Dual Pairs and Kostant–Sekiguchi Correspondence. II. Classification of nilpotent elements

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1 Introduction

Let G_0 and G_1 be an irreducible real reductive dual pair acting on a symplectic vector space W (see [H1],[H2]). Let $K_0 \subseteq G_0$, $K_1 \subseteq G_1$ be maximal compact subgroups with the complexifications $K_{0,\mathbb{C}}$, $K_{1,\mathbb{C}}$. The groups K_0, K_1 centralize a positive definite, compatible complex structure J on W. Let $W_{\mathbb{C}}^+$ denote an i-eigenspace of J in $W_{\mathbb{C}}$, the complexification of W. Let $\mathfrak{g}_0$, $\mathfrak{g}_1$ denote the Lie algebras of G_0 and G_1, with the Cartan decompositions $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$, $\mathfrak{g}_1 = \mathfrak{k}_1 \oplus \mathfrak{p}_1$. Let $\nu_0 : W \to \mathfrak{g}_0$, $\nu_1 : W \to \mathfrak{g}_1$ be the moment maps, as defined in [DP1], and let $\mu_0 : W_{\mathbb{C}} \to \mathfrak{g}_{0,\mathbb{C}}$, $\mu_1 : W_{\mathbb{C}} \to \mathfrak{g}_{1,\mathbb{C}}$ be the the analogous moment maps of the complexifications. Then $\mu_0(W_{\mathbb{C}}^+) \subseteq \mathfrak{p}_{0,\mathbb{C}}$ and $\mu_1(W_{\mathbb{C}}^+) \subseteq \mathfrak{p}_{1,\mathbb{C}}$. We have the following pair of diagrams

$$\begin{array}{ccccc} \mathfrak{g}_0 & \xleftarrow{\nu_0} & W & \xrightarrow{\nu_1} & \mathfrak{g}_1 \\ \mathfrak{p}_{0,\mathbb{C}} & \xleftarrow{\mu_0} & W_{\mathbb{C}}^+ & \xrightarrow{\mu_1} & \mathfrak{p}_{1,\mathbb{C}} \end{array} \tag{1}$$

We call an element $w \in W$ nilpotent if $\nu_0(w)$ is nilpotent in $\mathfrak{g}_0$ (equivalently if $\nu_1(w)$ is nilpotent in $\mathfrak{g}_1$). Similarly we define nilpotent elements in $W_{\mathbb{C}}$ as elements mapped by either of μ_0, μ_1 onto nilpotent elements of the complexified Lie algebras.

In this paper we provide a complete combinatorial description of the set of the nilpotent $G_0 \times G_1$-orbits in W and the set of the nilpotent $K_{0,\mathbb{C}} \times K_{1,\mathbb{C}}$-orbits in $W_{\mathbb{C}}^+$.

Both classification problems, as well as other classification problems such as the problem of classifying nilpotent orbits in $\mathfrak{g}$ and in $\mathfrak{p}_{\mathbb{C}}$ arise as special instances of a more general problem of classifying homogeneous nilpotent orbits in certain color Lie algebras. In section 3 we solve the general problem. The techniques we use are adapted from [BC], in contrast to the usual approach to the classification of the nilpotent orbits in a semisimple Lie algebra via the Jacobson-Morozov theorem (see [CM]).

In section 4 we show how one can apply the general results of section 3 to the classification problems described above in the case of a dual pair of type II, and in section 5 we do the same for pairs of type I.

The Kostant-Sekiguchi correspondence for G_0 is a bijection $\mathcal{S}$ from the set of nilpotent G_0-orbits in $\mathfrak{g}_0$ onto the set of nilpotent $K_{0,\mathbb{C}}$-orbits in $\mathfrak{p}_{0,\mathbb{C}}$, and similarly for G_1 ([S2]).

In section 6 we show that the Kostant-Sekiguchi correspondence is compatible with our classification of orbits in $\mathfrak{g}$ and in $\mathfrak{p}_{\mathbb{C}}$ for dual pairs of type I. As a main tool we use the description of the Cayley transform of a Cayley triple in $\mathfrak{g}$ by the conjugation by a special element of the complex group.

2 Sesqui-linear Forms

Let $\mathbb{D} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ (the quaternions), and let ι be a possibly trivial anti-involution on $\mathbb{D}$. Fix a positive integer n and let

$$V = V_0 \oplus V_1 \oplus .. \oplus V_{n-1} \tag{2}$$

be a direct sum of left $\mathbb{D}$-vector spaces V_0, V_1, ..., V_{n-1}. We view V as a $\mathbb{Z}/n\mathbb{Z}$-graded vector space, via the above decomposition, where $\mathbb{Z}/n\mathbb{Z}$ is realized as the set $\{0, 1, 2, ..., n-1\}$ with addition modulo n. The subspace $V_k \subseteq V$ is called the subspace of degree k $(= 0, 1, 2, ..., n-1)$.

A sesqui-linear form on V is a map $\tau : V \times V \to \mathbb{D}$ such that for all $u, v, u', v' \in V$ and for all $a \in \mathbb{D}$,

$$\begin{aligned} &\tau(au, v) = a\tau(u, v), \quad \tau(u, av) = \tau(u, v)\iota(a) \\ &\tau(u + u', v) = \tau(u, v) + \tau(u', v), \quad \tau(u, v + v') = \tau(u, v) + \tau(u, v'). \end{aligned}$$

For $\sigma = \pm 1$, we will say that the form τ is σ-hermitian if

$$\tau(u, v) \quad = \quad \sigma\iota(\tau(v, u))$$

for all $u, v \in V$.

We'll say that two subspaces $V', V'' \subseteq V$ are orthogonal (with respect to τ) if $\tau(v', v'') = 0$ for all $v' \in V'$ and all $v'' \in V''$. In this case we shall write $V' \perp V''$.

We shall say that the space V, or more precisely, the pair (V, τ) is decomposable if there are two non-zero graded subspaces $V', V'' \subseteq V$ such that $V = V' \oplus V''$ and $V' \perp V''$. Otherwise the space V, or the pair (V, τ) is called indecomposable. The form τ is called non-degenerate if the following two implications hold:

$$\begin{gathered} \text{if } \tau(u, v) = 0 \text{ for all } v \in V, \text{ then } u = 0, \text{ and} \\ \text{if } \tau(u, v) = 0 \text{ for all } u \in V, \text{ then } v = 0. \end{gathered}$$

Two formed spaces (V, τ) and (V', τ') are called isometric if there is a linear bijection $g : V \to V'$ such that

$$\begin{aligned} \tau(V_k) \subseteq V'_k \qquad & (k = 0, 1, 2, ..., n-1) \\ \tau(u, v) = \tau'(gu, gv) \qquad & (u, v \in V) \end{aligned}$$

In that case we shall write $(V, \tau) \approx (V', \tau')$ and say that the map g is a graded isometry.

3 Classification of homogeneous nilpotent elements in certain color Lie algebras

Let n be an even positive integer and let

$$V = V_0 \oplus V_1 \oplus \cdots \oplus V_{n-1}$$

be a $\mathbb{Z}/n\mathbb{Z}$-graded left vector space over $\mathbb{D}$. The dimension vector of V is the sequence

$$\underline{\dim}(V) = (dim_{\mathbb{D}}(V_0), dim_{\mathbb{D}}(V_1), \dots, dim_{\mathbb{D}}(V_{n-1})).$$

For $a \in \mathbb{Z}/n\mathbb{Z}$ let

$$\mathrm{E}nd(V)_a = \{X \in \mathrm{E}nd(V);\ X(V_b) \subseteq V_{a+b},\ \text{for } b \in \mathbb{Z}/n\mathbb{Z}\ \}.$$

Define a bilinear bracket

$$[-,-] : \mathrm{E}nd(V) \times \mathrm{E}nd(V) \to \mathrm{E}nd(V)$$

by the formula

$$[X, Y] = XY - (-1)^{ab} YX, \ \text{for } X \in \mathrm{E}nd(V)_a,\ Y \in \mathrm{E}nd(V)_b. \qquad (3)$$

Then $\mathrm{E}nd(V)$ becomes a color Lie algebra with respect to the symmetric bicharacter $\beta(a, b) = (-1)^{ab}$ on the group $\mathbb{Z}/n\mathbb{Z}$ (see [BLR]).

Fix $a \in \mathbb{Z}/n\mathbb{Z}$ and let $N \in \mathrm{E}nd(V)_a$ be nilpotent. Recall that the height of N, or the height of (N, V), is the integer $m \geq 0$ such that $N^m \neq 0$ and $N^{m+1} = 0$. We shall write $m = ht(N) = ht(N, V)$. The pair (N, V) of height m is called uniform if $Ker(N^m) = NV$.

Lemma 3.1 *Suppose the pair (N, V) is uniform of height m. Then for any graded subspace $E \subseteq V$, complementary to $Ker(N^m)$,*

$$V = E \oplus NE \oplus N^2E \oplus ... \oplus N^mE.$$

Moreover, $dim(E) = dim(NE) = ... = dim(N^mE)$.

Proof: By the choice of E, we have $V = E \oplus NV$. Hence, $NV \subseteq NE + N^2V$. Thus $V \subseteq E + NE + N^2V$. But $N^2V \subseteq N^2E + N^3V$. Hence, inductively,

$$V \subseteq E + NE + N^2E + ... + N^mE.$$

If for some $i_0 < i_1 < \ldots < i_k$ the intersection $N^{i_0}E \cap N^{i_1}E + \ldots + N^{i_k}E$ were nonzero, then $N^{i_0}e_0 = N^{i_1}e_1 + \ldots + N^{i_k}e_k$ for some $e_j \in E$, with $N^{i_0}e_0 \neq 0$, but then $N^m e_0 = N^m(e_0 - N^{i_1-i_0}e_1 - \ldots - N^{i_k-i_1}e_k) = 0$, hence $e_0 = 0$ by the injectivity of N^m on E. Thus

$$V = E \oplus NE \oplus N^2E \oplus ... \oplus N^mE.$$

It remains to show the equality of dimensions. Suppose $v \in E$, $0 \leq i \leq m-1$ and $NN^iv = 0$. Then $N^mv = 0$. Hence $v = 0$, and therefore $N^iv = 0$. Thus the linear map

$$N^iE \ni u \mapsto Nu \in N^{i+1}E$$

is injective. Since this map is obviously surjective, we are done.□

It is clear that if E is a graded subspace fulfilling the conditions of Lemma 3.1 then we can reconstruct the dimension vector of V from the dimension vector of E, since $\underline{\dim}(NE)$ is obtained from $\underline{\dim}(E)$ with a shift in grading by a. More formally, we can identify dimension vectors with functions on $\mathbb{Z}/n\mathbb{Z}$ and let η be the left regular representation of $\mathbb{Z}/n\mathbb{Z}$ i.e.

$$(\eta_b f)(c) = f(c-b) \quad (b, c \in \mathbb{Z}/n\mathbb{Z}). \tag{4}$$

Then

$$\underline{\dim}(V) = (1 + \eta_a + \eta_a^2 + \ldots + \eta_a^m)\underline{\dim}(E). \tag{5}$$

3.1 A general linear color Lie algebra

The group $GL(V)_0 = GL(V) \cap \mathrm{E}nd(V)_0$ acts by conjugation on nilpotent elements in $\mathrm{E}nd(V)_a$ for every $a \in \mathbb{Z}/n\mathbb{Z}$. In this subsection we will classify the nilpotent orbits of this action.

Lemma 3.2 *Let $N, N' \in \mathrm{E}nd(V)_a$ be nilpotent. Assume that the pairs (N, V), (N', V) are uniform of height m. Then the elements N, N' are in the same $GL(V)_0$-orbit if and only if $\underline{dim}(V/Ker(N^m)) = \underline{dim}(V/Ker(N'^m))$.*

Proof: There is only one non-trivial implication here which requires a proof. Suppose $dim(V/Ker(N^m))_a = dim(V/Ker(N'^m))_a$ for all $a = 0, 1, 2, ..., n-1$. According to Lemma 3.1 we have decompositions

$$V = E \oplus NE \oplus N^2E \oplus ... \oplus N^mE,$$
$$V = E' \oplus N'E' \oplus N'^2E' \oplus ... \oplus N'^mE'.$$

Since by our assumption $dim(E_a) = dim(E'_a)$ for all a, there is a graded linear isomorphism $g : E \to E'$. We extend g to a graded linear isomorphism $g : V \to V$ by

$$gN^i v = N'^i gv \qquad (v \in E;\ i = 0, 1, 2..., m).$$

Clearly $gNg^{-1} = N'$. □

Lemma 3.3 *If the pair (N, V) is uniform of height m, then there exist graded N-invariant subspaces $V^j \subseteq V$ such that*

$$V = V^1 \oplus V^2 \oplus ...,$$

where each pair (N, V^j) is uniform, and $dim(V^j / Ker(N|_{V^j})^m) = 1$. for each j.

Proof: This is clear from Lemma 3.1 via a decomposition of E into one dimensional subspaces. □

Lemma 3.4 *Let (N, V) be a pair of height m. Let $U \subseteq V$ be a subspace such that*
(a) U is N-invariant,
(b) (N, U) is uniform of height m.
Then there is a graded N-invariant subspace $U' \subseteq V$ such that
(c) $V = U \oplus U'$.

