of the building of $\mathrm{GL}_2(k)$

continuous injective map from $\mathbb{R}$ which maps integers to vertices.

Just as any two maximal split tori in G are conjugate, there is a natural action of G on $\mathcal{B}(G)$ with respect to which any apartment can be carried into any other. Moreover, a version of the two body problem can be solved: given any two points in the building, there is an apartment which contains both of them. Combining these two facts we have: for any two points x, y in $\mathcal{B}$, there exists a $g \in G$ such that $gx, gy \in \mathcal{A}(T)$. (Here gx denotes the image of x under the action of g on $\mathcal{B}(G)$.) Finally, the action of G on $\mathcal{B}$ is semisimple; that is, for $h \in G$, either there is a point in $\mathcal{B}$ which h fixes, or there is a line (i.e., a one-dimensional subspace of some apartment) on which h acts by nontrivial translation.

Suppose $x \in \mathcal{B}$ and $r \in \mathbb{R}$. We define the Moy-Prasad filtration lattices $\mathfrak{g}_{x,r}$ and $\mathfrak{g}_{x,r^+}$ as follows. Choose $g \in G$ so that $gx \in \mathcal{A}(T)$.

$$\mathfrak{g}_{x,r} := g^{-1} \cdot \mathfrak{g}_{gx,r} \cdot g \text{ and } \mathfrak{g}_{x,r^+} := g^{-1} \cdot \mathfrak{g}_{gx,r^+} \cdot g.$$

One can check that these definitions make sense. For $r \geq 0$ we define the subgroups $G_{x,r}$ and G_{x,r^+} in a similar fashion.

Finally, we note that $\mathcal{B}$ is endowed with a nontrivial invariant metric, denoted dist. Moreover, with respect to dist, $\mathcal{B}$ has nonpositive sectional curvature.

4.2. **G-domains for the Lie algebra.** We begin with a generalization of the result of Howe discussed above; the proof of this result (from [1]) nicely illustrates the benefit of working with the Moy-Prasad filtration lattices.

Lemma 4.2.1. *Let $x, y \in \mathcal{B}$, and let $r \in \mathbb{R}$. Then $\mathfrak{g}_{x,r} \subset \mathfrak{g}_{y,r} + \mathcal{N}$.*

Proof. Since $\mathfrak{g}_{x,r} \subset \mathfrak{g}_{y,r} + \mathcal{N}$ if and only if $\mathfrak{g}_{gx,r} \subset \mathfrak{g}_{gy,r} + \mathcal{N}$ for $g \in G$, we may assume that x and y are elements of $\mathcal{A}$.

Choose $\vec{v} \in \mathbf{X}_*(T) \otimes \mathbb{R}$ such that $x = y + \vec{v}$ (working modulo $\mathbf{X}_*(Z(G)) \otimes \mathbb{R}$). Let B' be a Borel subgroup determined by $\vec{v}$. That is, B' has a Levi decomposition $B' = TN'$ such that for all roots $\alpha \in \Phi^+_{N'}$, the set of roots which are positive with respect to N', we have $\langle \alpha, \vec{v} \rangle \geq 0$.

Let $\bar{B}' = T\bar{N}'$ denote the parabolic opposite $B = TN'$ and let $\mathfrak{g} = \bar{\mathfrak{n}}' + \mathfrak{t} + \mathfrak{n}'$ denote the associated Lie algebras. We have

$$\begin{aligned}
\mathfrak{g}_{x,r} &= \mathfrak{t}_r \oplus \sum_{\{\psi\in\Psi \mid \psi(x)\geq r\}} \mathfrak{g}_\psi \\
&= \mathfrak{t}_r \oplus \sum_{\{\psi\in\Psi \mid \dot\psi\in\Phi\smallsetminus\Phi^+_{N'} \text{ and } \psi(x)\geq r\}} \mathfrak{g}_\psi \oplus \sum_{\{\psi\in\Psi \mid \dot\psi\in\Phi^+_{N'} \text{ and } \psi(x)\geq r\}} \mathfrak{g}_\psi \\
&= \mathfrak{t}_r \oplus (\mathfrak{g}_{x,r} \cap \bar{\mathfrak{n}}') \oplus (\mathfrak{g}_{x,r} \cap \mathfrak{n}') \\
&\subset \mathfrak{g}_{y,r} + \mathfrak{n}' \subset \mathfrak{g}_{y,r} + \mathcal{N}. \quad \square
\end{aligned}$$

In fact, a kind of converse to the above lemma is true. Namely, if $X \in \mathfrak{g}$ belongs to $\mathfrak{g}_{x,r} + \mathcal{N}$ for all $x \in \mathcal{B}$, then there is a point $y \in \mathcal{B}$ such that $X \in \mathfrak{g}_{y,r}$. This result along with the above lemma give us the following theorem.

Theorem 4.2.2.

$$\bigcup_{x\in\mathcal{B}} \mathfrak{g}_{x,r} = \bigcap_{x\in\mathcal{B}} (\mathfrak{g}_{x,r} + \mathcal{N}).$$

In order to simplify our notation, we define $\mathfrak{g}_r := \cup_{x\in\mathcal{B}}\mathfrak{g}_{x,r}$. From its definition, $\mathfrak{g}_r$ is open and invariant. From the above theorem we have that $\mathfrak{g}_r$ is closed. Consequently, $\mathfrak{g}_r$ is a G-domain.

We now consider some examples. The set $\mathfrak{g}_0$ is usually referred to as the set of compact or integral elements of $\mathfrak{g}$. The set $\mathfrak{g}_{0^+}$ is usually called the set of topologically nilpotent elements in $\mathfrak{g}$; it consists of those $X \in \mathfrak{g}$ such that powers of X tend to zero in the p-adic topology. These G-domains have very natural interpretations; namely,

$$\mathfrak{g}_r = \{X \in \mathfrak{g} \mid \nu(e_X) \geq r \text{ for all eigenvalues } e_X \text{ of } X\ \}.$$

(If E is a finite extension of k, then there exists a unique extension of ν to E.) For $\mathrm{GL}_n(k)$ and $X \in \mathfrak{g}$, the value of $\nu(e_X)$ must lie in the set $\{k/n \mid k \in \mathbb{Z}\}$.

Of course, just as we have Moy-Prasad lattices of the form $\mathfrak{g}_{x,r^+}$, we also can define $\mathfrak{g}_{r^+} := \bigcup_{x\in\mathcal{B}} \mathfrak{g}_{x,r^+}$. These G-domains satisfy the obvious analogues of the results discussed above. From the previous paragraph, we have that $\mathfrak{g}_r \neq \mathfrak{g}_{r^+}$ implies that $r = k/n$ for some $k \in \mathbb{Z}$.

Exercise 4.2.3. Define the subspace D_r of $C_c^\infty(\mathfrak{g})$ by

$$D_r := \sum_{x\in\mathcal{B}} C(\mathfrak{g}/\mathfrak{g}_{x,r})$$

where the sum is interpreted as follows. A function f belongs to D_r if and only if f can be written as a finite sum $f = \sum_i f_i$ with $f_i \in C_c(\mathfrak{g}/\mathfrak{g}_{x_i,r})$ for some $x_i \in \mathcal{B}$. We can define D_{r^+} in a similar way. Show that the Fourier transform gives us bijective maps from D_{r^+} to $C_c^\infty(\mathfrak{g}_{-r})$ and from D_r to $C_c^\infty(\mathfrak{g}_{(-r)^+})$.

4.3. **G-domains in G.** We now turn our attention to the group-side of things. Unlike the Lie algebra, an element of the group can only belong to a Moy-Prasad filtration subgroup if it first belongs to a parahoric. It turns out that the interesting part of the story here rests in proving the equality

$$\bigcup_{x\in\mathcal{B}} G_{x,0} = \bigcap_{x\in\mathcal{B}} G_{x,0}\cdot\mathcal{U}$$

where $\mathcal{U}$ denotes the set of unipotent elements in G. Once this result is known to be true, it is very easy to establish the equality

$$G_r = \bigcap_{x\in\mathcal{B}} G_{x,r}\cdot\mathcal{U}$$

where $r\geq 0$ and $G_r = \bigcup_{x\in\mathcal{B}} G_{x,r}$. As above, it follows immediately that G_r is a G-domain.

We show how to prove the equality when $r=0$.

Lemma 4.3.1.

$$G_0 = \bigcap_{x\in\mathcal{B}} G_x\cdot\mathcal{U}.$$

Proof. We first show that the right-hand side is a subset of the left-hand side. We will argue by contradiction. Suppose that $g\in\bigcap_{x\in\mathcal{B}} G_{x,0}\cdot\mathcal{U}$ does not belong to G_0.

Since the action of G on $\mathcal{B}$ is semisimple, we either have that there is a point x in $\mathcal{B}$ which g fixes or a line ℓ in an apartment $\mathcal{A}'$ of $\mathcal{B}$ on which g acts by nontrivial translation.

In the first case, we can write $g=h\cdot u$ with $h\in G_{x,0}$ and $u\in\mathcal{U}$. Since u is unipotent, it must live in $G_{y,0}$ for some $y\in\mathcal{B}$. But, from a result of Eugene Kushnirsky this implies that $u\in G_{x,0}$ [10, Lemma 4.5.1].

In the latter case, there exists a facet F' in $\mathcal{A}'$ such that $F'\cap\ell$ is open in ℓ. For all $x,y\in F'$ we have $G_{y,0}=G_{x,0}$. By hypothesis, there exist elements $h\in G_{x,0}$ and $u\in\mathcal{U}$ such that $g=uh$. Since u is unipotent, there exists $w\in\mathcal{B}$ which is fixed by u. We have that for all $y\in F'\cap\ell$

$$\operatorname{dist}(w,y)=\operatorname{dist}(w,uy)=\operatorname{dist}(w,uhy)=\operatorname{dist}(w,gy).$$

Thus, we have , for $y,y'\in F'\cap\ell$ a picture something like that described in Figure 7.

However, since $\mathcal{B}$ has nonpositive sectional curvature, the line segment from x to w must be shorter than the line segments from y and gy to w. Similarly, the segments from gy to w must be shorter than the segments from x and gx to w. Consequently, we have

$$\operatorname{dist}(x,w)<\operatorname{dist}(gy,w)<\operatorname{dist}(x,w),$$

a contradiction.

We now show that the left-hand side is a subset of the right-hand side.

We need to show that for $x,y\in\mathcal{B}$, we have $G_x\subset\mathcal{U}\cdot G_y$. As in the proof for the Lie algebra, we may assume that x and y both belong to $\mathcal{A}$. Let $B'=TN'$ be a Borel subgroup of G so that the (spherical) chamber in $\mathcal{A}$ determined by N' is invariant under translation by the vector $(y-x)$.

