THETA LIFTING OF UNITARY LOWEST WEIGHT MODULES AND THEIR ASSOCIATED CYCLES

KYO NISHIYAMA AND CHEN-BO ZHU

Abstract. We consider a reductive dual pair (G, G') in the stable range with G' the smaller member and of Hermitian symmetric type. We study theta lifting of (holomorphic) nilpotent $K'_{\mathbb{C}}$-orbits in relation to theta lifting of unitary lowest weight modules of G'. We determine the associated cycles of all such representations. In particular, we prove that the multiplicity in the associated cycle is preserved under theta lifting.

Introduction

Let (G, G') be a reductive dual pair in a symplectic group $\mathbb{G} = Sp(2N, \mathbb{R})$, where N denotes the rank of $\mathbb{G}$. Let $\widetilde{\mathbb{G}} = Mp(2N, \mathbb{R})$ be the metaplectic two-fold cover of $\mathbb{G}$, and Ω be a fixed oscillator representation of $\widetilde{\mathbb{G}}$ ([8]). Often when no confusion should arise, we shall not distinguish Ω with its Harish-Chandra module.

Let $\mathbb{Z}_2 = \{\pm 1\}$ be the kernel of the projection from $\widetilde{\mathbb{G}}$ to $\mathbb{G}$. For a subgroup L of $\mathbb{G}$, we denote the pullback of L in $\widetilde{\mathbb{G}}$ by $\widetilde{L}$. Then $\widetilde{L}$ projects onto L with kernel $\mathbb{Z}_2$. We call an irreducible admissible representation of $\widetilde{L}$ *genuine* if its restriction to $\mathbb{Z}_2$ is a multiple of the unique non-trivial character ϵ of $\mathbb{Z}_2$.

Using the oscillator representation Ω, Howe associates a given irreducible admissible genuine representation π' of $\widetilde{G}'$ with an irreducible admissible genuine representation π of $\widetilde{G}$, called the *theta lift* of π' ([8, 11]). We shall write $\pi = \theta(\pi')$, as usual. Roughly saying, π is the theta lift of π' if and only if there is a non-trivial morphism

$$\Omega \longrightarrow \pi \otimes \pi' \quad \text{as a } (\mathfrak{g}, \widetilde{K}) \times (\mathfrak{g}', \widetilde{K}')\text{-module},$$

where $\mathfrak{g}$ (resp. $\mathfrak{g}'$) is the complexification of the Lie algebra of G (resp. G') and K (resp. K') is a maximal compact subgroup of G (resp. G').

Throughout this paper, we will assume that (G, G') is of type I, and it is in the stable range with G' the smaller member (cf. [8, 17]). According to [17], theta correspondence

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gives rise to an injection

$$\theta : \widehat{\widetilde{G'}}_\epsilon \hookrightarrow \widehat{\widetilde{G}}_\epsilon,$$

where $\widehat{\widetilde{G}}_\epsilon$ (resp. $\widehat{\widetilde{G'}}_\epsilon$) denotes the set of equivalent classes of irreducible unitary genuine representations of $\widetilde{G}$ (resp. $\widetilde{G'}$).

We will also assume that G'/K' is an irreducible Hermitian symmetric space. By the classification of irreducible dual pairs, our restriction amounts to saying that (G, G') is from the following Table 1 (cf. [11]). Note that $G = G(p, q)$ is the group of isometries of a non-degenerate Hermitian form of signature (p, q) over $D = \mathbb{R}, \mathbb{C}, \mathbb{H}$, respectively, and G' is the group of isometries of a non-degenerate skew-Hermitian form over D. We have thus labeled the three cases as Case $\mathbb{R}$, Case $\mathbb{C}$ and Case $\mathbb{H}$, respectively. Note also that we have excluded the equality $2n = \min(p, q)$ in the first case, to avoid some (small) technicalities.