Proof: We proceed by induction on m. If $m = 0$ then $N = 0$ and (c) is obvious. Suppose then that $m \geq 1$. We know from Lemma 3.1 that there is a graded subspace E such that

$$U = E \oplus NE \oplus N^2E \oplus ... \oplus N^m E.$$

Hence, $U \cap Ker(N^m) = NE \oplus N^2E \oplus ... \oplus N^m E = NU$. This space is N-invariant. The pair $(N, U \cap Ker(N^m))$ is uniform of height $m-1$. Moreover,

$$U \cap Ker(N^m) \cap Ker(N|_{Ker(N^m)})^{m-1} = U \cap Ker(N^m) \cap Ker(N^{m-1})$$
$$= U \cap Ker(N^{m-1}) = N^2E \oplus N^3E \oplus ... \oplus N^m E \neq U \cap Ker(N^m).$$

Hence, $U \cap Ker(N^m) \not\subseteq Ker(N|_{Ker(N^m)})^{m-1}$. Thus the pair of spaces $U \cap Ker(N^m) \subseteq Ker(N^m)$ satisfies the conditions (a) and (b), but the height of $(N, Ker(N^m))$ is $m-1$. Therefore, by induction, there exists a graded N-invariant subspace $U' \subseteq Ker(N^m)$ such that

$$Ker(N^m) = U \cap Ker(N^m) \oplus U'.$$

Hence,

$$U + Ker(N^m) = U \oplus U'. \tag{6}$$

If $V = U + Ker(N^m)$, we are done. Otherwise, choose a graded subspace $F \subseteq V$ such that $V = F \oplus (U + Ker(N^m))$. Let

$$W = F + NF + N^2F + ... + N^mF.$$

Then W is N-invariant and $ht(N, W) = m$. We claim that

$$U \cap W = 0. \tag{7}$$

Indeed, if this is not the case then there exist $e \in E \setminus \{0\}$ and $u \in N^{i+1}U$ for some i, $0 \leq i \leq m$, such that $N^ie + u \in W$. Hence $N^me \in W$. Since $NN^me = 0$, there exists $f \in F$ such that $N^me = N^mf$. Hence, $f \in U + Ker(N^m)$ and therefore $f \in F \cap (U + Ker(N^m))$. But this last space is zero. Thus $0 = f = N^mf = N^me = e$, a contradiction. Therefore (7) holds. If $V = U \oplus W$, we are done. Otherwise, notice that $V = (U \oplus W) + Ker(N^m)$. Moreover, the pair of spaces $U \oplus W \subset V$ satisfies the conditions (a), (b). Hence, by (6), $V = U \oplus (W \oplus U')$. □

Lemma 3.5 *If the pair (N, V) is indecomposable, then it is uniform.*

Proof: In order to avoid trivialities we assume $N \neq 0$ and $V \neq 0$. Let E be a graded subspace of V such that $V = E \oplus Ker(N^m)$, where $m = ht(N, V)$. Let $U = E \oplus NE \oplus N^2E \oplus ... \oplus N^mE$. It is easy to see that the subspace $U \subseteq V$ satisfies the conditions (a), (b) of Lemma 3.4. Hence, there is an N-invariant graded subspace $U' \subseteq V$ such that $V = U \oplus U'$. Since (N, V) is indecomposable, $U' = 0$. Thus $(N, V) = (N, U)$ is uniform. □

Corollary 3.6 *If $V \neq 0$ and if the pair (N, V) is indecomposable, then it is uniform and $dim(V/Ker(N^m)) = 1$, where $m = ht(N, V)$.*

Theorem 3.7 *Let $N \in \mathrm{End}(V)_a$ be nilpotent. Then there exist graded N-invariant subspaces $V^j \subseteq V$ such that*
(a) $V = V^1 \oplus V^2 \oplus \cdots \oplus V^s$,
(b) each (N, V^j) is indecomposable,
(c) $ht(N, V^1) \geq ht(N, V^2) \geq ....$
The decomposition (a) having the properties (b) and (c) is unique up to the action of $GL(V)_0^N$, the centralizer of N in $GL(V)_0$. Thus the above decomposition determines the $GL(V)_0$-orbit of N in $\mathrm{End}(V)_a$.

Proof: Let $E \subseteq V$ be a graded subspace complementary to the kernel of N^m, where m is the height of (N, V). Set $U = E \oplus NE \oplus N^2E \oplus ... \oplus N^mE$. Then the pair (N, U) is uniform. If $V = U$, the theorem follows from Lemmas 3.2 and 3.3. Otherwise, notice that the spaces $U \subseteq V$ satisfy the conditions (a), (b) of Lemma 3.4. Moreover, $V = U \oplus Ker(N^m)$. Hence, as in (6), there exists a graded N-invariant subspace $U' \subseteq Ker(N^m)$ such that $V = U \oplus U'$. In particular $ht(N, U') < ht(N, V)$. Hence we may proceed inductively. $\square$

For $b \in \mathbb{Z}/n\mathbb{Z}$, let $\mathbb{D}[b]$ denote $\mathbb{Z}/n\mathbb{Z}$-graded space which is concentrated in homogeneous degree b and isomorphic to $\mathbb{D}$ as a non-graded space. If (N, V) is indecomposable with N of degree a and height m then

$$\begin{aligned} V &= \mathbb{D}[b] \oplus N\mathbb{D}[b] \oplus \cdots \oplus N^m\mathbb{D}[b] \\ &= \mathbb{D}[b] \oplus \mathbb{D}[b+a] \oplus \cdots \oplus \mathbb{D}[b+ma] \end{aligned}$$

for some $b \in \mathbb{Z}/n\mathbb{Z}$.

Corollary 3.8 *Nilpotent orbits of the group $GL(V)_0$ in $\mathrm{End}(V)_a$ are parametrized by pairs of sequences*

$$\begin{aligned} m_1 \geq m_2 \geq \cdots \geq m_s \geq 0, \quad m_j \in \mathbb{N}; \\ (b_1, b_2, \ldots, b_s), \quad 0 \leq b_j \leq n-1, \end{aligned}$$

such that

1. $\underline{dim}(V) = \sum_{j=1}^{s}(1 + \eta_a + \eta_a^2 + \cdots + \eta_a^{m_j})\underline{dim}(\mathbb{D}[b_j])$;
2. *if* $m_j = m_{j+1}$ *then* $b_j \leq b_{j+1}$.

3.2 The color Lie algebra of a formed space

Now we consider hermitian analogue of orthosymplectic color Lie algebras of [BLR]. For the rest of this section fix $\sigma = \pm 1$.

Define a map $S \in \mathrm{E}nd(V)_0$ by

$$S(v) = (-1)^a v \qquad (v \in V_a;\ a \in \mathbb{Z}/n\mathbb{Z}), \tag{8}$$

and let τ be a non-degenerate sesqui-linear form on V such that

$$\tau(u, v) = \sigma\iota(\tau(v, Su)) \qquad (u, v \in V). \tag{9}$$

We assume that the form τ provides a non-degenerate pairing between V_b and V_{-b} for each $b \in \mathbb{Z}/n\mathbb{Z}$, and that $V_b \perp V_c$ if $b + c \neq 0$. Let

$$\mathfrak{g}(V, \tau)_a = \{X \in \mathrm{E}nd(V)_a;\ \tau(Xu, v) + \tau(S^a u, Xv) = 0,\ u, v \in V\}$$

and let

$$\mathfrak{g}(V, \tau) = \bigoplus_{a \in \mathbb{Z}/n\mathbb{Z}} \mathfrak{g}(V, \tau)_a.$$

Then $\mathfrak{g}(V,\tau)$ is closed under the bracket defined in (3) and it is a color Lie subalgebra of $\mathrm{E}nd(V)$.

Let $G(V,\tau)_0$ denotes the isometry group

$$G(V,\tau)_0 = \{g \in \mathrm{E}nd(V)_0;\ \tau(gu,gv) = \tau(u,v),\ u,v \in V\}. \tag{10}$$

The goal of this subsection is to classify the orbits of the group $G(V,\tau)_0$ in the set of nilpotent elements in each homogeneous component $\mathfrak{g}(V,\tau)_a$ of the algebra $\mathfrak{g}(V,\tau)$.

In the following lemma we collect several easy to check properties of a homogeneous elements of $\mathfrak{g}(V,\tau)$.

Lemma 3.9 *Let $X \in \mathfrak{g}(V,\tau)_a$. Then*

$$\begin{aligned} SX &= (-1)^a XS, \\ \tau(u,Sv) &= \tau(Su,v), \\ \tau(Xu,v) &= -\tau(u,S^aXv), \\ \tau(u,Xv) &= -\tau(XS^au,v). \end{aligned} \tag{11}$$

Define

$$\delta(k) = (-1)^{k(k-1)/2}.$$

Then for $k,l \geq 0$

$$\begin{aligned} (SX)^k &= \delta(k)^a S^k X^k, \\ S^l X^k &= (-1)^{akl} X^k S^l, \\ \tau(X^k u,v) &= (-1)^k \delta(k)^a \tau(u, S^{ak} X^k v), \\ \tau(u,X^k v) &= (-1)^k \delta(k+1)^a \tau(S^{ak} X^k u, v). \end{aligned} \tag{12}$$

Let, from now on, $N \in \mathfrak{g}(V,\tau)_a$ be nilpotent.

Lemma 3.10 *Let $m = ht(N,V)$. Then the formula*

$$\tilde{\tau}(\tilde{u},\tilde{v}) = \tau(u,N^m v) \quad (\tilde{u} = u + Ker(N^m),\ \tilde{v} = v + Ker(N^m);\ u,v \in V)$$

defines a non-degenerate sesqui-linear form on the $\mathbb{Z}/n\mathbb{Z}$-graded space $\tilde{V} = V/Ker(N^m)$ and

$$\tilde{\tau}(\tilde{u},\tilde{v}) = (-1)^m \delta(m+1)^a \sigma\iota\tilde{\tau}(S^{ma+1}\tilde{v},\tilde{u}), \tag{13}$$

$$\tilde{\tau}(S\tilde{u},\tilde{v}) = (-1)^{ma}\tilde{\tau}(\tilde{u},S\tilde{v}), \tag{14}$$

$$\tilde{V}_b \text{ and } \tilde{V}_c \text{ are } \tilde{\tau}\text{-orthogonal, unless } b+c+ma = 0. \tag{15}$$

Proof: The fact that the form is well defined and the statements (13), (14) and (15) follow easily from (9) and (12). We will show that the form $\tilde{\tau}$ is non-degenerate. Due to (14), it is enough to show that for every non-zero $\tilde{v} \in \tilde{V}$ there exists $\tilde{u} \in \tilde{V}$ such that $\tilde{\tau}(\tilde{u}, \tilde{v}) \neq 0$. Let $\tilde{v} = v + Ker(N^m)$. Then $N^m v \neq 0$ and there exists $u \in V$ such that $\tau(u, N^m v) \neq 0$ and for $\tilde{u} = u + Ker(N^m)$ we have $\tilde{\tau}(\tilde{u}, \tilde{v}) \neq 0$.□

Theorem 3.11 *Let (N, V) be uniform of height m. Then there is a graded subspace $F \subseteq V$, complementary to $Ker(N^m)$, such that*

$$V = F \oplus NF \oplus N^2F \oplus ... \oplus N^mF \tag{a}$$

and

$$N^kF \perp N^lF \quad \textit{for } k + l \neq m. \tag{b}$$

Remark 3.12 *It is clear from (12) that (b) is equivalent to*

$$F \perp N^kF \quad \textit{for } 0 < k \leq m - 1. \tag{b$'$}$$

Proof of Theorem 3.11. We will define inductively a sequence $F^{(0)}, F^{(1)}, \ldots, F^{(m-1)}$ of subspaces of V such that for all $k = 0, 1, \ldots, m-1$ the following conditions hold:
(i) $F^{(k)}$ is graded and $V = F^{(k)} \oplus NF^{(k)} \oplus N^2F^{(k)} \oplus ... \oplus N^mF^{(k)}$,
(ii) $F^{(k)} \perp N^{m-k}F^{(k)} + N^{m-k+1}F^{(k)} + \ldots + N^{m-1}F^{(k)}$.
Then $F = F^{(m-1)}$ fulfills the conditions of the theorem.

It follows from Lemma 3.1 that for $F^{(0)}$ we can take any graded subspace complementary to $Ker(N^m)$. Assume that $k > 0$ and that the space $F^{(k-1)}$ has already been constructed. Set $E = F^{(k-1)}$ and let $E^* = \mathrm{H}om_{\mathbb{D}}(E, \mathbb{D})$. Define two maps

$$\hat{\tau}_0,\, \hat{\tau} : E \to E^*,$$
$$\hat{\tau}_0(v)(u) = \tau(u, N^m v), \qquad \hat{\tau}(v)(u) = \tau(u, N^{m-k}v), \qquad (u, v \in E).$$

We know from Lemma 3.10 that $\hat{\tau}_0$ is a bijection.

Notice that

$$\tau(u, N^m \hat{\tau}_0^{-1}\hat{\tau}(v)) = \tau(u, N^{m-k}v) \qquad (u, v \in E). \tag{16}$$

Indeed, the left hand side of (16) is equal to

$$\tau(u, N^m \tau_0^{-1}\hat{\tau}(v)) = \hat{\tau}_0\hat{\tau}_0^{-1}\hat{\tau}(v)(u) = \hat{\tau}(v)(u) = \tau(u, N^{m-k}v).$$

Let

$$\rho = \rho_k = \frac{1}{2}\hat{\tau}_0^{-1}\hat{\tau} \tag{17}$$

and define the space $F^{(k)}$ as $F^{(k)} = (1 - N^k\rho)F^{(k-1)}$.

First we will show that the space $F^{(k)}$ is graded. We see from (16) that for any $v \in E$, $N^k\hat{\tau}_0^{-1}\hat{\tau}(v)$ coincides with the unique element $x \in N^kE$ such that for all $u \in E$, $\tau(u, N^{m-k}(x - v)) = 0$. Hence, if $v \in E_b := E \cap V_b$, then $\tau(u, N^{m-k}x) = 0$ for all $u \in \sum_{c \neq -c_0} E_c$, where $c_0 = (m - k)a + b$. Thus,

$$N^{m-k}x \in (N^mE) \cap \left(\sum_{c\neq -c_0} E_c\right)^{\perp} = (N^mE) \cap \bigcap_{c\neq -c_0} E_c^{\perp}$$

$$= \bigcap_{c\neq -c_0} (N^mE) \cap E_c^{\perp} = \bigcap_{c\neq -c_0} (N^mE) \cap V_c^{\perp} = (N^mE) \cap V_{(m-k)a+b}.$$

Let us write $x = x_b + x'$, where $x_b \in V_b \cap N^kE$ and $x' \in \sum_{c\neq b} V_c \cap N^kE$. Since $N^{m-k}x \in V_{(m-k)a+b}$, we have $N^{m-k}x = N^{m-k}x_b$. Therefore, by the uniqueness of x, $x' = 0$. Hence, $x \in V_b \cap N^kE$. Thus $N^k\rho(E_b) \subseteq V_b \cap N^kE$ for all $b \in \mathbb{Z}/n\mathbb{Z}$ and the space $F^{(k)} = (1 - N^k\rho)E$ is graded.

It follows from the construction that $dim(F^{(k)}) \leq dim(E)$ and $E \subset F^{(k)} + N^kE$. Since $N^{m+1}F^{(k)} = 0$, we have

$$V = E \oplus NE \oplus N^2E \oplus ... \oplus N^mE \subset F^{(k)} + NF^{(k)} + N^2F^{(k)} + \cdots + N^mF^{(k)},$$

so the sum on the right hand side is also a direct sum.