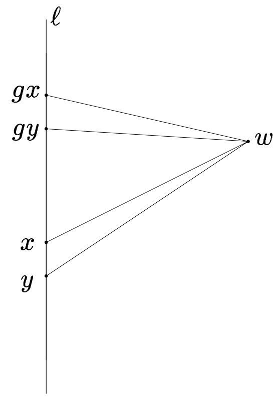

FIGURE 7

Let $\bar{N}'$ be the unipotent radical of the parabolic opposite $B = TN'$. From [3] we can write $G = N' \cdot \bar{N}' \cdot N' \cdot T$. It follows that we can write

$$G_{x,0} = N'_x \cdot \bar{N}'_x \cdot N'_x \cdot T_0$$

where $N'_x = N' \cap G_{x,0}$ and $\bar{N}'_x = \bar{N}' \cap G_{x,0}$. Because of the way in which N' was chosen, we have $N'_x \subset G_{y,0}$. Thus, if $g \in G_x$, then there exist $n_1, n_2 \in N_x$, $\overline{n} \in \overline{N}_x$, and $t \in T_0$ such that

$$\begin{aligned} g &= n_1 \cdot \overline{n} \cdot n_2 \cdot t \\ &= (n_1 \cdot \overline{n} \cdot n_1^{-1}) \cdot (\cdot n_1 \cdot n_2 \cdot t) \\ &\in \mathcal{U} \cdot G_y. \quad \square \end{aligned}$$

4.4. **A neighborhood basis for the nilpotent cone.** We remark that

$$\mathcal{N} = \bigcap_r \mathfrak{g}_r$$

and

$$\mathcal{U} = \bigcap_r G_r.$$

It follows that the G-domains we have defined provide us with a neighborhood basis of the nilpotent cone (resp., unipotent variety) consisting of open, invariant, closed neighborhoods.

5. NILPOTENT ORBITAL INTEGRALS AS DISTRIBUTIONS

A distribution on $\mathfrak{g}$ is any element of the linear dual of $C_c^\infty(\mathfrak{g})$. The aim of this section is to convince ourselves that integrating against a nilpotent orbit defines a distribution on $\mathfrak{g}$. We follow the argument of Ranga-Rao [29].

5.1. **Orbital integrals.** Fix an element $Y \in \mathfrak{g}$. Let $\mathcal{O}_Y = {}^G Y := \{gYg^{-1} \mid g \in G\}$ denote the G-orbit of Y. We identify the orbit of Y with the homogeneous space $G/C_G(Y)$. Thus, the tangent space to the orbit of Y at the point Y is identified with $\mathfrak{g}/C_{\mathfrak{g}}(Y)$.

We define an alternating bilinear form $\langle\,,\,\rangle_Y$ on $\mathfrak{g}$ by $\langle A, B\rangle = \mathrm{tr}(Y \cdot [A,B])$ for $A, B \in \mathfrak{g}$. Fix an element $B \in \mathfrak{g}$. A calculation shows that $\langle A, B\rangle = 0$ for all $A \in \mathfrak{g}$ if and only if $B \in C_{\mathfrak{g}}(Y)$. Since we have a similar statement when we switch the roles of A and B, it follows that $\langle\,,\,\rangle_Y$ induces a nondegenerate alternating form on $\mathfrak{g}/C_{\mathfrak{g}}(Y)$. Thus, the dimension of the orbit is even, say $2m$.

Similarly, for each ${}^gY = gYg^{-1} \in {}^G Y$, we have a nondegenerate alternating symmetric form on $\mathrm{Tan}_{{}^gY}(\mathcal{O}_Y)$. Consequently, there exists a nondegenerate, invariant two-form ω for $\mathcal{O}_Y$ and from this we can form a nonzero, left-invariant volume form $\omega \wedge \omega \wedge \cdots \wedge \omega$ (m times) on $\mathcal{O}_Y$. Let $X_1, X_2, \ldots, X_{2m}$ be coordinates for $\mathfrak{g}/C_{\mathfrak{g}}(Y)$. For each $1 \leq i \leq 2m$, fix a one-form dX_i and associated measure $|dX_i|$ normalized so that for all $f \in C_c^\infty(\mathfrak{g}/C_{\mathfrak{g}}(Y))$, we have $\hat{\hat{f}}(X) = f(-X)$, where the Fourier transform is taken with respect to the measure $\prod_{i=1}^{2m} |dX_i|$. There exists a locally convergent power series $f(X_1, X_2, \ldots, X_{2m})$ so that (locally)

$$\omega = f(X_1, X_2, \ldots X_{2m})\, dX_1 \wedge dX_2 \wedge \cdots \wedge dX_{2m}.$$

Define

$$|\omega| := |f(X_1, X_2, \ldots, X_{2m})| \prod_{i=1}^{2m} |dX_i|\,.$$

Thus, we have an invariant measure on $\mathcal{O}_Y$.

Exercise 5.1.1. Check that this definition is invariant of the various choices we have made.

5.2. **A framing of the problem.** If Y is a semisimple element of $\mathfrak{g}$, then $\mathcal{O}_Y$ is closed in $\mathfrak{g}$. Consequently, if $f \in C_c^\infty(\mathfrak{g})$, then the restriction of f to $\mathcal{O}_Y$ is an element of $C_c^\infty(\mathcal{O}_Y)$. So it makes sense to define the distribution $\mathcal{O}_Y \colon C_c^\infty(\mathfrak{g}) \to \mathbb{C}$ by setting

$$\mathcal{O}_Y(f) := \int_{G/C_G(Y)} f({}^gY)\, dg^*.$$

for $f \in C_c^\infty(\mathfrak{g})$. Here dg^* denotes the invariant measure on $G/C_G(Y)$ defined above.

What if Y is not semisimple? In particular, what if Y is nilpotent? In this case, we need to do some work to show that for arbitrary $f \in C_c^\infty(\mathfrak{g})$ the integral

$$\int_{G/C_G(Y)} f({}^gY)\, dg^*$$

makes sense.

5.3. **The basic notation associated to nilpotent orbital integrals.** We begin by recalling the familiar parameterization of nilpotent orbits in $\mathfrak{g}$. We then discuss various properties of nilpotent orbits which will be important in the sequel.

From basic linear algebra we know that the nilpotent orbits in $\mathfrak{g}$ can be parameterized by $\mathcal{P}(n)$, the set of ordered partitions of n. To $\mu \in \mathcal{P}(n)$ we associate the nilpotent element $X_\mu \in \mathcal{N} \cap \mathfrak{g}_{x_0,0}$ having Jordan canonical form corresponding to μ. That is, if

$\mu = (\mu_1, \mu_2, \ldots, \mu_k)$, then in the i^{th} block of size $\mu_i \times \mu_i$ we put the matrix with ones on the superdiagonal and zeroes elsewhere. For example, for $\mu = (2, 2, 1) \in \mathcal{P}(5)$ we have

$$X_\mu = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Note that the nilpotent orbits in $\mathfrak{g}_{x_0,0}/\mathfrak{g}_{x_0,0^+} \cong \mathrm{M}_n(\mathfrak{f})$ are also indexed by $\mathcal{P}(n)$ and the map taking X_μ to the image of X_μ in $\mathrm{M}_n(\mathfrak{f})$ gives a bijective correspondence between $\mathcal{O}(0)$, the set of nilpotent orbits in $\mathfrak{g}$, and the set of nilpotent orbits in $\mathrm{M}_n(\mathfrak{f})$.

We let $\mathcal{O}_\mu$ denote the G-orbit of X_μ. Note that if $\mu < \mu'$ in the usual partial order on $\mathcal{P}(n)$, then $\mathcal{O}_\mu \subset \bar{\mathcal{O}}_{\mu'}$.

Exercise 5.3.1. Show that if $\mathcal{O}$ is a nilpotent orbit such that $\mathcal{O} \cap (X_\mu + \mathfrak{g}_{x_0,0^+}) \neq \emptyset$, then $\mathcal{O}_\mu$ is contained in the p-adic closure of $\mathcal{O}$. Moreover, $\mathcal{O}_\mu \cap (X_\mu + \mathfrak{g}_{x_0,r^+}) = {}^{G_{x_0,r^+}} X_\mu$ for all non-negative r.

Note that if $X \in \mathcal{N}$, and $t \in k^\times$, then $tX \in \mathcal{N}$ and in fact, tX and X are $\mathrm{GL}_n(k)$ conjugate. (In general, it follows from Jacobson-Morosov that t^2X and X are conjugate, but in GL_n we can do better.) So, $\mathcal{N}$ and each nilpotent orbit are closed with respect to scaling.

We can associate a parabolic subgroup $P(\mu)$ of G to μ as follows: For all positive integers j, the matrix X_μ^j acts on $V = k^n$, and we define $V_j = \ker(X_\mu^j) \subset V$. Define $P(\mu) = \{g \in G \,|\, g \cdot V_j \subset V_j \text{ for all positive } j\,\}$. Then X_μ lies in the nilradical $\mathfrak{n}(\mu)$ of the Lie algebra of $P(\mu)$, and the $P(\mu)$-orbit of X_μ is dense in $\mathfrak{n}(\mu)$ and equal to $\mathcal{O}(\mu) \cap \mathfrak{n}(\mu)$.

We need most of the basic facts about $\mathfrak{sl}_2(k)$-triples. For a good reference, see [4]. We can complete X_μ to an $\mathfrak{sl}_2(k)$-triple (Y_μ, H_μ, X_μ). Explicitly, the i^{th} block of H_μ is given by the element $\mathrm{diag}((\mu_i - 1), (\mu_i - 3), \ldots, (1 - \mu_i))$ and the i^{th} part of Y_μ is given by certain entries on the super-subdiagonal. We let $\lambda_\mu \in \mathbf{X}_*(G)$ denote the associated one-parameter subgroup, that is, the i^{th} block of $\lambda_\mu(t)$ looks like $\mathrm{diag}(t^{(\mu_i-1)}, t^{(\mu_i-3)}, \ldots, t^{(1-\mu_i)})$. For $i \in \mathbb{Z}$, we define

$$\mathfrak{g}(i) = \{X \in \mathfrak{g} \,|\, {}^{\lambda(t)}X = t^i X\}.$$

We have $\mathfrak{g} = \sum_i \mathfrak{g}(i)$. For $j \in \mathbb{Z}$, define $\mathfrak{g}(\geq j) := \sum_{i \geq j} \mathfrak{g}(i)$. We let $\mathfrak{p}_\mu$ denote the parabolic subalgebra $\mathfrak{g}(\geq 0)$ and let $\mathfrak{m}_\mu = \mathfrak{g}(0)$ while the nilradical of $\mathfrak{p}_\mu$ is denoted by $\mathfrak{n}_\mu$ $(= \mathfrak{g}(\geq 1))$. We let P_μ denote the corresponding parabolic subgroup with Levi decomposition $M_\mu N_\mu$. The P_μ-orbit of X_μ is equal to ${}^{M_\mu}X_\mu + \mathfrak{g}(\geq 3)$ and ${}^{M_\mu}X_\mu$ is an open and dense subset of $\mathfrak{g}(2)$. Finally, it is true that $C_G(X_\mu) \subset P_\mu$.

5.4. **A sketch of the proof that nilpotent orbital integrals define distributions.** Fix $\mu \in \mathcal{P}(n)$. Let $f \in C_c^\infty(\mathfrak{g})$.