TABLE 1. The dual pairs treated in this paper

	the pair (G, G')	stable range condition
Case $\mathbb{R}$:	$(O(p, q), Sp(2n, \mathbb{R}))$	$2n < \min(p, q)$
Case $\mathbb{C}$:	$(U(p, q), U(m, n))$	$m + n \leq \min(p, q)$
Case $\mathbb{H}$:	$(Sp(p, q), O^*(2n))$	$n \leq \min(p, q)$

A particularly interesting subset of $\widehat{\widetilde{G'}}_\epsilon$ is the set of (genuine) unitary lowest weight modules of $\widetilde{G'}$. Note that this subset includes unitary characters, singular unitary lowest weight modules and holomorphic discrete series, and there is a vast amount of literature on the subject begining from the fundamental work of Harish-Chandra [7] and Wallach [31]. Its classification is also well-known [6, 12]. Their theta lifts will then constitute a collection of rather interesting singular unitary representations of $\widetilde{G}$.

The purpose of this paper is to investigate this collection of singular unitary representations. We will be especially interested in geometries underlying these representations (§2). In particular, we will show the strong interaction between their $\widetilde{K}$-structures and the geometry of the associated varieties (cf. [32, 23, 24]). The main application here is the determination of the associated cycles of this collection of singular unitary representations. In some sense our results demonstrate that stable range theta lifting commutes with Vogan's philosophy of orbit method [33].

We remark that lifting of discrete series representations was investigated by many researchers, in [30, 1, 18], etc.. Among them we mention two works which are more relevant to our work. In [1, 2], Adams considers the theta lifting of discrete series under the stable range condition, and shows that most of them are derived functor modules $A_{\mathfrak{q}}(\lambda)$. Thus in principle their $\widetilde{K}$-structures can be obtained via Blattner type formula. We shall not take this approach as it is well-known that these $\widetilde{K}$-type formulas are not very practical (at least for our purpose). In [19], Li describes a method for computing the $\widetilde{K}$-types of $\theta(\pi')$ for $\pi' \in \widehat{\widetilde{G'}}_\epsilon$, which is effective when π' is holomorphic. We shall discuss this method in parts of §3.

We introduce some notations. Fix a choice of the maximal compact subgroup K of G (resp. K' of G'). This determines a Cartan decomposition of the complexified Lie algebra $\mathfrak{g}$ (resp. $\mathfrak{g}'$) of G (resp. G'):

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}, \qquad \mathfrak{g}' = \mathfrak{k}' \oplus \mathfrak{s}',$$

where $\mathfrak{k}$ (resp. $\mathfrak{k}'$) is the complexified Lie algebra of K (resp. K'), and $\mathfrak{s}$ (resp. $\mathfrak{s}'$) is the orthogonal complement of $\mathfrak{k}$ (resp. $\mathfrak{k}'$) with respect to the Killing form.

Let $\mathbb{O}' \subset \mathfrak{s}'$ be a nilpotent $K'_\mathbb{C}$-orbit. Then we can define the *theta lift* of $\mathbb{O}'$, which is a certain nilpotent $K_\mathbb{C}$-orbit $\mathbb{O}$ in $\mathfrak{s}$ (see §2.2). We shall denote $\mathbb{O} = \theta(\mathbb{O}')$.

Now let π' be a unitary lowest weight representation of $\widetilde{G}'$. It is well-known that the associated variety of π', denoted by $\mathcal{AV}(\pi')$, is the closure of a nilpotent $K'_\mathbb{C}$-orbit $\mathbb{O}'^{\text{hol}} \subseteq \mathfrak{s}'_-$, which will be called holomorphic. Here $\mathfrak{s}'$ breaks up into irreducible pieces $\mathfrak{s}'_\pm$ under the action of $K'_\mathbb{C}$. It turns out that the associated variety of $\theta(\pi')$ is $\overline{\mathbb{O}^{\text{hol}}}$, where $\mathbb{O}^{\text{hol}} = \theta(\mathbb{O}'^{\text{hol}})$ is the theta lift of $\mathbb{O}'^{\text{hol}}$.