It remains to prove that $F^{(k)}$ fulfills the orthogonality property (ii). For $u, v \in E$

$$\begin{aligned} &\tau((1 - N^k\rho)u, N^{m-k}(1 - N^k\rho)v) \\ &= \tau(u, N^{m-k}v) - \tau(u, N^m\rho v) - \tau(N^k\rho u, N^{m-k}v), \end{aligned} \tag{18}$$

because $N^kN^m = 0$. Furthermore, by (16),

$$\tau(u, N^m\rho v) = \frac{1}{2}\tau(u, N^{m-k}v),$$

and by (9), (12) and (16)

$$\begin{aligned} &\tau(N^k\rho u, N^{m-k}v) = \sigma\iota\,\tau(N^{m-k}v, SN^k\rho u) \\ &= (-1)^{m-k}\delta(m-k)^a\sigma\iota\,\tau(v, S^{(m-k)a}N^{m-k}SN^k\rho u) \\ &= (-1)^{(m-k)(a+1)}\delta(m-k)^a\sigma\iota\tau(S^{(m-k)a+1}v, N^m\rho u) \\ &= \tfrac{1}{2}(-1)^{(m-k)(a+1)}\delta(m-k)^a\sigma\iota\tau(S^{(m-k)a+1}v, N^{m-k}u) \\ &= \tfrac{1}{2}(-1)^{(m-k)(a+1)}\delta(m-k)^a\tau(N^{m-k}u, S^{(m-k)a}v) \\ &= \tfrac{1}{2}(-1)^{(m-k)a}\tau(u, S^{(m-k)a}N^{m-k}S^{(m-k)a}v) \\ &= \tfrac{1}{2}\tau(u, N^{m-k}v). \end{aligned}$$

Hence, the quantity (18) is zero and $F^{(k)}$ is orthogonal to $N^{m-k}F^{(k)}$. Notice also that for $i > 0$

$$\tau((1-N^k\rho)u, N^{m-k+i}(1-N^k\rho)v)$$
$$= \tau(u, N^{m-k+i}v) - \tau(u, N^{m+i}\rho v) - \tau(N^k\rho u, N^{m-k+i}v) = \tau(u, N^{m-k+i}v).$$

Hence $F^{(k)}$ is also orthogonal to $N^lF^{(k)}$, $(l = m-k+1, \ldots, m-1)$, due to condition (ii) for $F^{(k-1)}$. □

Corollary 3.13 *Let the pairs (N,V), (N',V) be uniform of height m with N, N' homogeneous of the same degree. Then the elements N, N' are in the same $G(V,\tau)_0$-orbit if and only if the spaces $(V/Ker(N^m), \tilde{\tau})$, $(V/Ker(N'^m), \tilde{\tau})$ are isometric.*

Proof: It is easy to check that if N, N' are conjugate by an element of $G(V,\tau)_0$ then the above two graded spaces are isometric.

Conversely, suppose these two spaces are isometric. Let $N, N' \in \mathfrak{g}(V,\tau)_a$ and let $F, F' \subseteq V$ be as in Theorem 3.11 for N, N' respectively. By the assumption, we have a graded bijection $g : F \to F'$ such that

$$\tau(gu, N'^m gv) = \tau(u, N^m v) \qquad (u, v \in F).$$

Set

$$g(N^k v) = N'^k g(v) \qquad (v \in F;\ k = 0, 1, 2, ..., m-1).$$

Then $g \in \mathrm{E}nd(V)$ is bijective and intertwines N and N'. Furthermore, for $u, v \in F$ and for $k = 0, 1, 2, ..., m-1$ we have

$$\begin{aligned}
&\tau(gN^k u, gN^{m-k}v) = \tau(N'^k gu, N'^{m-k}gv)\\
&= (-1)^k \delta(k)^a \tau(gu, S^{ak}N'^m gv)\\
&= (-1)^{k+akm} \delta(k)^a \tau(gu, N'^m g S^{ak} v)\\
&= (-1)^{k+akm} \delta(k)^a \tau(u, N^m S^{ak} v)\\
&= (-1)^{k+ak} \delta(k)^a \tau(u, N^k S^{ak} N^{m-k} v)\\
&= (-1)^{ak} \delta(k)^a \delta(k+1)^a \tau(S^{ak}N^k u, S^{ak}N^{m-k}v)\\
&= \tau(N^k u, N^{m-k}v).
\end{aligned}$$

Hence, $g \in G(V,\tau)_0$. □

Definition 3.14 *The pair (N,V) is called indecomposable if V does not have any non- trivial orthogonal N-invariant direct sum decomposition into graded subspaces. Otherwise the pair (N,V) is called decomposable.*

Proposition 3.15 *If the pair (N,V) is indecomposable then it is uniform.*

Proof: Let $m = ht(N,V)$ and let $E \subseteq V$ be a graded subspace complementary to $Ker(N^m)$. Set $U = E + NE + N^2E + ... + N^mE$. Then U is a graded subspace of V preserved by N and it is easy to see that U is uniform. We will show that the restriction of the form τ to U is non-degenerate. Since, $U^\perp$ is N-invariant (by (11)) this will complete the proof.

Let $0 \leq k \leq i \leq m$ and let $u_i \in E$. Suppose

$$N^k u_k + N^{k+1} u_{k+1} + ... + N^m u_m \perp U.$$

Then $N^k u_k \perp N^{m-k}E$. Hence, by (12), $u_k \perp N^m E$. But $N^m E = N^m V$. Thus $u_k \perp N^m V$. Therefore $N^m u_k \perp V$. Hence, $u_k \in Ker(N^m) \cap E = \{0\}$. Similarly, $u_{k+1} = u_{k+2} = ... = u_m = 0$. □

Proposition 3.16 *Let the pair (N,V) be uniform of height m. Then (N,V) is indecomposable if and only if the formed space $(V/Ker(N^m), \tilde{\tau})$ is indecomposable.*

Proof: Clearly if (N,V) is decomposable then so is $(V/Ker(N^m), \tilde{\tau})$.

Conversely, suppose $(V/Ker(N^m), \tilde{\tau})$ is decomposable. Choose a subspace $F \subseteq V$ as in Theorem 3.11 and let

$$\tau_m(u,v) = \tau(u, N^m v) \quad (u, v \in F).$$

Then (F, τ_m) is isometric to $(V/Ker(N^m), \tilde{\tau})$, and hence is decomposable. Thus there exist two non-zero graded τ_m-orthogonal subspaces $F', F'' \subseteq F$ such that $F = F' \oplus F''$. Let $V' = F' + NF' + N^2F' + ... + N^mF'$ and let $V'' = F'' + NF'' + N^2F'' + ... + N^mF''$. Since the spaces F', F'' are τ_m-orthogonal, we have $F' \perp N^mF'$ and $F'' \perp N^mF''$, with respect to τ. Hence it is easy to see that $V = V' \oplus V''$ and $V' \perp V''$. □

It follows from Corollary 3.13 and Proposition 3.16 that in order to classify indecomposable nilpotent elements of height m in $\mathfrak{g}(V,\tau)_a$ up to the action of $G(V,\tau)_0$, we have to classify (up to grading-preserving isometry) indecomposable $\mathbb{Z}/n\mathbb{Z}$-graded formed spaces $(\tilde{V}, \tilde{\tau})$.

Proposition 3.17 *Let $(\tilde{V}, \tilde{\tau})$ be indecomposable $\mathbb{Z}/n\mathbb{Z}$-graded formed space satisfying (13), (14) and (15). Then $dim(\tilde{V}) = 1$ or 2 and the form is σ'-hermitian for suitable $\sigma' \in \{\pm 1\}$. More precisely, one of the following conditions holds for $(\tilde{V}, \tilde{\tau})$.*

1. *$\tilde{V} = \tilde{V}_b$ for some $b \in \mathbb{Z}/n\mathbb{Z}$ with $2b + ma = 0$ (which implies that ma is even), $\sigma' = (-1)^m(-1)^b\delta(m+1)^a\sigma$ and $(\tilde{V}_b, \tilde{\tau})$ is nondegenerate indecomposable as a (nongraded) formed space.*

2. *$\tilde{V} = \tilde{V}_b \oplus \tilde{V}_{-b-ma}$ for some $b \in \mathbb{Z}/n\mathbb{Z}$ with $2b + ma \neq 0$, and $\sigma' = (-1)^m(-1)^{(ma+1)b}\delta(m+1)^a\sigma$. In this case both summands are $\tilde{\tau}$-isotropic of dimension one and $\tilde{\tau}$ provides a pairing between them.*

Proof: Let

$$\tilde{V}_{even} = \tilde{V}_0 \oplus \tilde{V}_2 \oplus \ldots \oplus \tilde{V}_{n-2},$$
$$\tilde{V}_{odd} = \tilde{V}_1 \oplus \tilde{V}_3 \oplus \ldots \oplus \tilde{V}_{n-1}.$$

Assume first that $2|ma$. Then (14) is equivalent to the condition that $\tilde{V}_{even}$ and $\tilde{V}_{odd}$ are $\tilde{\tau}$-orthogonal. Indecomposability of $(\tilde{V}, \tilde{\tau})$ forces $\tilde{V}_{even} = 0$ or $\tilde{V}_{odd} = 0$. Then condition (13) says that $\tilde{\tau}$ is σ'-hermitian for appropriate σ' and once again from the indecomposability we see that we are in the case (1) or (2) of the proposition.

If $2 \not| ma$ then (14) is equivalent to the condition that $\tilde{V}_{even}$ and $\tilde{V}_{odd}$ are $\tilde{\tau}$-isotropic. Indecomposability and conditions (13) and (15) guarantee that we are in the case (2). □

Definition 3.18 *Let $a \in \mathbb{Z}/n\mathbb{Z}$ and $m \in \mathbb{N}$. An (a, m)-admissible space is an indecomposable $\mathbb{Z}/n\mathbb{Z}$-graded formed space $(F, \check{\tau})$, where the form $\check{\tau}$ is σ'-hermitian and σ' is determined by the pair (a, m) as in Proposition 3.17.*

Theorem 3.19 *Let $N \in \mathfrak{g}(V, \tau)_a$ be a nilpotent. Then there exist a sequence*

$$(F^{(1)}, F^{(2)}, \ldots, F^{(s)})$$

of graded subspaces of V and a sequence

$$m_1 \geq m_2 \geq \ldots \geq m_s \geq 0$$

of nonnegative integers such that

1. *for every $i = 1, 2, \ldots, s$, the space $F^{(i)}$ with the form $\tau_{(i)}$ given by the formula*
$$\tau_{(i)}(u, v) = \tau(u, N^{m_i} v), \quad (u, v \in F^{(i)})$$
is an (a, m_i)-admissible space;

2. *$V = \bigoplus_{i=1}^{s} F^{(i)} \oplus NF^{(i)} \oplus \cdots \oplus N^{m_i} F^{(i)}$.*

Let $N' \in \mathfrak{g}(V, \tau)_a$ be another nilpotent element and let $(F'^{(1)}, F'^{(2)}, \ldots, F'^{(s')})$ and $m'_1 \geq m'_2 \geq \ldots \geq m'_{s'}$ be the sequences corresponding to N'. Then N and N' are $G(V, \tau)_0$-conjugate if and only if $s = s'$, $m_i = m'_i$ for every $i = 1, 2, \ldots, s$ and, up to a permutation of indices i preserving the sequence $(m_i)_{i=1}^{s}$, the graded formed spaces $(F^{(i)}, \tau_{(i)})$ and $(F'^{(i)}, \tau'_{(i)})$ are isometric.

Proof: Let $E \subseteq V$ be a graded subspace complementary to $Ker(N^m)$, where $m = ht(N, V)$. Set $U = E + NE + N^2E + ... + N^mE$. As in the proof of Proposition 3.15 we verify that the restriction of τ to U is non-degenerate. Notice that

$$U^{\perp} \subseteq (N^m E)^{\perp} = (N^m V)^{\perp} = Ker(N^m).$$

Hence, $ht(N, U^{\perp}) < ht(N, V)$. After a finite number of steps we obtain

$$V = U^{(1)} \oplus U^{(2)} \oplus \ldots \oplus U^{(r)},$$

where the spaces $U^{(j)}$ are graded, N-invariant, mutually orthogonal, each pair $(N, U^{(j)})$ is uniform and

$$ht(N, U^{(1)}) > ht(N, U^{(2)}) > \ldots > ht(N, U^{(r)}).$$

Now we split each $(N, U^{(j)})$ into indecomposables and obtain

$$V = V^{(1)} \oplus V^{(2)} \oplus \ldots \oplus V^{(s)}.$$

Denote $m_i = ht(N, V^{(i)})$ and let $F^{(i)}$ be a graded subspace of $V^{(i)}$ fulfilling conditions of Theorem 3.11 for the restriction of N to $V^{(i)}$. Then the sequences $(F^{(1)}, F^{(2)}, \ldots, F^{(s)})$ and $(m_1, m_2, \ldots, m_s)$ satisfy conditions (1)–(2) of the theorem.

Consider a nilpotent element N' in the $G(V,\tau)_0$-orbit of N and assume that the sequences $(F'^{(1)}, F'^{(2)}, \ldots, F'^{(s')})$ and $m'_1 \geq m'_2 \geq \cdots \geq m'_{s'}$ satisfy conditions (1)–(3). Then $m_1 = ht(V, N) = ht(V, N') = m'_1$. Denote this number by m and let

$$\begin{aligned} \ell &= \max\{i : \ m_i = m\}, \\ \ell' &= \max\{i : \ m'_i = m\}. \end{aligned}$$

Then the spaces

$$\begin{aligned} V/N^m V &\approx F^{(1)} \oplus F^{(2)} \oplus \cdots \oplus F^{(\ell)}, \\ V/N'^m V &\approx F'^{(1)} \oplus F'^{(2)} \oplus \cdots \oplus F'^{(\ell')} \end{aligned}$$

are isomorphic and the isomorphism becomes an isometry when we equip the spaces with the forms $\tau_{(1)} \oplus \tau_{(2)} \oplus \cdots \oplus \tau_{(\ell)}$ and $\tau'_{(1)} \oplus \tau'_{(2)} \oplus \cdots \oplus \tau'_{(\ell')}$, respectively. Hence $\ell = \ell'$ and, up to permutation of indices i, the formed spaces $(F^{(i)}, \tau_{(i)})$ and $(F'^{(i)}, \tau'_{(i)})$ are isometric for $i = 1, 2, \ldots, \ell$. Now, the necessity of the condition follows by induction on dimension of V.

It is clear from Corollary 3.13 that the above argument may be reversed. Hence the proof is complete. □

Corollary 3.20 *Let $(F^{(1)}, \tau^{(1)})$, $(F^{(2)}, \tau^{(2)}), \ldots,$ $(F^{(t)}, \tau^{(t)})$ be a system of representatives for isometry classes of indecomposable, $\mathbb{Z}/n\mathbb{Z}$-graded formed spaces with hermitian or skew-hermitian forms. Nilpotent orbits of the group $G(V,\tau)_0$ in $\mathfrak{g}(V,\tau)_a$ are parameterized by pairs of sequences*

$$\begin{aligned} &m_1 \geq m_2 \geq \cdots \geq m_s \geq 0, \quad m_j \in \mathbb{Z}; \\ &(F^{(i_1)}, F^{(i_2)}, \ldots, F^{(i_s)}), \quad 1 \leq i_j \leq t, \end{aligned}$$

such that

1. *for every* $j = 1, 2, \ldots, s$, $(F^{(i_j)}, \tau^{(i_j)})$ *is an* (a, m_j)*-admissible space;*

2. $\underline{dim}(V) = \sum_{j=1}^{s}(1 + \eta_a + \eta_a^2 + \cdots + \eta_a^{m_j})\underline{dim}(F^{(i_j)})$;

3. *if* $m_j = m_{j+1}$ *then* $i_j \leq i_{j+1}$.

Proof: Due to Theorem 3.19, it remains to show that for every pair of sequences $(m_1, m_2, \ldots, m_s)$ and $(F^{(i_1)}, F^{(i_2)}, \ldots, F^{(i_s)})$ fulfilling conditions *1.–3.* of the corollary, there exists a corresponding nilpotent element $N \in \mathfrak{g}(V, \tau)_a$.