The Iwasawa decomposition (see [33]) tells us that we can write $G = G_{x_0,0}P_\mu$. Fix a Haar measure dk on K and a left Haar measure $d_\ell p$ on P_μ so that

$$\int_G h(g)\,dg = \int_{P_\mu} d_\ell p \int_{G_{x_0,0}} h(kp)\,dk$$

for all $h \in C_c^\infty(G)$, the space of complex-valued, compactly supported, locally constant functions on G. Because it will be important, we recall that for all $p_0 = m_0 n_0 \in P_\mu = M_\mu N_\mu$ we have $d_\ell(p_0 p p_0^{-1}) = \left|\det\left(m_0|_{\mathfrak{n}_\mu}\right)\right|^{-1} d_\ell p$.

Define $\bar{f} \in C_c^\infty(\mathfrak{g})$ by $\bar{f}(X) := \int_{G_{x_0,0}} f({}^k X)\,dk$. Let us assume for the moment that

$$(*) \qquad \int_{P/C_G(X_\mu)} \bar{f}({}^p X)\,d_\ell p^*$$

converges. (Here $d_\ell p^*$ is the quotient measure.) In this case, we have

$$(*) = \int_{P/C_G(X_\mu)} \int_K f({}^{kp}X)\,dk\,d_\ell p^* = \int_{G/C_G(X_\mu)} f({}^g X_\mu)\,dg^*,$$

and so we can define the invariant distribution $\mathcal{O}_\mu$ by

$$\mathcal{O}_\mu(f) = \int_{G/C_G(X_\mu)} f({}^g X_\mu)\,dg^*.$$

for $f \in C_c^\infty(\mathfrak{g})$.

We now sketch the proof of why (*) makes sense. Let dZ be a Haar measure on $\mathfrak{g}(\geq 2)$. We'd like to define a function F_0 on $\mathfrak{g}(\geq 2)$ so that

$$(*) = \int_{{}^{M_\mu}X_\mu + \mathfrak{g}(\geq 3)} \bar{f}(Z)\,|F_0(Z)|\;dZ.$$

For $p_0 = m_0 n_0$ as above, we have $d(p_0 Z p_0^{-1}) = \left|\det\left(m_0|_{\mathfrak{g}(\geq 2)}\right)\right|^{-1} dZ$. Consequently, it is sufficient to produce a nonzero function F_0 with the property $F_0({}^{p_0}Z) = \det\left(p_0|_{\mathfrak{g}(1)}\right)$.

The map $X \mapsto \big(Y \mapsto \operatorname{tr}(Y \cdot X)\big)$ induces an isomorphism of $\mathfrak{g}(1)$ with $\mathfrak{g}(-1)^*$. Just as at the beginning of this section, for $Z \in \mathfrak{g}(2)$, we can define an alternating bilinear form $\langle\,,\,\rangle_Z$ on $\mathfrak{g}(-1)$. When $Z = X_\mu$, the form is also nondegenerate. Consequently, both $\mathfrak{g}(1)$ and $\mathfrak{g}(-1)$ are even dimensional (of dimension $2n'$). Let a_Z be the matrix which represents $\langle\,,\,\rangle_Z$ with respect to some fixed basis of $\mathfrak{g}(-1)$. From [22, Theorem 6.4] there is a polynomial (Pfaffian) Pf of degree n' on $\mathfrak{g}(2)$ so that $\det(a_Z) = (\operatorname{Pf}(Z))^2$. Moreover, since $\det(a_{({}^m Z)}) = \det\left(m^{-1}|_{\mathfrak{g}(-1)}\right)^2 \cdot \det(a_Z)$, the polynomial Pf transforms in the manner we desire. For $Z \in \mathfrak{g}(\geq 2)$, set $F_0(Z) = \operatorname{Pf}(Z_2)$ where Z_2 denotes the image of Z under the projection map from $\mathfrak{g}(\geq 2)$ to $\mathfrak{g}(2)$.

Example 5.4.1. For the partition $(3, 2) \in \mathcal{P}(5)$, we have that the dimension of $\mathfrak{g}(-1)$ is four. The space $\mathfrak{g}(2)$ is spanned by the Chevalley basis elements $X_{\alpha_{12}}$, $X_{\alpha_{23}}$ and $X_{\alpha_{45}}$. If $Z = xX_{\alpha_{12}} + yX_{\alpha_{23}} + zX_{\alpha_{45}}$, then $F_0(Z) = xy$.

5.5. **An important calculation.** For $j > 0$ we calculate $\mathcal{O}_\mu([X_\mu + \mathfrak{k}_j])$.

Proposition 5.5.1.

$$\mathcal{O}_\mu([X_\mu + \mathfrak{k}_j]) = q^{((1-2j)\cdot \dim(\mathcal{O}_\mu)/2)}$$

Proof. Suppose that $n > j$ is very large. Let $\mathcal{O}_\mu^n := {}^{K_n}X_\mu = \{{}^kX_\mu \mid k \in K_n\}$. It follows from Exercise 5.3.1 that

$$\mathcal{O}_\mu([X_\mu + \mathfrak{k}_j]) = [K_j \,:\, K_n \cdot C_{K_j}(X_\mu)] \cdot \mathcal{O}_\mu([\mathcal{O}_\mu^n]).$$

Now, by definition, we have

$$\begin{aligned}
\mathcal{O}_\mu([\mathcal{O}_\mu^n]) &= \mathrm{meas}_{dg^*}(\mathcal{O}_\mu^n) \\
&= \mathrm{meas}_{|dX_1|\cdot|dX_2|\cdots|dX_{2m}|}\big((\mathfrak{k}_n + C_\mathfrak{g}(X_\mu))/C_\mathfrak{g}(X_\mu)\big) \\
&= [\mathfrak{k}_{1-n} + C_\mathfrak{g}(X_\mu) \,:\, \mathfrak{k}_n + C_\mathfrak{g}(X_\mu)]^{-1/2} \\
&= \frac{[C_{\mathfrak{k}_{(1-n)}}(X_\mu) \,:\, C_{\mathfrak{k}_n}(X_\mu)]^{1/2}}{[\mathfrak{k}_{(1-n)} \,:\, \mathfrak{k}_n]^{1/2}}.
\end{aligned}$$

On the other hand,

$$[K_j \,:\, K_n \cdot C_{K_j}(X_\mu)] = \frac{[K_j \,:\, K_n]}{[C_{K_j}(X_\mu) \,:\, C_{K_n}(X_\mu)]}.$$

After putting the two pieces together and doing a bit of calculation, we arrive at

$$\mathcal{O}_\mu([X_\mu + \mathfrak{k}_j]) = \frac{[C_{\mathfrak{k}_0}(X_\mu) \,:\, C_{\mathfrak{k}_j}(X_\mu)]}{[\mathfrak{k}_0 \,:\, \mathfrak{k}_j]} \cdot \frac{[\mathfrak{k}_0 \,:\, \mathfrak{k}_1]^{1/2}}{[C_{\mathfrak{k}_0}(X_\mu) \,:\, C_{\mathfrak{k}_1}(X_\mu)]^{1/2}}.$$

The proposition follows immediately. □

5.6. **Nilpotent orbital integrals are homogeneous.** Suppose $t \in k^\times$. For $f \in C_c^\infty(\mathfrak{g})$ we define the *dilation* $f_t \in C_c^\infty(\mathfrak{g})$ of f by t via $f_t(X) = f(tX)$. For $\mu \in \mathcal{P}(n)$ we have

$$\mathcal{O}_\mu(f_t) = \int_{G/C_G(X_\mu)} f({}^g(tX_\mu))\, dg^*.$$

Consequently, since nilpotent orbits are closed with respect to scaling and the invariant measure on a nilpotent orbit is unique up to a constant, $\mathcal{O}_\mu(f_t)$ and $\mathcal{O}_\mu(f)$ must differ by a constant. A calculation shows that

$$\mathcal{O}_\mu(f_t) = |t|^{\frac{\dim(\mathcal{O}_\mu)}{2}}\, \mathcal{O}_\mu(f).$$

For example, from Proposition 5.5.1 we have

$$\mathcal{O}_\mu([\varpi^{(-2j+1)}X_\mu + \mathfrak{k}_{1-j}]) = 1. \tag{2}$$

6. The Fourier transforms of invariant distributions

6.1. **Basics.** A *distribution* on $\mathfrak{g}$ is any element of the linear dual of $C_c^\infty(\mathfrak{g})$ (no topological restrictions). We denote the subspace of invariant distributions on $\mathfrak{g}$ by $J(\mathfrak{g})$[5].

Suppose $T \in J(\mathfrak{g})$. We define the Fourier transform $\hat{T} \in J(\mathfrak{g})$ of T by

$$\hat{T}(f) := T(\hat{f})$$

for $f \in C_c^\infty(\mathfrak{g})$.

It is a remarkable fact that $\hat{T}$ is represented by a locally integrable function; that is, there is a function $\hat{T} \in L^1_{\mathrm{loc}}(\mathfrak{g})$ such that for all $f \in C_c^\infty(\mathfrak{g})$,

$$\hat{T}(f) = \int_{\mathfrak{g}} \hat{T}(X) \cdot f(X)\, dX.$$

Unfortunately, describing this function is beyond our abilities in all but the simplest situations.

Example 6.1.1. Consider the trivial nilpotent orbital integral $\mathcal{O}_{(1,1,\dots 1)} \in J(\mathfrak{g})$. For $f \in C_c^\infty(\mathfrak{g})$ we have

$$\hat{\mathcal{O}}_{(1,1,\dots,1)}(f) = \hat{f}(0) = \int_{\mathfrak{g}} f(X) \cdot \Lambda(\mathrm{tr}(0 \cdot X))\, dX = \int_{\mathfrak{g}} f(X)\, dX$$

and we also have

$$\hat{\mathcal{O}}_{(1,1,\dots,1)}(f) = \int_{\mathfrak{g}} f(X) \cdot \hat{\mathcal{O}}_{(1,1,\dots,1)}(X)\, dX.$$

We therefore conclude that $\hat{\mathcal{O}}_{(1,1,\dots,1)} = 1$.

We let $\mathfrak{g}^{\text{r.s.s.}}$ denote the set of regular semisimple elements in $\mathfrak{g}$; this is a dense open subset of $\mathfrak{g}$. In our situation, $\mathfrak{g}^{\text{r.s.s.}}$ consists of those elements of $\mathfrak{g}$ having distinct eigenvalues.

The main idea of this section is to present a very elegant description of $\hat{T}(X)$ when $T \in J(\mathfrak{g})$ and $X \in \mathfrak{g}^{\text{r.s.s.}}$. (From this description, it will follow that $\hat{T}$ is represented by a locally constant function on $\mathfrak{g}^{\text{r.s.s.}}$.) The existence (of a form) of this expression was conjectured by Paul Sally, Jr. and proved by Reid Huntsinger [21]. The proof follows an argument of Harish-Chandra [13].