We assume that π' comes from the compact dual pair correspondence of $(G(k), G')$, for some k. For $G' = Sp(2n, \mathbb{R})$ or $U(m, n)$, all unitary lowest weight representations of $\widetilde{G}'$ arise this way [5]. For $G' = O^*(2n)$, this assumption is very minor and it amounts to excluding a (tiny) subset of unitary lowest weight modules with their levels of reduction all being 1. See the paper of Davidson, Enright and Stanke (§7, [4]) for a precise statement. This corrects an assertion made in passing in the classfication article [6]. For such a π', we have $\mathbb{O}'^{\text{hol}} = \mathbb{O}'^{\text{hol}}_r$, the $K'_\mathbb{C}$ orbit of "rank" r in $\mathfrak{s}'_-$. Here $r = \min(k, l)$ and l is the real rank of G'. See [23].

In our approach to the geometry of $\mathbb{O}^{\text{hol}}_r = \theta(\mathbb{O}'^{\text{hol}}_r)$, an important role is played by certain affine variety $X_{p,q+k}$, which fits into an affine quotient map $\zeta : X_{p,q+k} \to \overline{\mathbb{O}^{\text{hol}}_r}$. More precisely the variety $X_{p,q+k}$ is the closure of the theta lift of the trivial orbit of $\mathfrak{s}'$ with respect to a larger dual pair $(G(p, q+k), G')$. In fact $X_{p,q+k}$ is a spherical $G(p, \mathbb{C}) \times G(q+k, \mathbb{C})$-variety and one may consider the space of covarients

$$M[\sigma] = \left(\sigma \otimes \mathbb{C}[X_{p,q+k}]\right)^{G(k,\mathbb{C})},$$

where σ is an irreducible finite dimensional representation σ of $G(k, \mathbb{C})$.

To fix notations, let σ be an irreducible finite-dimensional representation of $G(k, \mathbb{C})$ which appears in $\mathbb{C}[W_k]$. Here W_k is a certain complex vector space canonically associated to the dual pair $(G(k), G')$, and for our three cases it may be identified with the space of complex matrices $M_{k,n}$, $M_{k,m+n}$ and $M_{2k,n}$, respectively. We assume that π' and σ are related by the formula:

$$\pi'\big|_{\widetilde{K'}} \simeq \chi_k \otimes L[\sigma], \qquad L[\sigma] = \big(\sigma \otimes \mathbb{C}[W_k]\big)^{G(k,\mathbb{C})}, \tag{0.1}$$

where χ_k is a certain character of $\widetilde{K'}$.

On the geometry side, our approach will yield the $K_{\mathbb{C}} = G(p, \mathbb{C}) \times G(q, \mathbb{C})$-module structure of $M[\sigma]$, in terms of $K'_{\mathbb{C}}$-module structure of $L[\sigma]$ (Theorem 2.4). This fits in a general notion of lifting of covariants (§1.4). On the representation side, we will show that the $\widetilde{K}$-types of $\theta(\pi')$ can be described by the $\widetilde{K}'$-types of π' in more or less the same way (Theorem 3.9). We highlight here a relevant key fact which may be of independent interest. Namely, Howe's maximal quotient for a unitary lowest weight module is irreducible under our assumption of stable range (Corollary 3.6). This result was obtained by H.-Y. Loke by a different method.

By comparing certain Poincaré series for $\pi = \theta(\pi')$ and $M[\sigma]$, we conclude that (the graded module of) π and $M[\sigma]$ have the same multiplicity along their common support $\overline{\mathbb{O}_r^{\rm hol}}$. We thus obtain the associated cycles of all such π's. The main result is the following

Theorem (Theorem 4.7)**.** *Let π' be a unitary lowest weight representation of $\widetilde{G}'$ arising from the compact dual pair correspondence of $(G(k), G')$, for some k. Assume that π is genuine with respect to the dual pair $(G(p, q), G')$, and let $\pi = \theta(\pi')$ be the theta lift of π'. If the associated cycle of π' is*

$$\mathcal{AC}\,(\pi') = m_{\pi'}[\overline{\mathbb{O}'^{\rm hol}}],$$

then we have

$$\mathcal{AC}\,(\pi) = m_{\pi'}[\overline{\mathbb{O}^{\rm hol}}], \qquad \textit{where} \qquad \mathbb{O}^{\rm hol} = \theta(\mathbb{O}'^{\rm hol}).$$