For an $a \in \mathbb{Z}/n\mathbb{Z}$ and a $\mathbb{Z}/n\mathbb{Z}$-graded vector space W, let $\eta_a W$ be a copy of W shifted in grading by a. For $j = 1, 2, \ldots, s$, let

$$V'^{(j)} = F^{(i_j)} \oplus \eta_a F^{(i_j)} \oplus \eta_a^2 F^{(i_j)} \oplus \cdots \oplus \eta_a^{m_j} F^{(i_j)}$$

and

$$V' = V'^{(1)} \oplus V'^{(2)} \oplus \cdots \oplus V'^{(s)}.$$

We equip V' with a sesquilinear form τ' such that the spaces $V'^{(j)}$ are mutually orthogonal and for $u \in F^{(i_j)}$, $v \in F_b^{(i_j)}$

$$\tau'(\eta_a^k u, \eta_a^l v) = \begin{cases} (-1)^k(-1)^{ak(m_j+b)}\delta(k)^a \tau^{(i_j)}(u, v), & \text{if } k + l = m_j, \\ 0, & \text{otherwise.} \end{cases} \tag{19}$$

Then it follows from (12) that the formed spaces (V, τ) and (V', τ') are isometric. Defining a nilpotent endomorphism N' of V' by

$$N'(\eta_a^k u) = \begin{cases} \eta_a^{k+1}(u), & \text{for } k < m_j, \\ 0, & \text{for } k = m_j, \end{cases} \quad (u \in F^{(i_j)}), \tag{20}$$

we obtain the orbit for which the pair of parameterizing sequences coincides with the pair $(m_1, m_2, \ldots, m_s)$, $(F^{(i_1)}, F^{(i_2)}, \ldots, F^{(i_s)})$. □

4 Dual pairs of type II

Let $V = V_0 \oplus V_1$ be a $\mathbb{Z}/2\mathbb{Z}$-graded vector space over $\mathbb{D}$. The group $GL(V)_0 = GL(V) \cap \mathrm{E}nd(V)_0$ of degree zero linear automorphisms of V is isomorphic to the direct product $GL(V_0) \times GL(V_1)$. The results of section 3.1 give the classification of nilpotent orbits of $GL(V)_0$ in $\mathrm{E}nd(V)_0 = \mathrm{E}nd(V_0) \times \mathrm{E}nd(V_1)$, which reduces to the well known classification of nilpotent orbits of $GL(V_j)$ in $\mathrm{E}nd(V_j)$ via the Jordan normal form, and the classification of nilpotent orbits of $GL(V)_0$ in $\mathrm{E}nd(V)_1 = \mathrm{H}om(V_0, V_1) \oplus \mathrm{H}om(V_1, V_0)$, which is also well known (see [DKP], Section 2).

The formula

$$\langle (A,B),(A',B')\rangle = \mathrm{Tr}_{\mathbb{D}/\mathbb{R}}(AB') - \mathrm{Tr}_{\mathbb{D}/\mathbb{R}}(BA') \quad (A,A' \in \mathrm{H}om(V_0,V_1), B,B' \in \mathrm{H}om(V_1,V_0)) \tag{21}$$

defines a non-degenerate symplectic form on the real vector space $W = \mathrm{H}om(V_0,V_1) \oplus \mathrm{H}om(V_1,V_0)$. The action of the group $GL(V_0) \times GL(V_1)$ on W preserves this form, hence the groups $GL(V_0)$, $GL(V_1)$ form a dual pair of type II in the symplectic group $Sp(W)$.

The maps

$$\mathrm{E}nd(V)_1 \ni N \to N^2|_{V_0} \in \mathrm{E}nd(V_0), \tag{22}$$

$$\mathrm{E}nd(V)_1 \ni N \to N^2|_{V_1} \in \mathrm{E}nd(V_1) \tag{23}$$

coincide with the moment maps

$$\mu_k : W \longrightarrow \mathrm{E}nd(V_k), \tag{24}$$

$$\mathrm{H}om(V_0,V_1) \oplus \mathrm{H}om(V_1,V_0) \ni (A,B) \to BA \in \mathrm{E}nd(V_0), \tag{25}$$

$$\mathrm{H}om(V_0,V_1) \oplus \mathrm{H}om(V_1,V_0) \ni (A,B) \to AB \in \mathrm{E}nd(V_1). \tag{26}$$

4.1 Nilpotent orbits in $\mathfrak{gl}_n(\mathbb{D})$

Theorem 3.7 and Corollary 3.8 give the following classification of nilpotent orbits of $GL(V)_0$ in $\mathrm{E}nd(V)_0 = \mathrm{E}nd(V_0) \times \mathrm{E}nd(V_1)$ in terms of the sizes of the blocks of the Jordan normal form of the restriction of a nilpotent endomorphism of V to V_k:

Corollary 4.1 *The nilpotent orbits of the group $GL(V)_0$ in $\mathrm{E}nd(V)_0 = \mathrm{E}nd(V_0) \times \mathrm{E}nd(V_1)$ are parametrized by pairs of sequences*

$$m_1 \geq m_2 \geq \cdots \geq m_s > 0, \quad m_j \in \mathbb{N};$$
$$(b_1, b_2, \ldots, b_s), \quad 0 \leq b_j \leq 1,$$

such that

1. $\dim(V_k) = \sum_{b_j=k}(m_j+1)$,
2. *if* $m_j = m_{j+1}$ *then* $b_j \leq b_{j+1}$,

where for each $j = 1, \ldots, s$ m_j+1 is the size of the appropriate block in V_{a_j}.

In particular, the nilpotent orbits of $GL(V_k)$ in $\mathrm{E}nd(V_k)$ are parametrized by sequences $m_1 \geq m_2 \geq \cdots \geq m_s > 0$ satisfying $\sum_j(m_j+1) = \dim(V_k)$.

4.2 Nilpotent orbits in W

Theorem 3.7 and Corollary 3.8 give the classification of nilpotent orbits of the group $GL(V)_0 = GL(V_0) \times GL(V_1)$ in $W = \mathrm{E}nd(V)_1$ in terms of the parameters of the Jordan normal form. For an alternate description of the parametrization of orbits the reader may consult [DKP], Section 2. Let

$$d_k(b, m) = \left[\frac{m+1}{2}\right] + r, \quad k = 0, 1, \tag{27}$$

where $r = 0$ unless m is even, in which case $r = \delta_{k,b}$ is the Kronecker delta.

Corollary 4.2 *The nilpotent orbits of the group $GL(V)_0$ in $W = \mathrm{E}nd(V)_1$ are parametrized by pairs of sequences*

$$m_1 \geq m_2 \geq \cdots \geq m_s \geq 0, \quad m_j \in \mathbb{Z};$$
$$(b_1, b_2, \ldots, b_s), \quad 0 \leq b_j \leq 1,$$

such that

1. $\dim(V_k) = \sum_j d_k(b_j, m_j)$,
2. *if* $m_j = m_{j+1}$ *then* $b_j \leq b_{j+1}$.

We can now describe the action of the moment maps (24) on nilpotent orbits.

Corollary 4.3 *Let $\mathcal{O} \subseteq W$ be the nilpotent orbit which corresponds to the pair $(m_1, \ldots, m_s)$, $(b_1, \ldots, b_s)$, then the image $\mu_k(\mathcal{O})$ is equal to the nilpotent orbit in $\mathrm{E}nd(V_k)$ corresponding to the sequence $(d_k(m_1, b_1), \ldots, d_k(m_s, b_s))$.*

We end this section by giving a detailed description of all non-zero nilpotent indecomposable elements (N, V), $N \in \mathrm{E}nd(V)_1$.

Proposition 4.4 *The following is a complete list of all non-zero nilpotent indecomposable elements (N, V), $N \in \mathrm{E}nd(V)_1$.*

(a) $V = \bigoplus_{k=0}^{m} \mathbb{D}v_k, \ v_{even} \in V_0, \ v_{odd} \in V_1, \quad v_k = N^k v_0 \neq 0, 0 \leq k \leq m, N v_m = 0;$

(b) $V = \bigoplus_{k=1}^{m+1} \mathbb{D}v_k, \ v_{even} \in V_0, \ v_{odd} \in V_1, \quad v_{k+1} = N^k v_1 \neq 0, 0 \leq k \leq m, N v_{m+1} = 0.$

4.3 Nilpotent orbits in $\mathfrak{p}_{\mathbb{C}}$

Let $\mathfrak{gl}_n(\mathbb{D}) = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition of the general Lie algebra. The complexification $K_{\mathbb{C}}$ of the maximal compact subgroup $K \subseteq GL_n(\mathbb{D})$ acts on the complexification $\mathfrak{p}_{\mathbb{C}}$ of $\mathfrak{p}$. The classification of the nilpotent orbits in this case is well known ([S1]). Both in the case $\mathbb{D} = \mathbb{R}$ and in the case $\mathbb{D} = \mathbb{H}$ a nilpotent $K_{\mathbb{C}}$-orbit in $\mathfrak{p}_{\mathbb{C}}$ is uniquely determined by the Jordan canonical form of any of its members, hence by the corresponding partition of n if $\mathbb{D} = \mathbb{H}$ and of $2n$ if $\mathbb{D} = \mathbb{H}$. In the first case all partitions arise, in the second case a partition arises if and only if each of its parts occurs with an even multiplicity.

4.4 Nilpotent orbits in $W_{\mathbb{C}}^{+}$

The complete classification of the nilpotent orbits in $W_{\mathbb{C}}^{+}$ has been given in [DKP] (in fact as noted in [DKP], in the context of symmetric spaces it has been described earlier by Ohta). It turns out that it can be also obtained as a special case of the classification of section 3.

Consider first the case $\mathbb{D} = \mathbb{R}$. Let $U = U_0 \oplus U_1 \oplus U_2 \oplus U_3$ be the $\mathbb{Z}/4\mathbb{Z}$-graded complex vector space defined by $U_0 = V_0 \otimes \mathbb{C}$, $U_2 = V_1 \otimes \mathbb{C}$, $U_1 = U_3 = 0$, endowed with a nondegenerate symmetric form φ with U_0 orthogonal to U_2. Then the group $G(U, \varphi)_0$ of homogeneous isometries of U, equal to $O(U_0) \times O(U_2)$, is isomorphic to $K_{\mathbb{C}} \times K'_{\mathbb{C}}$, and its conjugation action on $\mathfrak{g}(U, \varphi)_2$ can be identified with the action of $K_{\mathbb{C}} \times K'_{\mathbb{C}}$ on $W_{\mathbb{C}}^{+}$ (see Section 3 of [DKP]). It is easy to see that Theorem 3.19 and Corollary 3.20 describe a classification of nilpotent orbits in $W_{\mathbb{C}}^{+}$ equivalent to theorem 3.6 in [DKP].

The case $\mathbb{D} = \mathbb{H}$ is similar. Let $U = U_0 \oplus U_1 \oplus U_2 \oplus U_3$ be the $\mathbb{Z}/4\mathbb{Z}$-graded complex vector space, defined by $U_0 = U_2 = 0$, $U_1 = V_0|_{\mathbb{C}}$, $U_3 = V_1|_{\mathbb{C}}$, with the complex structures being the restriction of the structures of vector spaces over $\mathbb{H}$. Let φ be a nondegenerate skew-symmetric form on U with U_1 orthogonal to U_3. Then $\mathfrak{g}(U, \varphi)_2$ can be identified with $W_{\mathbb{C}}^{+}$. The group $G(U, \varphi)_0$ of homogeneous isometries of U is equal to $Sp(U_1) \times Sp(U_3)$, and it is isomorphic to $K_{\mathbb{C}} \times K'_{\mathbb{C}}$. Its conjugation action on $\mathfrak{g}(U, \varphi)_2$ can be identified with the action of $K_{\mathbb{C}} \times K'_{\mathbb{C}}$ on $W_{\mathbb{C}}^{+}$ (see Section 4 of [DKP]). Theorem 3.19 and Corollary 3.20 give a classification of nilpotent orbits in $W_{\mathbb{C}}^{+}$ equivalent to Theorem 4.5 in [DKP].

5 Dual Pairs of type I

Now we consider the case of the color Lie algebra of a $\mathbb{Z}/2\mathbb{Z}$-graded formed space $V = V_0 \oplus V_1$. Let τ be the form considered in (9) with $\sigma = 1$. Then

Table 1: List of indecomposable σ'-hermitian forms

$(\mathbb{D},\iota)$	$\sigma'=1$	$\sigma'=-1$
$(\mathbb{R},\ id)$	$(\mathbb{R},+)\quad (x,y)\mapsto xy$ $(\mathbb{R},-)\quad (x,y)\mapsto -xy$	$(\mathbb{R}^2,\ sk)\quad (x,y)\mapsto x\left[\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right]y^t$
$(\mathbb{C},\ id)$	$(\mathbb{C},\ sym)\quad (x,y)\mapsto xy$	$(\mathbb{C}^2,\ sk)\quad (x,y)\mapsto x\left[\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right]y^t$
$(\mathbb{C},\ \bar{\cdot})$	$(\mathbb{C},+)\quad (x,y)\mapsto x\bar{y}$ $(\mathbb{C},-)\quad (x,y)\mapsto -x\bar{y}$	$(\mathbb{C},\ +i)\quad (x,y)\mapsto xi\bar{y}$ $(\mathbb{C},\ -i)\quad (x,y)\mapsto -xi\bar{y}$
$(\mathbb{H},\ \bar{\cdot})$	$(\mathbb{H},+)\quad (x,y)\mapsto x\bar{y}$ $(\mathbb{H},-)\quad (x,y)\mapsto -x\bar{y}$	$(\mathbb{H},\ sk)\quad (x,y)\mapsto xj\bar{y}$

$\tau=\tau_0\oplus\tau_1$, where τ_0 is a non-degenerate hermitian form on V_0 and τ_1 is a non-degenerate skew-hermitian form on V_1.

The group $G(V,\tau)_0$, defined in (10), is isomorphic to the direct product $G(V_0,\tau_0)\times G(V_1,\tau_1)$, by restriction. Similarly, the component of degree 0 of $\mathfrak{g}(V,\tau)$ is a Lie algebra isomorphic to the direct sum of of the corresponding Lie algebras

$$\mathfrak{g}(V,\tau)_0=\mathfrak{g}(V_0,\tau_0)\oplus\mathfrak{g}(V_1,\tau_1).$$

Hence the classification of $G(V,\tau)_0$ orbits in $\mathfrak{g}(V,\tau)_0$ is equivalent to the classification of $G(V_0,\tau_0)$ orbits in $\mathfrak{g}(V_0,\tau_0)$ and $G(V_1,\tau_1)$ orbits in $\mathfrak{g}(V_1,\tau_1)$. We consider this case in subsection 5.1.

The action of $G(V,\tau)_0$ on $\mathfrak{g}(V,\tau)_1$ may be viewed in terms of dual pairs of type I. We explain it in subsection 5.2.