Originally, Huntsinger and Sally were only interested in studying the behavior of the Fourier transform of a nilpotent orbital integral. We shall temporarily restrict our attention

[5]According to Howe, he chose the letter J because I (for invariant) was already taken, and J follows the letter I in the alphabet.

to this situation. Suppose $\mu \in \mathcal{P}(n)$. For $f \in C_c^\infty(\mathfrak{g})$ we have

$$\begin{aligned}\int_{\mathfrak{g}} f(X) \cdot \hat{\mathcal{O}}_\mu(X)\, dX = \hat{\mathcal{O}}_\mu(f) = \mathcal{O}_\mu(\hat{f}) \\ &= \int_{G/C_G(X_\mu)} \hat{f}({}^g X_\mu)\, dg^* \\ &= \int_{G/C_G(X_\mu)} \int_{\mathfrak{g}} f(X) \cdot \Lambda\big(\mathrm{tr}({}^g X_\mu \cdot X)\big)\, dX\, dg^* \\ &\text{``}=\text{''} \int_{\mathfrak{g}} f(X) \int_{G/C_G(X_\mu)} \Lambda\big(\mathrm{tr}({}^g X_\mu \cdot X)\big)\, dg^*\, dX.\end{aligned}$$

So, if we could justify the equality in quotation marks, we'd have

$$\begin{aligned}\hat{\mathcal{O}}_\mu(X) &= \int_{G/C_G(X_\mu)} \Lambda\big(\mathrm{tr}({}^g X_\mu \cdot X)\big)\, dg^* \\ &= \mathcal{O}_\mu\big(Y \mapsto \Lambda(\mathrm{tr}(Y \cdot X))\big).\end{aligned}$$

This is nearly correct; we now describe what is true.

Theorem 6.1.2 (Reid Huntsinger)**.** *Let K be any compact open subgroup of G and let dk denote the normalized Haar measure on K. For all $X \in \mathfrak{g}^{\text{r.s.s.}}$ we have*

$$\hat{\mathcal{O}}_\mu(X) = \mathcal{O}_\mu\big(Y \mapsto \int_K \Lambda(\mathrm{tr}(Y \cdot {}^k X))\, dk\big).$$

Remark 6.1.3. For reasons which will become clear later, we remark that the function $\hat{\mathcal{O}}_\mu$ is canonical. On the face of things, it depends on two choices: the additive character Λ and the choice of a measure on $\mathcal{O}_\mu$. However, our choice of the measure on $\mathcal{O}_\mu$ was not arbitrary; it depended on Λ. As one can verify, this dependence makes $\hat{\mathcal{O}}_\mu$ canonical.

Remark 6.1.4. In our situation, since every nilpotent orbit is Richardson, there is a very nice description, due to Howe, of the function $\hat{\mathcal{O}}_\mu$. Namely, $\hat{\mathcal{O}}_\mu$ can be related to the character of the representation obtained by inducing the trivial representation on $P(\mu)$ up to G.

6.2. **A more general statement.** More generally, following a suggestion of Bob Kottwtiz, Reid Huntsinger proved the following statement.

Theorem 6.2.1 (Reid Huntsinger)**.** *Fix $r \in \mathbb{R}$. If $T \in J(\mathfrak{g}_r)$, then $\hat{T}$ is represented on $\mathfrak{g}^{\text{r.s.s.}}$ by*

$$X \mapsto T(\eta_X).$$

Here, for $Y \in \mathfrak{g}$, $\eta_X(Y) := \int_K \big(\Lambda(\mathrm{tr}(Y \cdot {}^k X))\big)\, dk$. (As before, K is a compact open subgroup, and dk is the normalized Haar measure on K.)

At the heart of the proof of this theorem lies the statement that the map from $\mathfrak{g}^{\text{r.s.s.}}$ to $C^\infty(\mathfrak{g})$ sending X to $\eta_{X,r} := \eta_X \cdot [\mathfrak{g}_r]$ is locally constant. Since the verification of this statement requires some fairly detailed analysis, we shall skip the proof. This statement immediately implies that the function $X \mapsto T(\eta_{X,r}) = T(\eta_X)$ from $\mathfrak{g}^{\text{r.s.s.}}$ to $\mathbb{C}$ is locally constant.

Suppose $f \in C_c^\infty(\mathfrak{g}^{\text{r.s.s.}})$. We need to show

$$\hat{T}(f) = \int_{\mathfrak{g}} f(X) \cdot T(\eta_X)\, dX.$$

From the previous paragraph, there exists a finite collection $\{\omega_i\}_{i=1}^m$ of compact open disjoint subsets of $\mathfrak{g}^{\text{r.s.s.}}$ such that both $X \mapsto T(\eta_X)$ and $X \mapsto f(X)$ are locally constant on ω_i and the support of f is contained in $\cup\omega_i$. Using Fubini's theorem and the invariance of T, we then have[6]

$$\begin{aligned}
\int_{\mathfrak{g}} f(X) \cdot T(\eta_X)\, dX &= \sum_{i=1}^m \text{meas}_{dX}(\omega_i) \cdot f(X_i) \cdot T(\eta_{X_i}) \\
&= T\big(\sum_{i=1}^m \text{meas}_{dX}(\omega_i) \cdot f(X_i) \cdot \eta_{X_i}\big) \\
&= T\big(Y \mapsto \int_{\mathfrak{g}} f(X) \cdot \eta_X(Y)\, dX\big) \\
&= T\big(Y \mapsto \int_{\mathfrak{g}} f(X) \int_K \Lambda(\text{tr}({}^k X \cdot Y))\, dk\, dX\big) \\
&= T\big(Y \mapsto \int_K \hat{f}({}^k Y)\, dk\big) \\
&= T(\hat{f}).
\end{aligned}$$

7. Characters

In this section we define the character of an admissible representation and discuss some properties of characters.

7.1. **Admissible representations.** Fix an admissible representation (π, V) of G. Recall, from Gordan Savin's lectures [33], that (π, V) is a representation for which

(1) for all $v \in V$ there exists a compact open subgroup K such that

$$v \in V^K := \{v \in V \mid \pi(k)v = v \text{ for all } k \in K\}$$

and

(2) for all compact open subgroups K of G we have $\dim_{\mathbb{C}} V^K < \infty$.

The second condition is equivalent to saying that for all compact open subgroups K of G and all irreducible representations σ of K, the multiplicity of σ in π is finite.

Example 7.1.1. Suppose $\bar{\sigma}$ is a cuspidal representation of $\text{GL}_n(\mathfrak{f}) \cong K_0/K_1$. Inflate $\bar{\sigma}$ to a representation of K_0 and extend this inflation to a representation σ of $Z(G) \cdot K_0$ where $Z(G)$ denotes the center of G. The representation (π_σ, V_σ) obtained by (compact) induction of σ from $Z(G) \cdot K_0$ up to G is an (irreducible) admissible representation.

[6]This part of the proof supports David Vogan's adage: "The p-adics: if you can add, you can integrate."

Example 7.1.2. Suppose $m > 0$ and $\bar{\chi}$ is a character of T/T_m. Inflate $\bar{\chi}$ to a character χ of T. Since the unipotent radical N of B is normal in B, we may extend χ to a character of B. The representation (π_χ, V_χ) obtained by (compact) parabolic induction of χ from B to G is an admissible representation of G.

7.2. **The character distribution.** We define the *character distribution*, Θ_π, associated to the representation (π, V) as follows.

For $f \in C_c^\infty(G)$, we define $\pi(f) \in \mathrm{End}(V)$ by

$$\pi(f)v = \int_G f(g) \cdot \pi(g)v \, dg$$

for $v \in V$. Since f is compactly supported and locally constant and (π, V) is admissible, this integral is really just a finite sum. In fact, if K is a compact open subgroup of G such that $f(k_1 g k_2) = f(g)$ for all $k_1, k_2 \in K$ and $g \in G$, then we have

$$\pi(f)v = \pi([K] * f * [K])v = \pi([K])\pi(f)\pi([K])v.$$

(Here $*$ denotes the usual convolution operation.) Thus, $\pi(f)$ is a map from V to V^K. If $e_K := \mathrm{meas}_{dg}(K)^{-1} \cdot \pi([K])$, then e_K is the projection operator from V to V^K, and we have

$$V = V^K \oplus (1 - e_K)V.$$

Consequently, since $\dim_{\mathbb{C}}(V^K) < \infty$, it follows that $\pi(f)$ is a finite-rank operator. We can therefore define the *character distribution*

$$\Theta_\pi \colon C_c^\infty(G) \to \mathbb{C}$$

by sending f to $\mathrm{tr}(\pi(f))$.

Just as the Fourier transform of an invariant distribution on $\mathfrak{g}$ is represented on $\mathfrak{g}^{\text{r.s.s.}}$ by a function in $C^\infty(\mathfrak{g}^{\text{r.s.s.}})$, so too is the character distribution [15]. We abuse notation and denote this function by $\Theta_\pi \in C^\infty(G^{\text{r.s.s}})$. ($G^{\text{r.s.s}}$ denotes the open, dense subset of G consisting of regular semisimple elements, that is, those elements of G whose eigenvalues are distinct.) We call this function the *character* of π.

Remark 7.2.1. Fix $\gamma \in G^{\text{r.s.s}}$ and $n > 0$ large enough so that $\gamma K_n \subset G^{\text{r.s.s}}$. Define $f_n = \mathrm{meas}_{dg}(K_n)^{-1} \cdot [\gamma K_n]$. We then have

$$\Theta_\pi(f_n) = \int_G f_n(g) \cdot \Theta_\pi(g) \, dg = \mathrm{meas}_{dg}(K_n)^{-1} \int_{K_n} \Theta_\pi(\gamma k) \, dk.$$

Since the function Θ_π is locally constant on $G^{\text{r.s.s}}$, the right-hand side becomes $\Theta_\pi(\gamma)$ for all n sufficiently large. Thus, “in the limit” the function Θ_π looks like a character.

7.3. **A calculation.** Fix $j \geq 1$.

Suppose that $\bar{\tau}$ is an irreducible representation of K_j/K_{2j}. Since K_j/K_{2j} is abelian, $\bar{\tau}$ is a character. Let τ denote the corresponding character of K_j; we regard τ as an element of $C_c^\infty(G)$ in the obvious way.

Let $\mathrm{res}_{K_j} \pi$ denote the restriction of π to K_j. Suppose that

$$\mathrm{res}_{K_j} \pi = \oplus_{\sigma \in \widehat{K_j}} m(\sigma)\sigma$$

where $m(\sigma)$ denotes the multiplicity of σ in $\mathrm{res}_{K_j}\pi$. We have

$$\begin{aligned}\Theta_\pi(\tau) &= \mathrm{tr}\int_G \tau(g)\cdot\pi(g)\,dg \\ &= \mathrm{tr}\int_{K_j}\tau(g)\cdot\pi(g)\,dg \\ &= \mathrm{tr}\int_{K_j}\sum_{\sigma\in\widehat{K_j}}\tau(g)\cdot m(\sigma)\sigma(g)\,dg \\ &= \mathrm{tr}\int_{K_j}\sum_{\bar\sigma\in\widehat{K_j/K_{2j}}}\tau(g)\cdot m(\sigma)\sigma(g)\,dg \\ &= m(\tau^{-1},\pi)\cdot\mathrm{meas}_{dg}(K_j).\end{aligned}$$

In other words, the character picks out the multiplicity of τ^{-1} in $\mathrm{res}_{K_j}\pi$ (up to a constant).

7.4. **Depth.** The depth of a representation was introduced in the fundamental papers of Allen Moy and Gopal Prasad [28, 27]. Essentially, the depth of a representation is a rational number which measures the first occurrence of fixed vectors with respect to all filtration subgroups of G that arise naturally from Bruhat-Tits theory.

Recall from §3.4.2 that $\mathcal{O}$ denotes the set of optimal points. Up to conjugation, the set of optimal points is finite, and so the subset $\{r\in\mathbb{R}\,|\,\mathfrak{g}_{x,r}\neq\mathfrak{g}_{x,r^+}$ for some $x\in\mathcal{O}\}$ is discrete (and in fact, is a subset of $\mathbb{Q}$). For an admissible representation (π,V), we define $\rho(\pi)$, the depth of (π,V), by

$$\rho(\pi) := \min\{r\in\mathbb{Q}_{\geq 0}\mid \text{ there is an } x\in\mathcal{O} \text{ such that } V^{G_{x,r^+}}\neq 0\}.$$

Proposition 7.4.1. *$\rho(\pi)$ is the unique rational number satisfying the following statement. If $(x,r)\in\mathcal{A}\times\mathbb{R}_{\geq 0}$ with $V^{G_{x,r^+}}\neq\{0\}$, then $r\geq\rho(\pi)$.*

Proof. Suppose $(x,r)\in\mathcal{A}\times\mathbb{R}_{\geq 0}$ with $V^{G_{x,r^+}}\neq\{0\}$. From the group version of Lemma 3.4.3 there exists $y\in\mathcal{O}$ such that $G_{x,r^+}\supset G_{y,r^+}$. Consequently, $\{0\}\neq V^{G_{x,r^+}}\subset V^{G_{y,r^+}}$. □

All the various things you would want to be true about the depth of a representation are true. For example, If σ is an irreducible representation of a parabolic subgroup of G, and π is an irreducible subquotient of the induced representation, then $\rho(\pi)=\rho(\sigma)$.

Example 7.4.2. Any representation with Iwahori (that is, B_0) fixed vectors has depth zero.

Example 7.4.3. The representation (π_σ,V_σ) defined in Example 7.1.1 does not have Iwahori fixed vectors, but it does have depth zero.

Example 7.4.4. The representation (π_χ,V_χ) defined in Example 7.1.2 has depth $(m-1)$.

7.5. **Elementary Kirillov theory.** Fix an irreducible representation (π, V).

For nilpotent real groups, Kirillov [24] established that the representations were parameterized by coadjoint orbits in the linear dual of the Lie algebra. In our context, the term *Kirillov theory* is used to describe the connection between representations of compact open subgroups occurring in π and coadjoint orbits in the linear dual of $\mathfrak{g}$. Via our nondegenerate trace form, we have identified $\mathfrak{g}$ with its dual. The type of result discussed below was first studied, I believe, by Howe [17].

Fix $r, s \in \mathbb{R}_{\geq 0}$ and $x, y \in \mathcal{B}$.

The map $g \mapsto (g-1)$ induces an isomorphism of the abelian groups $G_{x,r^+}/G_{x,(2r)^+}$ and $\mathfrak{g}_{x,r^+}/\mathfrak{g}_{x,(2r)^+}$. Moreover, every character of $\mathfrak{g}_{x,r^+}/\mathfrak{g}_{x,(2r)^+}$ is of the form $\bar{Y} \mapsto \Lambda(\mathrm{tr}(X \cdot Y)$ for some $\bar{X} \in \mathfrak{g}_{x,-2r}/\mathfrak{g}_{x,-r}$. (That is, the Pontrjagin dual of $G_{x,r^+}/G_{x,(2r)^+} \cong \mathfrak{g}_{x,r^+}/\mathfrak{g}_{x,(2r)^+}$ is isomorphic to $\mathfrak{g}_{x,-2r}/\mathfrak{g}_{x,-r}$.)

Fix a character $\bar{\sigma}$ of $G_{x,r^+}/G_{x,(2r)^+}$ and a character $\bar{\tau}$ of $G_{y,s^+}/G_{y,(2s)^+}$ Let $X_\sigma + \mathfrak{g}_{x,-r} \in \mathfrak{g}_{x,-2r}/\mathfrak{g}_{x,-r}$ represent $\bar{\sigma}$ and $X_\tau + \mathfrak{g}_{y,-s} \in \mathfrak{g}_{y,-2s}/\mathfrak{g}_{y,-s}$ represent $\bar{\tau}$.

Proposition 7.5.1. *If σ and τ both occur in π, then there exists a $g \in G$ such that*

$$^g(X_\tau + \mathfrak{g}_{y,-s}) \cap (X_\sigma + \mathfrak{g}_{x,-r}) \neq \emptyset.$$

Proof. Let $V_\sigma \subset V$ (resp. $V_\tau \subset V$) denote a one-dimensional subspace of V on which $\mathrm{res}_{G_{x,r^+}} \pi$ (resp. $\mathrm{res}_{G_{y,s^+}} \pi$) acts by σ (resp. τ). Since (π, V) is irreducible, there exists a $g \in G$ such that the image of $\pi(g)V_\sigma$ under the projection of V onto V_τ is nonzero. This implies that for all $h \in G_{x,r^+} \cap {}^gG_{y,s^+}$, we have

$$\sigma(h) = \tau(g^{-1}hg).$$

Thus, for all $H \in \mathfrak{g}_{x,r^+} \cap {}^g\mathfrak{g}_{y,s^+}$, we have

$$\Lambda(\mathrm{tr}(X_\sigma \cdot H)) = \Lambda(\mathrm{tr}({}^gX_\tau \cdot H)).$$

This implies that for all $H \in \mathfrak{g}_{x,r^+} \cap {}^g\mathfrak{g}_{y,s^+}$, we have $\mathrm{tr}((X_\sigma - {}^gX_\tau) \cdot H) \in \wp$. Consequently, $(X_\sigma - {}^gX_\tau) \in (\mathfrak{g}_{x,r^+} \cap {}^g\mathfrak{g}_{y,s^+})^* = \mathfrak{g}_{x,-r} + {}^g\mathfrak{g}_{y,-s}$. The proposition follows. $\square$

7.6. **Understanding the distribution** $\mathrm{res}_{C_c^\infty(G_{\rho(\pi)^+})} \Theta_\pi$**.** Let (π, V) be an irreducible admissible representation.

From our discussion of depth, we know that we can find an $x \in \mathcal{A}$ such that $V^{G_{x,\rho(\pi)^+}} \neq \{0\}$. Thus, the trivial representation of $G_{x,\rho(\pi)^+}$ occurs in π; the associated coset in $\mathfrak{g}$ is $\mathfrak{g}_{x,-\rho(\pi)}$.

Now fix $s > \rho(\pi)$, $y \in \mathcal{B}$, and $\bar{\tau} \in \widehat{G_{y,s}/G_{y,s^+}}$ such that the character τ of $G_{y,s}$ occurs in π. Let $X_\tau + \mathfrak{g}_{y,(-s)^+} \in \mathfrak{g}_{y,(-s)}/\mathfrak{g}_{y,(-s)^+}$ be the coset corresponding to $\bar{\tau}$.

From our discussion of Kirillov theory, there exists a $g \in G$ such that ${}^g\mathfrak{g}_{x,-\rho(\pi)} \cap (X_\tau + \mathfrak{g}_{y,(-s)^+}) \neq \emptyset$. However, from Lemma 4.2.1, we have

$${}^g\mathfrak{g}_{x,-\rho(\pi)} \subset \mathfrak{g}_{y,-\rho(\pi)} + \mathcal{N} \subset \mathfrak{g}_{y,(-s)^+} + \mathcal{N}.$$

Consequently, we can assume that X_τ is nilpotent!

We now use this fact to say something about the distribution $\mathrm{res}_{C_c^\infty(G_{\rho(\pi)^+})} \Theta_\pi$. For a function $h \in C_c^\infty(\mathfrak{g}_{\rho(\pi)^+})$, we define $\tilde{h} \in C_c^\infty(G_{\rho(\pi)^+})$ by $\tilde{h}(g) = h(g-1)$. We define the distribution $\Theta_{\pi,\mathfrak{g}}$ on $\mathfrak{g}$ by $\Theta_{\pi,\mathfrak{g}}(f) = \Theta_\pi(\tilde{f}_{\rho(\pi)})$ where $f_{\rho(\pi)} = f \cdot [\mathfrak{g}_{\rho(\pi)+}]$. We let $\hat{\Theta}_\pi$ denote the Fourier transform of $\Theta_{\pi,\mathfrak{g}}$.

From Exercise 4.2.3, we know that the Fourier transform of an element of $D_{-\rho(\pi)}$ belongs to $C_c^\infty(\mathfrak{g}_{\rho(\pi)+})$. Suppose $x \in \mathcal{B}$ and $s > \rho(\pi)$. If $f \in C(\mathfrak{g}_{x,-s}/\mathfrak{g}_{x,-\rho(\pi)})$, then we have $\hat{\Theta}_\pi(f) = \Theta_\pi(\tilde{\hat{f}})$. It follows from our discussion above and a few lines of calculation that

$$\hat{\Theta}_\pi(f) \neq 0 \text{ implies } \operatorname{supp}(f) \cap (\mathfrak{g}_{x,(-s)^+} + \mathcal{N}) \neq \emptyset.$$

This last statement is equivalent to the statement

$$\hat{\Theta}_\pi(f) \neq 0 \text{ implies } \operatorname{supp}(f) \cap \mathfrak{g}_{(-s)^+} \neq \emptyset. \tag{3}$$

7.7. **The Harish-Chandra–Howe local character expansion.** Considerations similar to those above led Roger Howe to make his finiteness conjectures [19] (which we will discuss later) and establish the following remarkable connection between the character of an irreducible representation of G and the Fourier transforms of nilpotent orbital integrals.

Theorem 7.7.1 (Harish-Chandra–Howe local character expansion)**.** *If π is an irreducible admissible representation of G, then there exist constants $c_\mu(\pi)$ indexed by $\mu \in \mathcal{P}(n)$ such that*

$$\Theta_\pi(1+X) = \sum_{\mu \in \mathcal{P}(n)} c_\mu(\pi) \cdot \hat{\mathcal{O}}_\mu(X) \tag{4}$$

for all $X \in \mathfrak{g}^{\text{r.s.s.}}$ sufficiently near zero.

Remark 7.7.2. Howe [16] proved this result without any restrictions on the characteristic of k. Later, under the assumption that the characteristic of k is zero, Harish-Chandra [13] generalized the proof to all connected reductive groups. The above theorem and its analogues (local expansions about any semisimple point) play a crucial role in Harish-Chandra's proof that characters are locally integrable on G (not just on $G^{\text{r.s.s}}$). (The question of integrability is still open for arbitrary groups in positive characteristic. From work of Rodier [30] and Lemaire [25], it is known to be true for $\mathrm{GL}_n(k)$ when k has positive characteristic.)