5.1 Nilpotent orbits in the Lie algebra of an isometry group

In order to describe $G(V,\tau)_0$ orbits in $\mathfrak{g}(V,\tau)_0$ we begin with a classification of $(0,m)$-admissible spaces. It follows from Proposition 3.17 that if the space $(\tilde{V},\tilde{\tau})$ is $(0,m)$-admissible, then $\tilde{V}=\tilde{V}_b$ for some $b=0,1$ and $\tilde{\tau}$ is σ'-hermitian nondegenerate indecomposable form with $\sigma'=(-1)^{b+m}$. Such forms are well known. We collect them in Table 1. A complete classification of the orbits is given by Theorem 3.19 (with $a=0$).

5.2 Nilpotent orbits in W

Let $W=\mathrm{H}om(V_0,V_1)$. The groups $G(V_0,\tau_0)$, $G(V_1,\tau_1)$ act on the space W by the formula:

$$\begin{gathered} g_0(w)=wg_0^{-1},\quad g_1(w)=g_1w \\ (g_0\in G(V_0,\tau_0),g_1\in G(V_1,\tau_1),w\in W). \end{gathered} \tag{28}$$

Define a map $\mathrm{H}om(V_0,V_1)\ni w\to w^*\in \mathrm{H}om(V_1,V_0)$ by

$$\tau_1(wv_0,v_1)=\tau_0(v_0,w^*v_1)\qquad (v_0\in V_0,v_1\in V_1). \tag{29}$$

Then the formula

$$\langle w, w' \rangle = -tr_{\mathbb{D}/\mathbb{R}}(w'^* w) \qquad (w, w' \in \mathrm{H}om(V_0, V_1)) \tag{30}$$

defines a non-degenerate symplectic form on the real vector space $\mathrm{H}om(V_0, V_1)$. It is easy to see that the action (28) preserves the form (30). Hence the groups $G(V_0, \tau_0)$, $G(V_1, \tau_1)$ form a dual pair of type I in the symplectic group defined by the form (30). Furthermore, the maps

$$\begin{aligned} \nu_0 &: \mathfrak{g}(V, \tau)_1 \ni N \to N^2|_{V_0} \in \mathfrak{g}(V_0, \tau_0), \\ \nu_1 &: \mathfrak{g}(V, \tau)_1 \ni N \to N^2|_{V_1} \in \mathfrak{g}(V_1, \tau_1) \end{aligned} \tag{31}$$

coincide with the moment maps

$$\begin{aligned} &\mathrm{H}om(V_0, V_1) \ni w \to -w^* w \in \mathfrak{g}(V_0, \tau_0), \\ &\mathrm{H}om(V_0, V_1) \ni w \to -w w^* \in \mathfrak{g}(V_1, \tau_1). \end{aligned} \tag{32}$$

Lemma 5.1 *The map $\mathfrak{g}(V, \tau)_1 \ni N \to N|_{V_0} \in \mathrm{H}om(V_0, V_1)$ is an $\mathbb{R}$-linear bijection which intertwines the adjoint action of $G(V, \tau)_0$ on $\mathfrak{g}(V, \tau)_1$ with the action (7.1) of $G(V_0, \tau_0) \times G(V_1, \tau_1)$ on $\mathrm{H}om(V_0, V_1)$.*

Proof: Let $N \in \mathfrak{g}(V, \tau)_1$. Then for $v_0 \in V_0$ and $v_1 \in V_1$

$$\tau_1(Nv_0, v_1) = \tau(Nv_0, v_1) = \tau(v_0, SNv_1) = -\tau_0(v_0, Nv_1).$$

Hence, $N|_{V_1} = -(N|_{V_0})^*$. Thus the $\mathbb{R}$-linear map $N \to N|_{V_0}$ is bijective.

For $g \in G(V, \tau)_0$ let $g_0 = g|_{V_0}$ and let $g_1 = g|_{V_1}$. Then

$$(gNg^{-1})|_{V_0} = g_1(N|_{V_0})g_0^{-1}.$$

□

Notice by the way, that in terms of Lemma (5.1), the symplectic form (30) coincides with the graded trace

$$\langle N, N' \rangle = \frac{1}{4} tr_{\mathbb{D}/\mathbb{R}}([SN, N']), \quad (N, N' \in \mathfrak{g}(V, \tau)_1), \tag{33}$$

where $[-, -]$ is the bracket defined in (3) i.e.

$$[SN, N'] = SNN' + N'SN \in \mathfrak{g}(V, \tau)_1. \tag{34}$$

We see from Lemma 5.1 that the problem of classifying the nilpotent orbits in the symplectic space $\mathrm{H}om(V_0, V_1)$ under the action of the dual pair $G(V_0, \tau_0)$, $G(V_1, \tau_1)$ is equivalent to the problem of classifying the nilpotent $G(V, \tau)_0$-orbits in $\mathfrak{g}(V, \tau)_1$, considered in section 2.2. In order to obtain a classification of the orbits we need a list of $(1, m)$-admissible spaces. Such a list is provided in Table 2. The following proposition gives an explicit description of indecomposable graded nilpotent morphisms. The proposition follows directly from Theorem 3.19 and Table 2.

Table 2: List of $\mathbb{Z}/2\mathbb{Z}$-graded $(1,m)$-admissible spaces

Case 1. m even

$(\mathbb{D},\iota)$	$m\equiv 0 \pmod 4$	$m\equiv 2 \pmod 4$
$(\mathbb{R},\ id)$	$(\tilde V_0,\tilde\tau) = (\mathbb{R},\ +)$ $(\tilde V_0,\tilde\tau) = (\mathbb{R},\ -)$ $(\tilde V_1,\tilde\tau) = (\mathbb{R}^2,\ sk)$	$(\tilde V_0,\tilde\tau) = (\mathbb{R}^2,\ sk)$ $(\tilde V_1,\tilde\tau) = (\mathbb{R},\ +)$ $(\tilde V_1,\tilde\tau) = (\mathbb{R},\ -)$
$(\mathbb{C},\ id)$	$(\tilde V_0,\tilde\tau) = (\mathbb{C},\ sym)$ $(\tilde V_1,\tilde\tau) = (\mathbb{C}^2,\ sk)$	$(\tilde V_0,\tilde\tau) = (\mathbb{C}^2,\ sk)$ $(\tilde V_1,\tilde\tau) = (\mathbb{C},\ sym)$
$(\mathbb{C},\bar{\cdot}\,)$	$(\tilde V_0,\tilde\tau) = (\mathbb{C},\ +)$ $(\tilde V_0,\tilde\tau) = (\mathbb{C},\ -)$ $(\tilde V_1,\tilde\tau) = (\mathbb{C},\ i)$ $(\tilde V_1,\tilde\tau) = (\mathbb{C},\ -i)$	$(\tilde V_0,\tilde\tau) = (\mathbb{C},\ i)$ $(\tilde V_0,\tilde\tau) = (\mathbb{C},\ -i)$ $(\tilde V_1,\tilde\tau) = (\mathbb{C},\ +)$ $(\tilde V_1,\tilde\tau) = (\mathbb{C},\ -)$
$(\mathbb{H},\bar{\cdot}\,)$	$(\tilde V_0,\tilde\tau) = (\mathbb{H},\ +)$ $(\tilde V_0,\tilde\tau) = (\mathbb{H},\ -)$ $(\tilde V_1,\tilde\tau) = (\mathbb{H},\ sk)$	$(\tilde V_0,\tilde\tau) = (\mathbb{H},\ sk)$ $(\tilde V_1,\tilde\tau) = (\mathbb{H},\ +)$ $(\tilde V_1,\tilde\tau) = (\mathbb{H},\ -)$

Case 2. m odd

$m\equiv 1 \pmod 4$	$m\equiv 3 \pmod 4$
$\tilde V = \mathbb{D}[0]\oplus\mathbb{D}[1]$ $\tilde\tau(x,y) = x\cdot\left[\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right]\cdot\iota(y)^t$	$\tilde V = \mathbb{D}[0]\oplus\mathbb{D}[1]$ $\tilde\tau(x,y) = x\cdot\left[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right]\cdot\iota(y)^t$

Proposition 5.2 *The following is a complete list of all non-zero nilpotent indecomposable elements (N, V), $N \in \mathfrak{g}(V,\tau)_1$, up to similarity.*

(a)
$$\begin{aligned}
&m \in 4\mathbb{Z};\\
&V = \textstyle\sum_{k=0}^{m} \mathbb{D}v_k,\ \ v_{even} \in V_0,\ \ v_{odd} \in V_1;\\
&v_k = N^k v_0 \neq 0, 0 \leq k \leq m, N v_m = 0;\\
&\tau(v_k, v_l) = 0 \ \textit{if}\ l \neq m-k, \tau(v_k, v_{m-k}) = (-1)^k \delta(k)\delta(\tfrac{m}{2}) sgn(\tau_0),\\
&\textit{where } sgn(\tau_0) = 1 \ \textit{if}\ \mathbb{D} = \mathbb{C} \ \textit{and}\ \iota = 1;
\end{aligned}$$

(b)
$$\begin{aligned}
&m \in 4\mathbb{Z}, \mathbb{D} \neq \mathbb{R}, \iota \neq 1;\\
&V = \textstyle\sum_{k=1}^{m+1} \mathbb{D}v_k,\ \ v_{even} \in V_0,\ \ v_{odd} \in V_1;\\
&v_{k+1} = N^k v_1 \neq 0, 0 \leq k \leq m, N v_{m+1} = 0;\\
&\tau(v_k, v_l) = 0 \ \textit{if}\ l \neq m+2-k, \tau(v_k, v_{m+2-k}) = \delta(k-1)\tau(v_1, v_{m+1}),\\
&\tau(v_1, v_{m+1}) = i\, sgn(-i\tau_1)\delta(1 + \tfrac{m}{2}) \ \textit{if}\ \mathbb{D} = \mathbb{C};\\
&\tau(v_1, v_{m+1}) = j \ \textit{if}\ \mathbb{D} = \mathbb{H};
\end{aligned}$$

(c)
$$\begin{aligned}
&m \in 4\mathbb{Z}, \mathbb{D} \neq \mathbb{H}, \iota = 1;\\
&V = \textstyle\sum_{k=1}^{m+1} (\mathbb{D}v_k \oplus \mathbb{D}v'_k),\ \ v_{even}, v'_{even} \in V_0,\ \ v_{odd}, v'_{odd} \in V_1;\\
&v_{k+1} = N^k v_1 \neq 0, v'_{k+1} = N^k v'_1 \neq 0, 0 \leq k \leq m, N v_{m+1} = 0, N v'_{m+1} = 0;\\
&\tau(v_k, v_l) = \tau(v'_k, v'_l) = 0,\ \ 1 \leq k, l \leq m+1,\\
&\tau(v_k, v'_l) = \tau(v'_k, v_l) = 0,\ \ l \neq m+2-k,\\
&\tau(v_k, v'_{m+2-k}) = -\tau(v'_k, v_{m+2-k}) = \delta(k-1),\ \ 1 \leq k \leq m+1;
\end{aligned}$$

(d)
$$\begin{aligned}
&m \in 2\mathbb{Z} \setminus 4\mathbb{Z}, \mathbb{D} \neq \mathbb{R}, \iota \neq 1;\\
&V = \textstyle\sum_{k=0}^{m} \mathbb{D}v_k,\ \ v_{even} \in V_0,\ \ v_{odd} \in V_1;\\
&v_k = N^k v_0 \neq 0, 0 \leq k \leq m, N v_m = 0;\\
&\tau(v_k, v_l) = 0 \ \textit{if}\ l \neq m-k, \tau(v_k, v_{m-k}) = \delta(k-1) i sgn(-i\tau_1),\\
&\textit{(here } -i\tau_1 \textit{ is hermitian)};
\end{aligned}$$

(e)
$$\begin{aligned}
&m \in 2\mathbb{Z} \setminus 4\mathbb{Z};\\
&V = \textstyle\sum_{k=1}^{m+1} \mathbb{D}v_k,\ \ v_{even} \in V_0,\ \ v_{odd} \in V_1;\\
&v_{k+1} = N^k v_1 \neq 0, 0 \leq k \leq m, N v_{m+1} = 0;\\
&\tau(v_k, v_l) = 0 \ \textit{if}\ l \neq m+2-k, \tau(v_k, v_{m+2-k}) = \delta(k)\tau(v_1, v_{m+1}),\\
&\tau(v_1, v_{m+1}) = -\delta(1 + \tfrac{m}{2}) sgn(\tau_0);
\end{aligned}$$

(f)
$$\begin{aligned}
&m \in 2\mathbb{Z} \setminus 4\mathbb{Z}, \mathbb{D} \neq \mathbb{H}, \iota = 1;\\
&V = \textstyle\sum_{k=0}^{m} (\mathbb{D}v_k \oplus \mathbb{D}v'_k),\ \ v_{even}, v'_{even} \in V_0,\ \ v_{odd}, v'_{odd} \in V_1;\\
&v_k = N^k v_0 \neq 0, v'_k = N^k v'_0 \neq 0, 0 \leq k \leq m, N v_m = 0, N v'_m = 0;\\
&\tau(v_k, v_l) = \tau(v'_k, v'_l) = 0,\ \ 0 \leq k, l \leq m,\\
&\tau(v_k, v'_l) = \tau(v'_k, v_l) = 0,\ \ l \neq m-k,\\
&\tau(v_k, v'_{m-k}) = -\tau(v'_k, v_{m-k}) = \delta(k-1),\ \ 0 \leq k \leq m;
\end{aligned}$$

$$
(g)\quad
\begin{array}{l}
m \in 2\mathbb{Z}+1;\\
V = \sum_{k=0}^{m} (\mathbb{D}v_k \oplus \mathbb{D}v'_{k+1}),\ \ v_{even}, v'_{even} \in V_0,\ \ v_{odd}, v'_{odd} \in V_1;\\
v_k = N^k v_0 \neq 0, v'_{k+1} = N^k v'_1 \neq 0, 0 \leq k \leq m, Nv_m = 0, Nv'_{m+1} = 0;\\
\tau(v_k, v_l) = \tau(v'_{k+1}, v'_{l+1}) = 0,\ \ 0 \leq k, l \leq m,\\
\tau(v_k, v'_{l+1}) = \tau(v'_{k+1}, v_l) = 0,\ \ l \neq m-k,\\
\tau(v_k, v'_{m+1-k}) = \delta(k)\delta(m), \tau(v'_{k+1}, v_{m-k}) = \delta(k-1),\ \ 0 \leq k \leq m;
\end{array}
$$

5.3 Nilpotent orbits in $\mathfrak{p}_{\mathbb{C}}$

We assume that if $\mathbb{D} = \mathbb{C}$ then the involution ι is nontrivial.