One of the difficulties with this theorem is that it provides no indication of where equation (4) ought to hold. The conjecture of Hales, Moy, and Prasad [28] is a precise statement about this issue:

Statement 7.7.3. *Equation (4) ought to be valid on $\mathfrak{g}^{\text{r.s.s}}_{\rho(\pi)+}$.*

For integral-depth representations this was proved by Waldspurger [35]. With some restrictions on k, it was proved for arbitrary depth representations in [9].

8. An introduction to Howe's conjectures and homogeneity

In his paper, *Two conjectures about reductive p-adic groups*, Howe proposed two remarkable finiteness conjectures that now bear his name. Howe's conjecture for the Lie algebra looks like:

$$\dim_{\mathbb{C}} \operatorname{res}_{C_c(\mathfrak{g}/L)} J(\omega) < \infty.$$

Here $\omega \subset \mathfrak{g}$ is any compactly generated, invariant, and closed subset of $\mathfrak{g}$, $J(\omega)$ denotes the space of invariant distributions supported on ω, and L is any lattice in $\mathfrak{g}$. For $T \in J(\omega)$,

$$\operatorname{res}_{C_c(\mathfrak{g}/L)} T$$

denotes the restriction of T to $C_c(\mathfrak{g}/L)$. Howe's conjecture for the group looks very similar:

$$\dim_{\mathbb{C}} \operatorname{res}_{C_c(G/K)} J(\omega) < \infty.$$

Here K is a compact open subgroup of G, and ω is a closed, compactly generated, invariant subset of G, and $J(\omega)$ denotes the space of invariant distributions which are supported on ω.

Of course, although we still refer to these statements as Howe's conjectures, both are known to be true. Howe's conjecture for the Lie algebra was proved in the 1970s by Howe [16] and later by Harish-Chandra [13] for general groups (but only in the characteristic zero setting). Waldspurger [37] has also given a proof. Howe's conjecture for the group was proved in the 1980s by Clozel [6, 7] in the characteristic zero setting. In the 1990s, a characteristic free proof of Howe's conjecture for the group was given by Barbasch and Moy in their beautiful paper [2].

8.1. **Homogeneity.**

Beginning with Waldspurger's paper [36], much energy has been spent trying to produce optimal versions of Howe's conjectures. By optimal, we mean that we'd like to choose ω and L in such a way so that we can not only describe the dimension of $\operatorname{res}_{C_c(\mathfrak{g}/L)} J(\omega)$, but we can also find a good basis for this space in terms of distributions with which we are very familiar, namely, nilpotent orbital integrals.

These optimal versions of Howe's conjectures are referred to as *homogeneity results*. This is an appropriate name, because, according to Webster's Ninth New Collegiate Dictionary, the word homogeneity means

> "the state of having identical distribution functions or values".

We restrict our attention to the Lie algebra. Fix $r \in \mathbb{R}$. The role of ω in Howe's conjecture will be played by the G-domain $\mathfrak{g}_{r+}$. Guided by Harish-Chandra's philosophy and Theorem 7.7.1, it appears that we want something like the following: For all $T \in J(\mathfrak{g}_{r+})$ and for all $f \in C_c^\infty(\mathfrak{g}_{(-r)})$

$$\hat{T}(f) = \sum_{\mu \in \mathcal{P}(n)} c_\mu(T) \cdot \hat{\mathcal{O}}_\mu(f).$$

This last displayed equation is equivalent to requiring

$$T(\hat{f}) = \sum_{\mu \in \mathcal{P}(n)} c_\mu(T) \cdot \mathcal{O}_\mu(\hat{f}).$$

From Exercise 4.2.3 we have $\hat{f} \in D_{r^+}$. So, the role of $C_c^\infty(\mathfrak{g}/L)$ will be played by D_{r^+}. Putting it all together, we arrive at the homogeneity statement

$$\operatorname{res}_{D_{r^+}} J(\mathfrak{g}_{r^+}) = \operatorname{res}_{D_{r^+}} J(\mathcal{N}).$$

The fact that $\mathfrak{g}_{r^+} \subset \mathfrak{g}_{x,r^+} + \mathcal{N}$ for all $x \in \mathcal{B}$ gives some feeling as to why this homogeneity statement ought to be true.

8.2. **From Howe's conjectures to the Harish-Chandra–Howe local character expansion.** The homogeneity result discussed in the previous section is actually not strong enough to prove Theorem 7.7.1. As in [16, 13, 9, 35] we need something a bit stronger, but, unfortunately, more complicated.

For $x \in \mathcal{B}$ and $s \leq r$, define

$$\tilde{J}_{x,s,r^+} := \{T \in J(\mathfrak{g}) \mid \text{for } f \in C(\mathfrak{g}_{x,s}/\mathfrak{g}_{x,r^+}), \text{ if } \operatorname{supp}(f) \cap (\mathfrak{g}_{s^+}) = \emptyset, \text{ then } T(f) = 0\}.$$

Note that $J(\mathfrak{g}_{r^+}) \subset \tilde{J}_{x,s,r^+}$.

Remark 8.2.1. Although this definition seems a bit unnatural, it arises naturally from our understanding of $\operatorname{res}_{C_c^\infty(G_{\rho(\pi)^+})} \Theta_\pi$. In fact, from Equation (3), we have that $\hat{\Theta}_\pi \in \tilde{J}_{x,s,r^+}$ if $-r > \rho(\pi)$.

The following is the stronger homogeneity statement that needs to be proved in order to recover Statement 7.7.3:

$$\operatorname{res}_{D_{r^+}} \tilde{J}_{r^+} = \operatorname{res}_{D_{r^+}} J(\mathcal{N})$$

where

$$\tilde{J}_{r^+} := \bigcap_{x \in \mathcal{B}} \bigcap_{s \leq r} \tilde{J}_{x,s,r^+}.$$

9. Proving Homogeneity results

We recall that for $r \in \mathbb{R}$ we have

$$\mathfrak{g}_{r^+} = \bigcup_{x \in \mathcal{B}} \mathfrak{g}_{x,r^+} = \bigcap_{x \in \mathcal{B}} (\mathfrak{g}_{x,r^+} + \mathcal{N})$$

and

$$D_{r^+} = \sum_{x \in \mathcal{B}} C_c(\mathfrak{g}/\mathfrak{g}_{x,r^+}) = \widehat{C_c^\infty(\mathfrak{g}_{-r})}.$$

We want to show

$$\operatorname{res}_{D_{r^+}} J(\mathfrak{g}_{r^+}) = \operatorname{res}_{D_{r^+}} J(\mathcal{N}). \tag{5}$$

We decided that knowing this, or, in truth, a stronger (but more complicated) statement would tell us that the Harish-Chandra–Howe local character expansion was valid on $\mathfrak{g}^{\text{r.s.s}}_{\rho(\pi)^+}$.

As we discussed before, $\mathfrak{g}_r = \mathfrak{g}_{r^+}$ unless $r = \frac{k}{n}$ for some $k \in \mathbb{Z}$. Consequently, we only need to verify Equation 5 for $r \in \{\frac{k}{n}\}$.

Moreover, by taking advantage of the fact that nilpotent orbital integrals are homogeneous, we can further restrict our attention to r of the form $\frac{k}{n}$ with $0 \leq k < n$. Indeed, pick

$m \in \mathbb{Z}$ such that $m + r \in [0, 1)$. For $T \in J(\mathfrak{g})$ and $t \in k^\times$, define T_t by $T_t(f) = T(f_t)$. Note that

$$f \in D_{r+} \text{ if and only if } f_{\varpi^{-m}} \in D_{(m+r)+}$$

and

$$T \in J(\mathfrak{g}_{r+}) \text{ if and only if } T_{\varpi^m} \in J(\mathfrak{g}_{(m+r)+}).$$

Consequently, if we know

$$\mathrm{res}_{D_{(r+m)+}} J(\mathfrak{g}_{(r+m)+}) = \mathrm{res}_{D_{(r+m)+}} J(\mathcal{N})$$

then for $T \in J(\mathfrak{g}_{r+})$ and $f \in D_{r+}$ we have

$$\begin{aligned} T(f) &= T_{\varpi^m}(f_{\varpi^{-m}}) \\ &= \sum c_\mu(T_{\varpi^m}) \cdot \mathcal{O}_\mu(f_{\varpi^{-m}}) \\ &= \sum c_\mu(T_{\varpi^m}) \cdot q^{\frac{m \cdot \dim(\mathcal{O}_\mu)}{2}} \cdot \mathcal{O}_\mu(f), \end{aligned}$$

and so we may assume $r \in (0, 1]$.

For the remainder of this section we discuss how to prove Equation (5). We first prove it for $\mathrm{GL}_1(k)$, we next prove it for $\mathrm{GL}_2(k)$, and finally we discuss a way to go about giving a general proof.

9.1. **A proof for $\mathrm{GL}_1(k)$.** From the reasoning above, we need to show

$$\mathrm{res}_{C_c(k/\wp)} J(\wp) = \mathrm{res}_{C_c(k/\wp)} J(\mathcal{N}).$$

The right-hand side is one-dimensional and spanned by the distribution $f \mapsto f(0)$ for $f \in C_c(k/\wp)$. Moreover, the right-hand side is a vector subspace of the left-hand side.

Since $\mathrm{GL}_1(k)$ is abelian, all distributions are invariant. Note that $J(\wp)$ consists of linear maps from $C_c^\infty(k)$ to $\mathbb{C}$ which are supported on $\wp$. If $f \in C_c(k/\wp)$, then we can write

$$f = \sum_{\bar{X} \in k/\wp} c_{\bar{X}} \cdot [X + \wp]$$

where the $c_{\bar{X}}$ are complex numbers which are almost always zero. For $T \in J(\wp)$, we have

$$T(f) = T(c_{\bar{0}} \cdot [\wp]) = T([\wp]) \cdot f(0).$$

Consequently, the left-hand side is also one-dimensional and so the equality is established.

9.2. **A proof for $\mathrm{GL}_2(k)$.** This is where things begin to become interesting. Thanks to the remarks at the beginning of this section, we only need to verify two statements:

$$\mathrm{res}_{D_{0+}} J(\mathfrak{g}_{0+}) = \mathrm{res}_{D_{0+}} J(\mathcal{N}) \tag{6}$$

and

$$\mathrm{res}_{D_{(1/2)+}} J(\mathfrak{g}_{(1/2)+}) = \mathrm{res}_{D_{(1/2)+}} J(\mathcal{N}). \tag{7}$$

We begin by considering the first statement, Equation (6). We note that D_{0+} may be thought of as an invariant version of $C_c(\mathfrak{g}/\mathfrak{k}_1)$, but, for our purposes, this is <u>not</u> a good way to think about the space D_{0+}.

9.2.1. *Descent and recovery.* Fix $T \in J(\mathfrak{g}_{0^+})$. We wish to show that $\mathrm{res}_{D_{0^+}} T$ is completely determined by $\mathrm{res}_{C(\mathfrak{k}_0/\mathfrak{k}_1)+C(\mathfrak{b}_0/\mathfrak{b}_{1/2})} T$.

Fix $f \in D_{0^+}$. We will demonstrate that $T(f)$ is completely determined by $\mathrm{res}_{C(\mathfrak{k}_0/\mathfrak{k}_1)+C(_0/\mathfrak{b}_{1/2})} T$. We write $f = \sum_i f_i$ with $f_i \in C_c(\mathfrak{g}/\mathfrak{g}_{x_i,0^+})$ for some $x_i \in \mathcal{B}$. Since T is linear, without loss of generality we may assume that $f \in C_c(\mathfrak{g}/\mathfrak{g}_{x,0^+})$ for some $x \in \mathcal{B}$. We can write

$$f = \sum_{\bar{X} \in \mathfrak{g}/\mathfrak{g}_{x,0^+}} c_{\bar{X}} \cdot [X + \mathfrak{g}_{x,0^+}]$$

with the $c_{\bar{X}}$ complex constants which are almost always zero. Again, since T is linear, without loss of generality we may assume that $f = [X + \mathfrak{g}_{x,0^+}]$.

Now, $T(f) = 0$ if the support of f does not intersect $\mathfrak{g}_{0^+}$. Consequently, since $\mathfrak{g}_{0^+} \subset \mathfrak{g}_{x,0^+} + \mathcal{N}$, we must have

$$(X + \mathfrak{g}_{x,0^+}) \cap \mathcal{N} \neq \emptyset.$$

So, without loss of generality, $X \in \mathcal{N}$.

Up to conjugacy, we have two choices for $\mathfrak{g}_{x,0^+}$; it is either $\mathfrak{k}_1$ or $\mathfrak{b}_{1/2}$. In what follows, the reader is encouraged to consult Figure 8 to get a more geometric understanding of what is happening.

We first deal with the $\mathfrak{k}_1$ case. Since x_0 is the only point x where $\mathfrak{g}_{x,0^+} = \mathfrak{k}_1$, we may suppose that $X \in \mathcal{N} \cap (\mathfrak{g}_{x_0,-m} \smallsetminus \mathfrak{g}_{x_0,(-m)^+})$ for some $m > 0$. In other words, $X \in \mathcal{N} \cap (\mathfrak{k}_{-m} \smallsetminus \mathfrak{k}_{1-m})$. Since we are free to conjugate by $G_{x_0,0} = K_0$, we may assume that

$$X = \left(\begin{smallmatrix} 0 & \varpi^{-m}u \\ 0 & 0 \end{smallmatrix}\right)$$

with $u \in R^\times$. For any point y in the chamber C_0 (that is, a point between x_0 and x' in Figure (8)), we have that X is "closer" to the origin with respect to the y filtration than it is in the x filtration. For example, $X \in \mathfrak{g}_{y_0,(1/2-m)^+}$ where y_0 is the barycenter of C_0. The problem is: at the point y we require local constancy not with respect to $\mathfrak{g}_{x_0,0^+}$ but with respect to $\mathfrak{g}_{y,0^+} = \mathfrak{b}_{1/2}$. In order to recover the proper type of local constancy, we take advantage of the invariance of T. We write

$$\begin{aligned} T([X + \mathfrak{k}_1]) &= \frac{1}{q^2} \cdot \sum_{\bar{t} \in T_m/T_{m^+}} ([{}^tX + \mathfrak{k}_1]) \\ &= \frac{1}{q^2} \cdot \sum_{\bar{\alpha},\bar{\beta} \in R/\wp} ([X + \left(\begin{smallmatrix} 0 & u(\alpha-\beta) \\ 0 & 0 \end{smallmatrix}\right) + \mathfrak{k}_1]) \\ &= \frac{1}{q^2} \cdot T([X + \mathfrak{b}_{1/2}]). \end{aligned}$$

We have succeeded in writing $T(f)$ in terms of T evaluated at a function $f' \in D_{0^+}$ which is supported closer to the origin with respect to some other point in the building.

We now examine the $\mathfrak{b}_{1/2}$ case. In this case, we are looking at the coset $X + \mathfrak{g}_{y,0^+}$ where $X \in \mathcal{N}$ and y is any point in C_0. Free to conjugate by $G_{y,0} = B_0$, we may assume that X is either

$$\left(\begin{smallmatrix} 0 & \varpi^{-m}u \\ 0 & 0 \end{smallmatrix}\right)$$

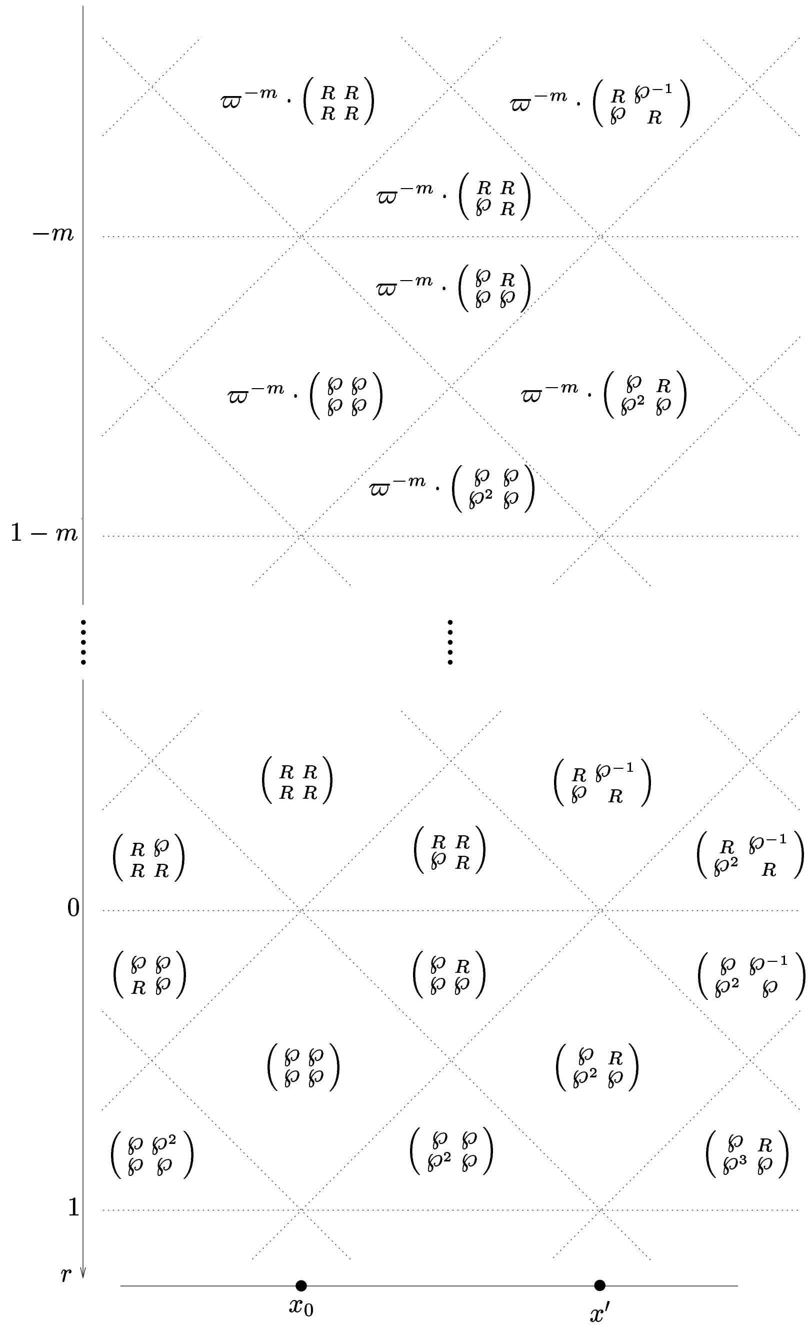

FIGURE 8. $(\mathfrak{gl}_2)_{x,r}$

or

$$\begin{pmatrix} 0 & 0 \\ \varpi^{(1-m)}u & 0 \end{pmatrix}$$

with $m > 0$ and $u \in R^\times$. In the former case, we can write

$$T([X + \mathfrak{g}_{y,0^+}]) = \sum_{\bar{\alpha} \in \wp/\wp^2} T([X + \left(\begin{smallmatrix} 0 & 0 \\ \alpha & 0 \end{smallmatrix}\right) + \mathfrak{g}_{x_1,0^+}])$$

where $x_1 = {}^{\mathrm{diag}(\varpi^{-1},1)}x_0$ is the other vertex of $\bar{C}_0$. We have

$$\mathfrak{g}_{x_1,0^+} = \left(\begin{smallmatrix} \wp & R \\ \wp^2 & \wp \end{smallmatrix}\right).$$

From Figure 8 it is clear that we have expressed T evaluated at $[X + \mathfrak{g}_{y,0^+}]$ in terms of T evaluated at f' where $f' \in D_{0^+}$ has support closer to the origin with respect to the x_1 filtration than $[X + \mathfrak{g}_{y,0^+}]$ had with respect to the y filtration.

Exercise 9.2.1. Do the analogous analysis for the latter case.

To summarize, the point of *descent and recovery* is as follows. We begin with a simple function $f \in C((\mathfrak{g}_{x,s} \smallsetminus \mathfrak{g}_{x,s^+})/\mathfrak{g}_{x,0^+})$ for some $x \in \mathcal{B}$. From this function, we find a point $y \in \mathcal{B}$ and a function $f' \in C(\mathfrak{g}_{y,s^+}/\mathfrak{g}_{y,0^+})$ so that $T(f) = T(f')$. After a finite number of steps, we will have shown that $T(f)$ is completely determined by $\mathrm{res}_{C(\mathfrak{k}_0/\mathfrak{k}_1)+C(\mathfrak{b}_0/\mathfrak{b}_{1/2})} T$.

9.2.2. *Counting.* We now know the following facts:

(1) Thanks to Harish-Chandra [13], the dimension of the complex vector space $\mathrm{res}_{D_{0^+}} J(\mathcal{N})$ is equal to the cardinality of $\mathcal{P}(n)$, which, in this case, is 2.
(2) From §4.4, we have $J(\mathcal{N}) \subset J(\mathfrak{g}_{0^+})$ and so $\mathrm{res}_{D_{0^+}} J(\mathcal{N}) \subset \mathrm{res}_{D_{0^+}} J(\mathfrak{g}_{0^+})$.
(3) From the previous section, we know that

$$\dim_{\mathbb{C}} \mathrm{res}_{D_{0^+}} J(\mathfrak{g}_{0^+}) = \dim_{\mathbb{C}} \mathrm{res}_{C(\mathfrak{k}_0/\mathfrak{k}_1)+C(\mathfrak{b}_0/\mathfrak{b}_{1/2})} J(\mathfrak{g}_{0^+}).$$

Consequently, we need only show that

$$\dim_{\mathbb{C}} \mathrm{res}_{C(\mathfrak{k}_0/\mathfrak{k}_1)+C(\mathfrak{b}_0)/\mathfrak{b}_{1/2})} J(\mathfrak{g}_{0^+}) \leq 2.$$

Since $\mathfrak{g}_{0^+} \subset \mathfrak{g}_{x,0^+} + \mathcal{N}$ for any $x \in \mathcal{B}$, we have that for $T \in J(\mathfrak{g}_{0^+})$, $\mathrm{res}_{C(\mathfrak{k}_0/\mathfrak{k}_1)+C(\mathfrak{b}_0)/\mathfrak{b}_{1/2})} T$ is completely determined by

$$T([\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) + \mathfrak{k}_1]), \quad T([\mathfrak{k}_1]), \quad \text{and} \quad T([\mathfrak{b}_{1/2}]).$$

Since

$$T([\mathfrak{b}_{1/2}]) = (q-1) \cdot T([\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) + \mathfrak{k}_1]) + T([\mathfrak{k}_1]),$$

we are done.