Lemma 5.3 *Up to conjugation by $G(V,\tau)_0$, there is a unique element $T \in G(V,\tau)_0$ such that $T^2 = S$ and such that the form $\tau(Tu, v)$ $(u, v \in V)$, is hermitian and positive definite. In particular $\theta = Ad(T)$ is a Cartan involution on $\mathfrak{g}(V,\tau)_0$.*

Furthermore, the following diagram

$$
\begin{array}{ccc}
\mathfrak{g}(V,\tau)_1 \ni N & \longrightarrow & N|_{V_0} \in \mathrm{Hom}(V_0, V_1)\\
T\big\downarrow & & J\big\downarrow\\
\mathfrak{g}(V,\tau)_1 \ni TNT^{-1} & \longrightarrow & (TNT^{-1})|_{V_0} \in \mathrm{Hom}(V_0, V_1)
\end{array}
$$

defines positive compatible complex structure J on the symplectic space $\mathrm{Hom}(V_0, V_1)$ (i.e. J preserves the symplectic form $\langle\ ,\ \rangle$ defined in (7.3), $J^2 = -1$ and the form $\langle J\ ,\ \rangle$ is symmetric and positive definite).

If we define a map $\mathfrak{g}(V,\tau)_1 \ni N \to N^\dagger \in \mathfrak{g}(V,\tau)_1$ by $\tau(TNu, v) = \tau(Tu, N^\dagger v)$, then $J(N) = SN^\dagger$.

Proof: We shall give an explicit construction of T. For integers $p \geq 0$, $q \geq 0$ $(p + q > 0)$ and for $n \geq 1$, let

$$
I_{p,q} = \begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix}, \quad J_{2n} = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}.
$$

There are three cases to consider.

$$
\text{(a)}\quad
\begin{array}{l}
\mathbb{D} = \mathbb{R},\ \ V_0 = \mathbb{R}^{p+q}, V_1 = \mathbb{R}^{2n},\\
\tau_0(u,v) = u^t I_{p,q} v \qquad (u, v \in V_0),\\
\tau_1(u,v) = u^t J_{2n} v \qquad (u, v \in V_1),\\
T|_{V_0} := I_{p,q},\ \ T|_{V_1} := J_{2n};
\end{array}
$$

$$
\text{(b)}\quad
\begin{array}{l}
\mathbb{D} = \mathbb{C},\ \ V_0 = \mathbb{C}^{p+q}, V_1 = \mathbb{C}^{r+s},\\
\tau_0(u,v) = u^t I_{p,q}\iota(v) \qquad (u, v \in V_0),\\
\tau_1(u,v) = u^t i I_{r,s}\iota(v) \qquad (u, v \in V_1),\\
T|_{V_0} := I_{p,q},\ \ T|_{V_1} := -iI_{r,s};
\end{array}
$$

$$\text{(c)}\quad \begin{array}{l} \mathbb{D}=\mathbb{H},\ V_0=\mathbb{H}^{p+q}, V_1=\mathbb{H}^n,\\ \tau_0(u,v)=u^tI_{p,q}\iota(v) \qquad (u,v\in V_0),\\ \tau_1(u,v)=u^tjI_n\iota(v) \qquad (u,v\in V_1),\\ T|_{V_0}:=I_{p,q},\quad T|_{V_1}:=\text{right multiplication by } j^{-1},\\ \text{where } j\in\mathbb{H} \text{ is such that } \mathbb{H}=\mathbb{C}\oplus j\mathbb{C},\ \iota(j)=-j=j^{-1}\\ \text{and } jzj^{-1}=\iota(z) \text{ for } z\in\mathbb{C}. \end{array}$$

For the last statement we notice that for $u, v \in V$ and for $N \in \mathfrak{g}(V,\tau)_1$ we have

$$\tau(TNu,v)=\tau(TNT^{-1}Tu,v)=\tau(Tu,STNT^{-1}v).$$

Hence, $N^\dagger = STNT^{-1}$, and therefore $J(N) = TNT^{-1} = SN^\dagger$, as claimed.

Via a case by case analysis we see that the condition $T^2 = S$ determines T up to a sign. But this sign is determined by the positivity of the form $\tau(Tu, v)$. Thus there is a one to one correspondence between the elements T and the maximal compact subgroups. Hence the T is unique up to conjugation. $\square$

Let $K = G(V,\tau)_0^T$ and $\mathfrak{k} = \mathfrak{g}(V,\tau)_0^T$. Since the form $\tau(Tu, v)$ $(u, v \in V)$ is hermitian and positive definite, K is a maximal compact subgroup of $G(V,\tau)_0$ corresponding to the Cartan involution θ and $\mathfrak{k}$ is the Lie algebra of K. Let $\mathfrak{p}$ be the (-1)-eigenspace of θ on $\mathfrak{g}(V,\tau)_0$ so that

$$\mathfrak{g}(V,\tau)_0 \quad = \quad \mathfrak{k}\oplus\mathfrak{p} \tag{35}$$

is the corresponding Cartan decomposition.

In order to describe the complexifications $\mathfrak{p}_\mathbb{C}$ and $K_\mathbb{C}$ of $\mathfrak{p}$ and K lets us define

$$U=V\otimes_\mathbb{R}\mathbb{C} \text{ if } \mathbb{D}=\mathbb{R},\ \text{ and } U=V|_\mathbb{C} \text{ if } \mathbb{D}=\mathbb{C} \text{ or } \mathbb{H}. \tag{36}$$

Then U is a vector space over $\mathbb{C}$ and the element T constructed in Lemma 5.3 acts on U. Since $T^4 = S^2 = 1$, T has at most four eigenvalues: $1, i, -1, -i$. Let

$$U_k=\{u\in U;\ Tu=i^ku\} \qquad (k=0,1,2,3). \tag{37}$$

Then

$$U=U_0\oplus U_1\oplus U_2\oplus U_3 \tag{38}$$

and U is a $\mathbb{Z}/4\mathbb{Z}$-graded vector space over $\mathbb{C}$.

Let us first analyze the case of $\mathbb{D} = \mathbb{C}$. In this case $\mathfrak{p}_\mathbb{C} = \mathrm{E}nd(U)_2$ and $K_\mathbb{C} = GL(U_0) \times GL(U_1) \times GL(U_2) \times GL(U_3)$. Since

$$\mathrm{E}nd(U)_2 \quad = \quad \mathrm{H}om(U_0,U_2)\oplus \mathrm{H}om(U_2,U_0)\oplus \mathrm{H}om(U_1,U_3)\oplus \mathrm{H}om(U_3,U_1),$$

the problem of the classification of nilpotent $K_{\mathbb{C}}$-orbits in $\mathfrak{p}_{\mathbb{C}}$ is equivalent to analogous problem for the nilpotent orbits of $GL(U_0) \times GL(U_2)$ in $\mathrm{H}om(U_0, U_2) \oplus \mathrm{H}om(U_2, U_0)$ and the nilpotent orbits of $GL(U_1) \times GL(U_3)$ in $\mathrm{H}om(U_1, U_3) \oplus \mathrm{H}om(U_3, U_1)$.

This is exactly the problem studied in section 4.2. As an immediate application of corollary 4.2 we get the description of the indecomposable nilpotents in $\mathfrak{p}_{\mathbb{C}}$ in this case. To make things simpler, we assume that $V = V_0$ (so that $\mathfrak{g}(V, \tau) = \mathfrak{u}_{p,q}$ for some p, q).

Proposition 5.4 *If $\mathbb{D} = \mathbb{C}$ and the anti-involution ι is nontrivial, there are two orbits of indecomposable nilpotent elements (N, U) in $\mathfrak{p}_{\mathbb{C}}$, determined by the condition $\tilde{U} = \mathbb{C}[b]i$, $b = 0, 2$.*

If $V = V_1$ (with $\mathfrak{g}(V, \tau) = \mathfrak{u}_{p,q}$ for some p, q as well), the description of orbits is the same, except now $b = 1, 3$.

From now on let $\mathbb{D} = \mathbb{R}$ or $\mathbb{H}$. We shall define a sesqui-linear form ϕ on the space U, as follows.

If $\mathbb{D} = \mathbb{R}$ we let ϕ be the unique complex linear extension of the form τ. Then $\phi|_{U_0+U_2}$ is symmetric, and $\phi|_{U_0}$, $\phi|_{U_2}$ are non-degenerate. Furthermore, $\phi|_{U_1+U_3}$ is skew-symmetric, and $\phi|_{U_1} = 0$, $\phi|_{U_3} = 0$.

If $\mathbb{D} = \mathbb{H}$ we define

$$\phi(u, v) = \left(\frac{1}{2i} \left(i\tau(u, v) + \tau(u, v)i \right) - \tau(u, v) \right) j \quad (u, v \in U). \qquad (39)$$

Lemma 5.5 *The form ϕ defined in* (39) *is $\mathbb{C}$-valued, $\mathbb{C}$-bilinear non-degenerate form on U. The restricted form $\phi|_{U_0+U_2}$ is skew-symmetric, and $\phi|_{U_0}$, $\phi|_{U_2}$ are non-degenerate. Similarly, $\phi|_{U_1+U_3}$ is symmetric, and $\phi|_{U_1} = 0$, $\phi|_{U_3} = 0$. Moreover, for $x \in \mathrm{End}_{\mathbb{H}}(V)_0$ and for $u, v \in V$, we have $\tau(xu, v) = -\tau(u, xv)$ if and only if $\phi(xu, v) = -\phi(u, xv)$.*

Proof: Notice that for any quaternion $q = a + bj$, where $a, b \in \mathbb{C}$, we have

$$\left(\frac{1}{2i} (iq + qi) - q \right) j = b.$$

Hence ϕ takes values in $\mathbb{C}$. Since $zj = j\overline{z}$, for $z \in \mathbb{C}$, it is easy to check that ϕ is bilinear over $\mathbb{C}$. Furthermore, $\iota(q) = \overline{a} - bj$, so

$$\left(\frac{1}{2i} (i\iota(q) + \iota(q)i) - \iota(q) \right) j = -b.$$

Thus if a restriction of τ to a subspace $U' \subset U$ is hermitian then $\phi|_{U'}$ is skew-symmetric, and if $\tau|_{U'}$ is skew-hermitian then $\phi|_{U'}$ is symmetric.

Suppose $\phi(u, v) = 0$ for all $v \in W$. Then $\tau(u, v) \in \mathbb{C}$ for all $v \in W$. But then $\tau(u, jv) = -\tau(u, v)j \in \mathbb{C}j$. Hence $\tau(u, v) = 0$ for all $v \in U$. Thus $u = 0$, so ϕ is non-degenerate.

The last claim follows from the equation (39) defining ϕ in terms of τ and from the following, easy to check, equation expressing τ in terms of ϕ:

$$\tau(u,v) = -\phi(u,jv) + \phi(u,v)j \qquad (u,v \in U).$$

□

Thus in any case, the form ϕ is non-degenerate sesqui-linear form over $\mathbb{C}$. Furthermore, ϕ satisfies condition (9) with $\sigma = 1$ if $\mathbb{D} = \mathbb{R}$ and with $\sigma = -1$ if $\mathbb{D} = \mathbb{H}$.

A straightforward argument shows that $\mathfrak{g}(U,\phi)_0$ coincides with the complexification $\mathfrak{k}_{\mathbb{C}}$ of $\mathfrak{k} = \mathfrak{g}(V,\tau)_0^T$ (the centralizer of T in $\mathfrak{g}(V,\tau)_0$), and that $K = G(V,\tau)_0^T$ is a maximal compact subgroup of $G(V,\tau)_0$. Thus

$$\begin{aligned} K_{\mathbb{C}} = G(U,\phi)_0^T &= \{g \in G(U,\phi);\ gT = Tg\} \\ &= \{g \in G(U,\phi);\ g(U_k) = U_k \text{ and for all } k\}, \qquad (40) \\ \mathfrak{p}_{\mathbb{C}} = \mathfrak{g}(U,\phi)_2 &= \{x \in g(U,\phi);\ x(U_k) \subseteq U_{k+2} \text{ for all } k\}. \end{aligned}$$

As explained in theorem 3.19, the orbits of $K_{\mathbb{C}}$ in $\mathfrak{p}_{\mathbb{C}}$ are determined by $(2,m)$-admissible spaces $(\tilde{U},\tilde{\phi})$. According to Proposition 3.17, they look as follows.

Proposition 5.6 *If $\mathbb{D} = \mathbb{R}$ then ϕ satisfies condition (9) with $\sigma = 1$ and $(2,m)$-admissible spaces $(\tilde{U},\tilde{\phi})$ are of the form:*

1. *$\tilde{U} = \mathbb{C}[b]$, where $b+m$ is even, and $\tilde{\phi}$ is symmetric;*
2. *$\tilde{U} = \mathbb{C}[b] \oplus \mathbb{C}[b+2]$, where $b+m$ is odd, and $\tilde{\phi}$ is antisymmetric.*

If $\mathbb{D} = \mathbb{H}$ then ϕ satisfies condition (9) with $\sigma = -1$ and $(2,m)$-admissible spaces $(\tilde{U},\tilde{\phi})$ are of the form:

1. *$\tilde{U} = \mathbb{C}^2[b]$, where $b+m$ is even, and $\tilde{\phi}$ is antisymmetric;*
2. *$\tilde{U} = \mathbb{C}[b] \oplus \mathbb{C}[b+2]$, where $b+m$ is odd, and $\tilde{\phi}$ is symmetric with maximal isotropic subspaces $\mathbb{C}[b]$ and $\mathbb{C}[b+2]$.*

5.4 Nilpotent orbits in $W_{\mathbb{C}}^+$

By definition, the space $W_{\mathbb{C}}^+$ is equal to the i-eigenspace of J acting on the complexification of W. In the case of $\mathbb{D} = \mathbb{C}$ we have $W_{\mathbb{C}}^+ = \mathrm{E}nd(U)_1$ and the problem of classification of $K_{\mathbb{C}}$-orbits in $W_{\mathbb{C}}^+$ is equivalent to the problem of classification of $GL(U)_0$-orbits in $\mathrm{E}nd(U)_1$.

This problem is analogous to the problem studied in section 4.2, we leave the details to the reader.

In the case of $\mathbb{D} = \mathbb{R}$ or $\mathbb{H}$ we have $W_{\mathbb{C}}^+ = \mathfrak{g}(U,\phi)_1$ and $K_{\mathbb{C}}$-orbits in $W_{\mathbb{C}}^+$ are $G(U,\phi)_0$-orbits in $\mathfrak{g}(U,\phi)_1$. In order to understand them in terms of Theorem 3.19, we need to list $(1,m)$-admissible spaces.

Proposition 5.7 *With σ as in Proposition 5.6, all $(1,m)$-admissible spaces $(\tilde{U},\tilde{\phi})$ are as follows:*

1. *$\tilde{U}=\tilde{U}_b$, m is even, and $\tilde{\phi}$ is σ-symmetric;*
2. *$\tilde{U}=\tilde{U}_b\oplus U_{-b-m}$, where $2b+m\neq 0$, and $\tilde{\phi}$ is σ'-symmetric with $\sigma'=(-1)^{(m+1)b}\delta(m)\sigma$.*

6 The Cayley transform and the Kostant-Sekiguchi correspondence

We begin by recalling the Kostant-Sekiguchi bijection.