Exercise 9.2.2. Using the above proof as a template, prove Equation (7).

9.3. **The general approach.** In general, the proof is very much like that produced above, only more complicated.

9.3.1. *Descent and recovery.* Suppose $T \in J(\mathfrak{g}_{r+})$. The main point is to show that

$$\operatorname{res}_{D_{r+}} T = 0 \text{ if and only if } \operatorname{res}_{D^r_{r+}} T = 0$$

where

$$D^r_{r+} := \sum_{x \in \mathcal{B}} C(\mathfrak{g}_{x,r}/\mathfrak{g}_{x,r^+}).$$

That is, we can find a very small space of functions from which we can choose a dual basis for $\operatorname{res}_{D_{r+}} J(\mathfrak{g}_{r+})$.

Suppose $f \in D_{r+}$. As before, without loss of generality we may assume that $f = [X + \mathfrak{g}_{x,r^+}]$ for some $X \in \mathcal{N}$ and some $x \in \mathcal{B}$. Under some hypotheses, we can use $\mathfrak{sl}_2(\mathfrak{f})$-theory to find a direction in the building in which to move so that, for some $y \in \mathcal{B}$ in that direction, the support of f is nearer the origin with respect to y than it was with respect to x. Moreover, if necessary, we can use the basics of $\mathfrak{sl}_2(\mathfrak{f})$-representation theory and the invariance of T to "beef-up" the coset $\mathfrak{g}_{x,r^+}$ so that we arrive at a function $f' \in C_c(\mathfrak{g}/\mathfrak{g}_{y,r^+})$ with the properties:

(1) $T(f') = T(f)$ and
(2) the support of f' is "closer" to the origin with respect to y than the support of f with respect to x.

Remark 9.3.1. Invoking $\mathfrak{sl}_2(\mathfrak{f})$-theory requires some restrictions. The theory only works well if the highest weights for the representations one wishes to consider are less than $(p-3)$ where p is the characteristic of $\mathfrak{f}$.

9.3.2. *Counting.* This is the real key. In the end, you need some type of correspondence between "suitable" elements of D^r_{r+} and $\mathcal{O}(0)$, the set of nilpotent orbits in $\mathfrak{g}$.

When r is zero, the situation is easy to understand. As discussed before, the conjugacy classes of facets in $\mathcal{B}$ are in one-to-one correspondence with the elements of $\mathcal{P}(n)$. Similarly, the nilpotent orbits are in one-to-one correspondence with the elements of $\mathcal{P}(n)$. For $\mu \in \mathcal{P}(n)$, let F_μ be the facet described in §3.4.1. The image of X_μ in $\mathfrak{g}_{F_\mu,0}/\mathfrak{g}_{F_\mu,0^+}$ is distinguished nilpotent (that is, it does not lie in a proper Levi subalgebra of $\mathfrak{g}_{F_\mu,0}/\mathfrak{g}_{F_\mu,0^+}$). We let $[X_\mu + \mathfrak{g}_{F_\mu,0^+}] \in D^0_{0+}$ denote the characteristic function of the coset $X_\mu + \mathfrak{g}_{F_\mu,0^+}$. This is the correspondence between "suitable" elements of D^0_{0+} and $\mathcal{O}(0)$ alluded to above. Indeed, for $T \in J(\mathfrak{g}_{0+})$, the restriction of T to D^0_{0+} is zero if and only if

$$T([X_\mu + \mathfrak{g}_{F_\mu,0^+}])$$

is zero for all $\mu \in \mathcal{P}(n)$.

When $r \neq 0$, life is much more complicated; but something beautiful is true. We refer the reader to [10].

10. A FEW COMMENTS ON THE $c_\mu(\pi)$S

Let (π, V) denote an irreducible representation of G. We recall that for all regular semisimple X in some neighborhood of zero we have the Harish-Chandra–Howe local character expansion:

$$\Theta_\pi(1+X) = \sum_{\mu\in\mathcal{P}(n)} c_\mu(\pi)\cdot\hat{\mathcal{O}}_\mu(X).$$

In his paper [16], Howe proves that for irreducible supercuspidal representations (that is, those representations that do not occur as subrepresentations of parabolically induced representations) the coefficients occurring in the Harish-Chandra–Howe local character expansion are all integers. In fact, given a bit of additional information, the same proof shows that this is true for all smooth irreducible representations. We first recall what we know about the $c_\mu(\pi)$s, and we then prove this result of Howe.

10.1. **The coefficient $c_{(1,1,\dots,1)}(\pi)$.** Suppose that (π, V) is a discrete series representation (that is, the matrix coefficients of (π, V) are square-integrable mod the center of G). In this case, Rogawski [32] extended a result of Harish-Chandra [13] to show that

$$c_{(1,1,\dots,1)}(\pi) = \frac{\deg(\pi)}{(-1)^\ell\cdot\deg(\mathrm{St})}.$$

Here $\deg(\pi)$ denotes the formal degree of π, St is the Steinberg representation (see, for example, [5]), and ℓ denotes the semisimple rank of G.

On the other hand, if (π, V) is a tempered representation (that is, it occurs in the Plancherel formula) which is not in the discrete series, then, using results of Kazhdan [23], Huntsinger [20] showed that $c_{(1,1,\dots,1)}(\pi) = 0$ (see also the paper of Schneider and Stuhler [34]).

10.2. **The leading coefficient.** According to Mœglin and Waldspurger [26], the set

$$\{\mu\in\mathcal{P}(n) \mid c_\mu(\pi)\neq 0\}$$

has a unique maximal element; call it μ_π. Moreover, according to Mœglin and Waldspurger [26] and Rodier [31], the coefficient $c_{\mu_\pi}(\pi)$ is an integer which is equal to the dimension of the degenerate Whittaker model corresponding to μ_π.

Example 10.2.1. Suppose $\mu = (n)\in\mathcal{P}(n)$. Recall that $X_{(n)}$ is regular nilpotent. Let $\bar{B}$ be the Borel subgroup consisting of lower triangular matrices in G. Let $\bar{N}$ denote the unipotent radical of $\bar{B}$; that is, the subgroup of $\bar{B}$ formed by the set of lower triangular matrices with ones on the diagonal. Define the character δ of $\bar{N}$ by $\bar{n}\mapsto\Lambda(\mathrm{tr}(X_{(n)}\cdot(\bar{n}-1))$. For an irreducible smooth representation (π, V) of G, let $V(\delta)$ denote the subspace of V generated by the set

$$\{\big(\pi(\bar{n})-\delta(\bar{n})\big)v \mid v\in V \text{ and } \bar{n}\in\bar{N}\}.$$

Define $V_{(n)} = V/V(\delta)$. Then according to the results quoted above, we have $c_{(n)}(\pi) = \dim_{\mathbb{C}}(V_{(n)}) = \dim_{\mathbb{C}}\mathrm{Hom}_G(V, \mathrm{Ind}_{\bar{N}}^G\,\delta)$.

10.3. **The remaining coefficients.** Not much is known about the remaining coefficients. Following an argument of Roger Howe [16], we show that they are all integers.

We begin by making Remark 6.1.4 precise. If $\mu' \in \mathcal{P}(n)$, then

$$\hat{\mathcal{O}}_{\mu'}(X) = \Theta_{\mathrm{Ind}_{P(\mu')}^{G} 1}(1+X)$$

for all $X \in \mathfrak{g}_{0+}$. Here $\mathrm{Ind}_{P(\mu')}^{G} 1$ is the representation of G obtained by inducing the trivial representation of $P(\mu')$ up to G.

Suppose $j > 0$, $\mu \in \mathcal{P}(n)$, and τ_μ is the character of K_j represented by the coset $\varpi^{(1-2j)} X_\mu + \mathfrak{k}_{(1-j)} \in \mathfrak{k}_{1-2j}/\mathfrak{k}_{1-j}$. From §7.3 we have

$$\Theta_\pi(\tau_\mu) = \mathrm{meas}_{dg}(K_j) \cdot m(\tau_\mu^{-1}, \pi) \tag{8}$$

and

$$\Theta_{\mathrm{Ind}_{P(\mu')}^{G} 1}(\tau_\mu) = \mathrm{meas}_{dg}(K_j) \cdot m(\tau_\mu^{-1}, \mathrm{Ind}_{P(\mu')}^{G} 1). \tag{9}$$

From the discussion above, Exercise 5.3.1, and Equation (2) we have

$$\begin{aligned} \hat{\mathcal{O}}_{\mu'}(\tau_\mu) &= \mathcal{O}_{\mu'}(\hat{\tau}_{\mu'}) = \mathrm{meas}_{dX}(\mathfrak{k}_j) \cdot \mathcal{O}_{\mu'}([-\varpi^{(1-2j)} \cdot X_\mu + \mathfrak{k}_{(1-j)}]) \\ &= \begin{cases} \mathrm{meas}_{dX}(\mathfrak{k}_j) \cdot m(\tau_\mu^{-1}, \mathrm{Ind}_{P(\mu')}^{G} 1) & \text{if } \mu < \mu', \\ \mathrm{meas}_{dX}(\mathfrak{k}_j) & \text{if } \mu = \mu', \text{ and} \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Thus, if j is sufficiently large, then

$$m(\tau_\mu^{-1}, \pi) = \sum_{\mu \leq \mu'} c_{\mu'}(\pi) \cdot m(\tau_\mu^{-1}, \mathrm{Ind}_{P(\mu')}^{G} 1).$$

We now proceed by induction. From Rodier [31] we have that $c_{(n)}$ is always an integer.

Suppose $\mu \in \mathcal{P}(n)$. By induction, if $\mu' > \mu$, then $c_{\mu'}(\pi) \in \mathbb{Z}$. We have

$$c_\mu(\pi) \cdot m(\tau_\mu^{-1}, \mathrm{Ind}_{P(\mu)}^{G} 1) = m(\tau_\mu^{-1}, \pi) - \sum_{\mu' > \mu} c_{\mu'}(\pi) \cdot m(\tau_\mu^{-1}, \mathrm{Ind}_{P(\mu')}^{G} 1) \in \mathbb{Z}.$$

Since $m(\tau_\mu^{-1}, \mathrm{Ind}_{P(\mu)}^{G} 1) = \mathcal{O}_\mu([-\varpi^{(1-2j)} \cdot X_\mu + \mathfrak{k}_{(1-j)}]) = 1$, we are done.

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