Let $\mathfrak{g}$ be a reductive Lie algebra over $\mathbb{R}$, and let $\mathfrak{g}_{\mathbb{C}}$ be its complexification. Recall that a standard triple (e,f,h) in $\mathfrak{g}$ is a triple of elements $e,f,h\in\mathfrak{g}$ satisfying conditions

$$[e,f]=h,\ [h,e]=2e,\ [h,f]=-2f.$$

Fix a Cartan involution θ of $\mathfrak{g}$, with Cartan decompositions $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$, $\mathfrak{g}_{\mathbb{C}}=\mathfrak{k}_{\mathbb{C}}\oplus\mathfrak{p}_{\mathbb{C}}$. A standard triple (e,f,h) in $\mathfrak{g}$ is a *Cayley* triple, if

$$f=-\theta(e), h=-\theta(h).$$

By definition, the Cayley transform of a Cayley triple (e,f,h) is the standard triple (e',f',h') in $\mathfrak{g}_{\mathbb{C}}$ defined by

$$\begin{aligned} e' &= \frac{1}{2}(e+f-ih),\\ f' &= \frac{1}{2}(e+f+ih),\\ h' &= -i(e-f). \end{aligned} \tag{41}$$

The Kostant-Sekiguchi bijection maps the G-orbit of e in $\mathfrak{g}$ into the $K_{\mathbb{C}}$-orbit of e' in $\mathfrak{p}_{\mathbb{C}}$, where G and $K_{\mathbb{C}}$ are the adjoint groups of $\mathfrak{g}$ and $\mathfrak{k}_{\mathbb{C}}$ respectively.

Proposition 6.1 *Let (e,f,h) be a standard triple in $\mathfrak{g}_{\mathbb{C}}$, and let*

$$\mathcal{C}=\exp(i\frac{\pi}{4}ad(e+f))\in Aut(\mathfrak{g}_{\mathbb{C}}). \tag{42}$$

Then

$$\begin{aligned} \mathcal{C}(e) &= \frac{1}{2}(e+f-ih),\\ \mathcal{C}(f) &= \frac{1}{2}(e+f+ih),\\ \mathcal{C}(h) &= -i(e-f). \end{aligned} \tag{43}$$

In particular, if (e,f,h) is a Cayley triple in $\mathfrak{g}$, then the triple

$$(\mathcal{C}(e), \mathcal{C}(f), \mathcal{C}(h))$$

is equal to the Cayley transform of (e,f,h).

Proof: The following formulas are easy to check:

$$\begin{aligned}
&ad(e+f)^{2k}e = 2^{2k-1}(e-f) \qquad (k \geq 1),\\
&ad(e+f)^{2k+1}e = -2^{2k}h \qquad (k \geq 0),\\
&ad(e+f)^{2k}f = -2^{2k-1}(e-f) \qquad (k \geq 1),\\
&ad(e+f)^{2k+1}f = 2^{2k}h \qquad (k \geq 0),\\
&ad(e+f)^{2k}h = 2^{2k}h \qquad (k \geq 0),\\
&ad(e+f)^{2k+1}h = -2^{2k+1}(e-f) \qquad (k \geq 0).
\end{aligned}$$

Hence, for any $z \in \mathbb{C}$,

$$\begin{aligned}
&\exp(z\, ad(e+f))e = \cosh^2(z)e - \sinh^2(z)f - \cosh(z)\sinh(z)h,\\
&\exp(z\, ad(e+f))f = -\sinh^2(z)e + \cosh^2(z)f + \cosh(z)\sinh(z)h,\\
&\exp(z\, ad(e+f))h = \cosh(2z)h - \sinh(2z)(e-f).
\end{aligned}$$

Substitution $z = i\frac{\pi}{4}$ ends the proof. □

Let V and τ be as in section 5.1, with additional assumption that either $V = V_0$ or $V = V_1$. Let T be as in section 5.3, let θ be the Cartan involution on $\mathfrak{g}(V,\tau)$ equal to the conjugation by T and assume that $e, f, h \in \mathfrak{g}(V,\tau)$ form a Cayley triple. Then $i(e+f)$ is in the complexification of $\mathfrak{g}(V,\tau)$ which coincides with $\mathfrak{g}(U,\phi)$ as in section 5.4. Set

$$c = \exp(i\frac{\pi}{4}(e+f)) \in G(U,\phi). \tag{44}$$

Then a standard Lie theory argument shows that the automorphism $\mathcal{C} \in \mathrm{Aut}(\mathfrak{g}(V,\tau)_{\mathbb{C}}) = \mathrm{Aut}(\mathfrak{g}(U,\phi))$ defined by (42) is equal to the conjugation by c, in particular proposition 6.1 implies that the Kostant-Sekiguchi bijection maps the orbit of e to the orbit of cec^{-1}.

6.1 The Kostant-Sekiguchi correspondence for indecomposable nilpotents

In this section we will compute the Kostant-Sekiguchi map $e \longmapsto \mathcal{C}(e) = cec^{-1}$ for indecomposable nilpotents $e \in \mathfrak{g}(V,\tau)$. We will do this by a case-by-case analysis, according to the classification of nilpotent orbits in $\mathfrak{g}(V,\tau)$ described in section 5.1. In each case we will
– define the space V,

– describe the action of e on an explicit basis of V,
– define the form τ in this basis,
– define the map $T : V \longrightarrow V$ in this basis,
– compute the element $c \in G(U, \phi)$ in the basis of U induced by the chosen basic of V,
– compute the formed space $(\tilde{U}, \tilde{\phi})$ corresponding to the nilpotent element (cec^{-1}, U) as in section 5.3.

We begin with a general construction, which will be used in all the cases.

Let ξ be a non-degenerate symplectic form on $\mathbb{R}^2$, and let $\epsilon_1, \epsilon_2 \in \mathbb{R}^2$ be a basis such that

$$\xi(\epsilon_1, \epsilon_1) = \xi(\epsilon_2, \epsilon_2) = 0, \quad \xi(\epsilon_1, \epsilon_2) = 1. \tag{45}$$

We extend the form ξ to the tensor algebra of $\mathbb{R}^2$, and restrict to the subspace $S^m\mathbb{R}^2$ of symmetric tensors homogeneous of degree $m = 0, 1, 2, \ldots.$ Then a straightforward calculation shows that

$$\xi(\epsilon_1^k\epsilon_2^{m-k}, \epsilon_1^l\epsilon_2^{m-l}) = \begin{cases} (-1)^{m-k}\dfrac{k!(m-k)!}{m!}, & \text{if } \ l = m-k, \\ 0, & \text{otherwise.} \end{cases} \tag{46}$$

Let $\mathbb{V} = S^m\mathbb{R}^2$. We set

$$\tau(u, v) = (-1)^m\xi(u, v) \qquad (u, v \in \mathbb{V}). \tag{47}$$

Let $v_k = \epsilon_1^k\epsilon_2^{m-k}$, $0 \leq k \leq m$. Then (46) may be rewritten as

$$\tau(v_k, v_l) = \begin{cases} (-1)^k\dfrac{k!(m-k)!}{m!}, & \text{if } \ l = m-k, \\ 0, & \text{otherwise.} \end{cases} \tag{48}$$

Let $\mathbb{T} : \mathbb{V} \to \mathbb{V}$ be a linear map defined by

$$\mathbb{T}v_k = (-1)^{m-k}v_{m-k} \qquad (0 \leq k \leq m). \tag{49}$$

Then (48) implies that $\mathbb{T} \in G(\mathbb{V}, \tau)$ and that the form $\tau(\mathbb{T}\ ,\ )$ is symmetric and positive definite. Let $E, F, H \in \mathfrak{g}(\mathbb{R}^2, \xi)$ (a Lie algebra isomorphic to $sl_2(\mathbb{R})$) be defined by

$$\begin{array}{lll} E(\epsilon_1) = 0, & F(\epsilon_1) = \epsilon_2, & H(\epsilon_1) = \epsilon_1, \\ E(\epsilon_2) = \epsilon_1, & F(\epsilon_2) = 0, & H(\epsilon_2) = -\epsilon_2. \end{array} \tag{50}$$

These elements act on $\mathbb{V}$ and it is easy to check that

$$E(v_k) = (m-k)v_{k+1}, \ F(v_k) = kv_{k-1}, \ H(v_k) = (-m+2k)v_k. \tag{51}$$

The formulas (49) and (51) imply that the elements E, F and H form a Cayley triple in $\mathfrak{g}(\mathbb{V},\tau)$ with respect to the Cartan involution $\theta = Ad(\mathbb{T})$.

The point of this construction is to avoid direct calculation of the exponential in (44). Instead we proceed as follows. Let $\mathbb{U} = \mathbb{C}\otimes\mathbb{V} = S^m\mathbb{C}^2$ be the complexification of $\mathbb{V}$. The complexification of the group $G(\mathbb{R}^2,\xi)$ (isomorphic to $SL(2,\mathbb{C})$) acts on the space $\mathbb{U}$. Consider E, F as elements of $\mathbb{C}^2$. It is easy to check that for $z\in\mathbb{C}$,

$$\begin{aligned}\exp(z(E+F))(\epsilon_1) &= \cosh(z)\epsilon_1 + \sinh(z)\epsilon_2\\ \exp(z(E+F))(\epsilon_2) &= \sinh(z)\epsilon_1 + \cosh(z)\epsilon_2,\end{aligned}$$

and hence, as an endomorphism of $\mathbb{U}$,

$$\exp(z(E+F))(v_k) = \sum_{l=0}^{k}\sum_{j=0}^{m-k}\binom{k}{l}\binom{m-k}{j}\cosh(z)^{m-k+l-j}\sinh(z)^{k-l+j}v_{l+j}.$$

Therefore, by taking $z = i\frac{\pi}{4}$,

$$c(v_k) = 2^{-m/2}\sum_{l=0}^{k}\sum_{j=0}^{m-k}\binom{k}{l}\binom{m-k}{j} i^{k-l+j}v_{l+j}. \qquad (52)$$

The formulas (49) and (52) imply that

$$\mathbb{T}(c(v_k)) = i^{m-2k}c(v_k) \qquad (0\le k\le m). \qquad (53)$$

Consider first the case $\mathbb{D} = \mathbb{R}$. According to section 5.1 there are six possibilities for the formed space $(\tilde{V},\tilde{\tau})$ corresponding to an indecomposable nilpotent element $e\in\mathfrak{g}(V,\tau)$ of height m, namely $(\mathbb{R}[b],+), (\mathbb{R}[b],-)$ with $b+m$ even, $(\mathbb{R}^2[b], sk)$ with $b+m$ odd, $b = 0,1$, where as usual $\mathbb{R}^k[b]$ denotes the graded vector space concentrated in degree b.

For an explicit realization of the first case let $(V,\tau) = (\mathbb{V}[b],\tau)$, $T = \mathbb{T}$ and $e = E$. The space V has a decomposition

$$V = \mathbb{R}v_0\oplus\mathbb{R}v_1\oplus ... \oplus\mathbb{R}v_m, \qquad (54)$$

with $e(\mathbb{R}v_k) = \mathbb{R}v_{k+1}$ and

$$\tau(v_0, e^m(v_0)) = m!, \qquad (55)$$

which implies that the space $(\tilde{V},\tilde{\tau})$ is of the form $(\mathbb{R}[b],+)$.

The complexification $U = \mathbb{U}$ has a decomposition

$$U = \mathbb{C}c(v_0)\oplus\mathbb{C}c(v_1)\oplus ... \oplus\mathbb{C}c(v_m), \qquad (56)$$

with $cec^{-1}(\mathbb{C}c(v_k)) = \mathbb{C}c(v_{k+1})$. Moreover, by (53),

$$T(c(v_0)) = i^m c(v_0).$$

This implies that the formed space $(\tilde{U}, \tilde{\phi})$ is of the form $(\mathbb{C}[\overline{m}], \tilde{\phi})$ with $\tilde{\phi}$ symmetric, $\overline{m} = m \mod 4$, so this is the formed space corresponding to the nilpotent element $(cec^{-1}, U) \in \mathfrak{g}(U, \phi)_2$

In the same way, if we consider the space $(V, \tau') = (\mathbb{V}[b], -\tau)$ with $T = -\mathbb{T}$, the indecomposable formed space corresponding to (e, V) will be of type $\mathbb{R}[b], -)$, and indecomposable formed space corresponding to (cec^{-1}, U) will be of type $(\mathbb{C}[\overline{m+2}], \tilde{\phi})$ with $\tilde{\phi}$ symmetric (the eigenvalue of T on $c(v_0)$ is equal to $-i^m = i^{m+2}$ in this case).

Finally we consider the space $V = \mathbb{V} \otimes \mathbb{R}^2$ with the form $\tau \otimes \xi$, where ξ is the skew-symmetric form on $\mathbb{R}^2$ defined in (45). Let R be the endomorphism of $\mathbb{R}^2$ defined by $R(\epsilon_1) = \epsilon_2$, $R(\epsilon_2) = -\epsilon_1$. Then the form $\xi(R\ ,\ )$ is symmetric and positive definite. Let $T = \mathbb{T} \otimes R : V \longrightarrow V$. The form $(\tau \otimes \xi)(T\ ,\ )$ is symmetric and positive definite. Moreover, $T \in G(V \otimes \mathbb{R}^2, \tau \otimes \xi)$, and it is clear that the elements $e = E \otimes 1$, $f = F \otimes 1$, $h = H \otimes 1$ form a Cayley triple in $\mathfrak{g}(V \otimes \mathbb{R}^2, \tau \otimes \xi)$ with respect to the Cartan involution $Ad(T)$. Let $U = V \otimes \mathbb{C}$. We have

$$(\tau \otimes \xi)(v_0 \otimes \epsilon_\mu, e^m(v_0 \otimes \epsilon_\nu)) = \tau(v_0, E^m(v_0))\xi(\epsilon_\mu, \epsilon_\nu) = m!\xi(\epsilon_\mu, \epsilon_\nu),$$

$$c(v_k \otimes \epsilon_\nu) = 2^{-m/2} \sum_{l=0}^{k} \sum_{r=0}^{m-k} \binom{k}{l} \binom{m-k}{r} i^{k-l+r} v_{l+r} \otimes \epsilon_\nu,$$

$$T(c(v_k \otimes \epsilon_\nu)) = i^{m-2k} c(v_k \otimes R(\epsilon_\nu)).$$

Hence, in particular,

$$\begin{array}{rcl} T(c(v_0 \otimes \epsilon_1) + ic(v_0 \otimes \epsilon_2)) & = & -i^{m+1}(c(v_0 \otimes \epsilon_1) + ic(v_0 \otimes \epsilon_2)), \\ T(c(v_0 \otimes \epsilon_1) - ic(v_0 \otimes \epsilon_2)) & = & i^{m+1}(c(v_0 \otimes \epsilon_1) - ic(v_0 \otimes \epsilon_2)). \end{array} \quad (57)$$

It follows that T has two eigenvalues i^{m+1}, i^{m+3} on the space $\tilde{U}$, so the formed space $(\tilde{U}, \tilde{\phi})$ is of type $(\mathbb{C}[\overline{m+1}] \oplus \mathbb{C}[\overline{m+3}], \tilde{\phi})$.

We can now sum up our computations in the following proposition.

Proposition 6.2 *The Kostant-Sekiguchi correspondence for the orbits of indecomposable nilpotent elements in Lie algebras of isometries of formed spaces over* $\mathbb{R}$ *maps the orbit* $\mathcal{O} = \mathcal{O}(\tilde{V}, \tilde{\tau}) \subseteq \mathfrak{g}(V, \tau)$ *of a nilpotent element of height* m *corresponding to a formed space* $(\tilde{V}, \tilde{\tau})$ *into the orbit* $\mathcal{S}(\mathcal{O}) \subseteq \mathfrak{g}(U, \phi)$ *corresponding to the formed space* $(\tilde{U}, \tilde{\phi})$ *as in the following table.*

m	$\mathcal{O}$	$\mathcal{S}(\mathcal{O})$
even	$(\mathbb{R}[0],+)$	$(\mathbb{C}[\overline{m}], sym)$
even	$(\mathbb{R}[0],-)$	$(\mathbb{C}[\overline{m+2}], sym)$
odd	$(\mathbb{R}^2[0], sk)$	$(\mathbb{C}[0]\oplus\mathbb{C}[2], sk)$
odd	$(\mathbb{R}[1],+)$	$(\mathbb{C}[\overline{m}], sym)$
odd	$(\mathbb{R}[1],-)$	$(\mathbb{C}[\overline{m+2}], sym)$
even	$(\mathbb{R}^2[1], sk)$	$(\mathbb{C}[1]\oplus\mathbb{C}[3], sk)$

Let now $\mathbb{D} = \mathbb{C}$ or $\mathbb{H}$. Let $\tau_{\mathbb{D}}$ be a hermitian form on $\mathbb{D}$ such that $\tau_{\mathbb{D}}(1,1) = 1$. Consider the left vector space $V = \mathbb{D}\otimes\mathbb{V}$ over $\mathbb{D}$, with the form $\tau_{\mathbb{D}}\otimes\tau$. Then the map $T = 1\otimes\mathbb{T}$ belongs to $G(V,\tau_{\mathbb{D}}\otimes\tau)$ and the form $(\tau_{\mathbb{D}}\otimes\tau)(T\ ,\ )$ is hermitian and positive definite. Furthermore, it is clear that the elements $e = 1\otimes E$, $f = 1\otimes F$, $h = 1\otimes H$ form a Cayley triple in $\mathfrak{g}(V,\tau_{\mathbb{D}}\otimes\tau)$ with respect to $Ad(T)$. The equality (55) implies that the indecomposable formed space $(\tilde{V},\tau_{\mathbb{D}}\tilde{\otimes}\tau)$ is of the form $(\mathbb{D}[b],+)$.

Moreover, the map c defined in (44), can be written as

$$c(1\otimes v_k) = 2^{-m/2}\sum_{l=0}^{k}\sum_{r=0}^{m-k}\binom{k}{l}\binom{m-k}{r} i^{k-l+r}\otimes v_{l+r},$$

and therefore

$$T(c(1\otimes v_k)) = i^{m-2k}c(1\otimes v_k) \qquad (0\leq k\leq m). \tag{58}$$

Assume now that $\mathbb{D} = \mathbb{C}$ with ι the complex conjugation. By the results of section 5.1 there are two possibilities for the formed space corresponding to a indecomposable nilpotent element of height m, namely $(\mathbb{C},+)$ and $(\mathbb{C},-)$ for even m and $(\mathbb{C},+i)$ and $(\mathbb{C},-i)$ for odd m (we assume that $\mathbb{C}$ is in degree zero in all cases). Similarly, by Proposition 5.4 the formed space $\tilde{U}$ corresponding to an orbit of an indecomposable nilpotent element in $\mathfrak{p}_{\mathbb{C}}$ is of the form $\mathbb{C}[k]$, $k = 0, 2$. In the above construction the space $(\tilde{V},\tau_{\mathbb{C}}\tilde{\otimes}\tau)$ is of the form $(\mathbb{C},+)$, and the space $\tilde{U}$ is equal to $\mathbb{C}[\overline{m}]$. When we replace the form $\tau_{\mathbb{C}}$ by $i^k\tau_{\mathbb{C}}$ and the map T by $i^{-k}T$, we will get the remaining cases of the following proposition.

Proposition 6.3 *The Kostant-Sekiguchi correspondence for the orbits of indecomposable nilpotent elements in Lie algebras of isometries of formed spaces over* $\mathbb{C}$ *maps the orbit* $\mathcal{O} = \mathcal{O}(\tilde{V},\tilde{\tau}) \subseteq \mathfrak{g}(V,\tau)$ *of a nilpotent element of height* m *corresponding to a formed space* $(\tilde{V},\tilde{\tau})$ *into the orbit* $\mathcal{S}(\mathcal{O}) \subseteq \mathfrak{g}(U,\phi)$ *corresponding to the space* $\tilde{U}$ *as in the following table.*

$\mathcal{O}$	$\mathcal{S}(\mathcal{O})$
$(\mathbb{C},+)$	$\mathbb{C}[\overline{m}]$
$(\mathbb{C},-)$	$\mathbb{C}[\overline{m+2}]$
$(\mathbb{C},+i)$	$\mathbb{C}[\overline{m+3}$
$(\mathbb{C},-i)$	$\mathbb{C}[\overline{m+1}]$

Assume now that $\mathbb{D} = \mathbb{H}$, so our formed space is equal to $(V, \tau_{\mathbb{H}} \otimes \tau)$. It follows that

$$T(c(1 \otimes v_0)) = i^m c(1 \otimes v_0),$$

and similarly

$$T(c(j \otimes v_0)) = i^m c(j \otimes v_0),$$

(where j is the element of the standard basis $\{1, i, j, k\}$ of $\mathbb{H}$) hence the formed space $(\tilde{U}, \tilde{\phi})$ is of the form $(\mathbb{C}^2[\overline{m}], \tilde{\phi})$ with $\tilde{\phi}$ skew-symmetric.

If we replace the form τ by $-\tau$ and the map T by $-T$, we will obtain the nilpotent element (e, V) whose corresponding indecomposable formed space $(\tilde{V}, \tau_{\mathbb{H}} \tilde{\otimes} \tau)$ is of the form $(\mathbb{D}[b], -)$, and the nilpotent element (cec^{-1}, U) whose corresponding indecomposable formed space $(\tilde{U}, \tilde{\phi})$ is of the form $(\mathbb{C}^2[\overline{m+2}], \tilde{\phi})$ with $\tilde{\phi}$ skew-symmetric.

The last case to consider is the case $(\tilde{V}, \tilde{\tau})$ of type $(\mathbb{H}[b], sk)$ with $b + m$ odd. We proceed as follows.

For $a, b \in \mathbb{H}$ set $\tau_{\mathbb{H}}(a, b) = a j \iota(b) \in \mathbb{H}$. Then $\tau_{\mathbb{H}}$ is a skew-hermitian form on left vector space $\mathbb{H}$. Consider the space $V = \mathbb{H} \otimes \mathbb{V}$ with the form $\tau_{\mathbb{H}} \otimes \tau$. Let $e = 1 \otimes E$, as before. Since

$$(\tau_{\mathbb{H}} \otimes \tau)(1 \otimes v_0, e^m(1 \otimes v_0)) = \tau_{\mathbb{H}}(1, 1)\tau(v_0, E^m(v_0)) = m!\, j,$$

the formed space $(\tilde{V}, \tau_{\mathbb{H}} \tilde{\otimes} \tau)$ is of the form $(\mathbb{H}[b], sk)$.

Let $R_{j^{-1}}$ denote the right multiplication by j^{-1} on $\mathbb{H}$. Then the map $T = R_{j^{-1}} \otimes \mathbb{T}$ belongs to $G(\mathbb{H} \otimes V, \tau_{\mathbb{H}} \otimes \tau)$ and the form $(\tau_{\mathbb{H}} \otimes \tau)(T\ ,\ )$ is hermitian and positive definite. Furthermore the elements $e = 1 \otimes E$, $f = 1 \otimes F$, $h = 1 \otimes H$ form a Cayley triple in $\mathfrak{g}(\mathbb{H} \otimes V, \tau_{\mathbb{H}} \otimes \tau)$ with respect to the Cartan involution $Ad(T)$. Furthermore, a straightforward calculation using (49) and (52), shows that

$$\begin{array}{rcl} T(c(1 \otimes v_0)) & = & -i^m c(j \otimes v_0), \\ T(c(j \otimes v_0)) & = & i^m c(1 \otimes v_0). \end{array} \tag{59}$$

Hence the restriction of T to the space $c(\mathbb{H} \otimes v_0)$ has two eigenvalues i^{m+1} and $-i^{m+1}$. The eigenspaces are isotropic with respect to the form $\tilde{\phi}$ if ϕ is the form defined in 39 with τ in 39 replaced by $\tau_{\mathbb{H}} \otimes \tau$. It follows that the indecomposable formed space $(\tilde{U}, \tilde{\phi})$ corresponding to cec^{-1} is of the form $(\mathbb{C}[\overline{m}] \oplus \mathbb{C}[\overline{m+2}], \tilde{\phi})$ with $\tilde{\phi}$ symmetric.

We can sum up our computations in the following proposition.

Proposition 6.4 *The Kostant-Sekiguchi correspondence for the orbits of indecomposable nilpotent elements in Lie algebras of isometries of formed spaces over* $\mathbb{H}$ *maps the orbit* $\mathcal{O} = \mathcal{O}(\tilde{V}, \tilde{\tau}) \subseteq \mathfrak{g}(V, \tau)$ *of a nilpotent element of height* m *corresponding to a formed space* $(\tilde{V}, \tilde{\tau})$ *into the orbit* $\mathcal{S}(\mathcal{O}) \subseteq \mathfrak{g}(U, \phi)$ *corresponding to the formed space* $(\tilde{U}, \tilde{\phi})$ *as in the following table.*

m	$\mathcal{O}$	$\mathcal{S}(\mathcal{O})$
even	$(\mathbb{H}[0],+)$	$(\mathbb{C}[\overline{m}], sym)$
even	$(\mathbb{H}[0],-)$	$(\mathbb{C}[\overline{m+2}], sym)$
odd	$(\mathbb{H}[0],sk)$	$(\mathbb{C}[1]\oplus\mathbb{C}[3], sk)$
odd	$(\mathbb{H}[1],+)$	$(\mathbb{C}[\overline{m}], sym)$
odd	$(\mathbb{H}[1],-)$	$(\mathbb{C}[\overline{m+2}], sym)$
even	$(\mathbb{H}[1],sk)$	$(\mathbb{C}[0]\oplus\mathbb{C}[2], sk)$

We'll say that an orbit $\mathcal{O} \subseteq \mathfrak{g}(V,\tau)$ is indecomposable if for every (or equaivalently some) element $N \in \mathcal{O}$ the element (N,V) is indecomposable. Similarly we define indecomposable orbits in $\mathfrak{g}(U,\phi)$.

Corollary 6.5 *The image of an indecomposable nilpotent orbit in* $\mathfrak{g}(V,\tau)$ *under the Kostant-Sekiguchi correspondence is also indecomposable.*

6.2 The Kostant-Sekiguchi correspondence for general nilpotent elements

Let $N \in \mathfrak{g}(V,\tau)$ be an arbitrary nilpotent element, and let

$$(V,\tau) = (V^{(1)},\tau^{(1)}) \oplus \ldots \oplus (V^{(s)},\tau^{(s)}) \tag{60}$$

be an orthogonal decomposition such that each of the spaces $V^{(k)}$ is N-invariant and each of the restrictions $N^{(k)} = N|_{V^{(k)}}$ is an idecomposable nilpotent element in $\mathfrak{g}(V^{(k)},\tau^{(k)})$. Let $U^{(k)} = V^{(k)} \otimes \mathbb{C}$ if $\mathbb{D} = \mathbb{R}$, and $U^{(k)} = V^{(k)}|_{\mathbb{C}}$ if $\mathbb{D} = \mathbb{C},\mathbb{H}$. By the results of the previous subsection for each k there exists $T^{(k)} \in G(V^{(k)},\tau^{(k)})$ such that the form $\tau^{(k)}(T^{(k)}\ ,\ )$ is hermitian and positive definite, and that $e^{(k)} = N^{(k)},\ f^{(k)} = -T^{(k)}e^{(k)}(T^{(k)})^{-1}, h^{(k)} = [e^{(k)},f^{(k)}]$ is a Cayley triple in $\mathfrak{g}(V^{(k)},\tau^{(k)})$ with respect to the Cartan involution $\theta^{(k)} = Ad(T^{(k)})$.

Moreover, if $c^{(k)}$ denotes the element $c^{(k)} = \exp(i\frac{\pi}{4}(e^{(k)}+f^{(k)})) \in G(U^{(k)},\phi^{(k)})$ (with $\phi^{(k)}$ defined by $\tau^{(k)}$ as in (39)), then the $G(U^{(k)},\phi^{(k)})$-orbit through $c^{(k)}e^{(k)}(c^{(k)})^{-1}$ is the image of the $G(V^{(k)},\tau^{(k)})$-orbit through $e^{(k)}$ by the Kostant-Sekiguchi correspondence.

Define $T \in G(V,\tau)$ as $T = T^{(1)} \oplus \ldots \oplus T^{(s)}$, then the form $\tau(T\ ,\ )$ is hermitian and positive definite, and the triple $e = N, f = -TeT^{-1}, h = [e,f]$ is a Cayley triple in $\mathfrak{g}(V,\tau)$ with respect to the Cartan involution $\theta = Ad(T) = \theta^{(1)} \oplus \ldots \oplus \theta^{(s)}$. The element $c \in G(U,\phi)$ defined in (44) is equal to the product of commuting elements $c = c^{(1)} \cdot \ldots \cdot c^{(s)}$ and the nilpotent element $(cec^{-1},U) \in \mathfrak{g}(U,\phi)_2$, whose orbit in $\mathfrak{g}(U,\phi)_2$ is equal to the image of the $G(V,\tau)$-orbit of e under the Kostant-Sekiguchi correspondence, is equal to the orthogonal sum

$$(cec^{-1},U) = (c^{(1)}e^{(1)}(c^{(1)})^{-1},U^{(1)}) \oplus \ldots \oplus (c^{(s)}e^{(s)}(c^{(s)})^{-1},U^{(s)}).$$

We can conclude that the Kostant-Sekiguchi correspondence is compatible with orthogonal decompositions of nilpotent elements in $\mathfrak{g}(V,\tau)$.

Let us remark that, granted the standard conjugacy results about Cayley triples and their Cayley transforms (see e.g. [CM], Chapter 9), our results can be considered as an alternate proof of the bijectivity of the Kostant-Sekiguchi correspondence for the Lie algebras of isometry groups of formed spaces.

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