Moreover we have

$$m_{\pi'} = \begin{cases} \dim \sigma & \textit{if } k \le l, \\ \dim \sigma^{G(k-l,\mathbb{C})} & \textit{if } k > l \textit{ and } G'/K' \textit{ is of tube type}, \end{cases}$$

where σ is related to π' through the formula (0.1)*.*

The multiplicity for the non-tube case is given in Theorem 4.7 as well. We remark that for π' a holomorphic discrete series representation, the multiplicity $m_{\pi'}$ may also be given by the dimension of its lowest $\widetilde{K}'$-type. For example this will be so if $k > 2n$ for Case $\mathbb{R}$ ($k \ge m+n$ for Case $\mathbb{C}$, and $k \ge n$ for Case $\mathbb{H}$, respectively), and so one gets an equality of dimension of certain space of fixed vectors of σ with the dimension of the lowest $\widetilde{K}'$-type of π'. In fact more is true. See Proposition 4.10 immediately after Theorem 4.7.

Our theorem generalizes considerably results of a previous paper [24] for π' a unitary character (see also [28]) or a holomorphic discrete series with a scalar minimal $\widetilde{K}'$-type. In any of these cases, the associated cycle of $\theta(\pi')$ is multiplicity-free.

Here are the outline of contents and some words on the organization of this paper. In §1, we review some standard results on affine quotients and introduce a general notion of lifting of coherent sheaves and covariants in the context of double fiberation (by affine quotients). In particular we prove a result on preservation of multiplicity (Proposition 1.9), which may of be some independent interest. In §2, we investigate geometries underlying theta lifting of holomorphic nilpotent orbits. We lift the affine quotient map $\zeta' : W_k \to \overline{\mathbb{O}'^{\mathrm{hol}}_k}$ to an affine quotient map $\zeta : X_{p,q+k} \to \overline{\mathbb{O}^{\mathrm{hol}}_k}$ and we determine the multiplicity and the $K_{\mathbb{C}}$-module structure of the space of covariants $M[\sigma]$ over $\overline{\mathbb{O}^{\mathrm{hol}}_k}$. In §3, we develop the representation side. The main result is the description of $\widetilde{K}$-structure of the theta lift $\theta(\pi')$ (actually the maximal quotient) of all unitary lowest weight modules π', in terms of the $\widetilde{K}'$-structure of π'. We also show (Proposition 3.12) that the associated variety of $\theta(\pi')$ is contained in the closure of the theta lift $\mathbb{O} = \theta(\mathbb{O}')$. Here π' is any (genuine) admissible representation of $\widetilde{G}'$ and the associated variety $\mathcal{AV}(\pi')$ of π' is assumed to be irreducible, hence the closure of a single nilpotent $K'_{\mathbb{C}}$-orbit $\mathbb{O}'$. In §4, we investigate certain Poincaré series of $\theta(\pi')$, and we then apply the results of §2 and §3 to obtain the formula for the associated cycle of $\theta(\pi')$. The computations of the generic fibers of $\zeta : X_{p,q+k} \to \overline{\mathbb{O}^{\mathrm{hol}}_r}$ are done in the Appendix.

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1. Generalities on affine quotients

1.1. **Preliminaries on affine quotients.** Let H be a reductive linear algebraic group over $\mathbb{C}$ which acts on an affine variety X. As usual let $X/\!/H$ denote the affine quotient of X by the action of H, which is by definition $\operatorname{Spec}\mathbb{C}[X]^H$. Here and throughout this article, $\mathbb{C}[X]$ denotes the regular function ring on X and $\mathbb{C}[X]^H$ the ring of H invariants. The natural inclusion $\mathbb{C}[X]^H \hookrightarrow \mathbb{C}[X]$ then induces a projection map $\zeta : X \to X/\!/H$, called an affine quotient map.

We have the following well-known

Proposition 1.1. (1) *For any $y \in X/\!/H$, the fiber $\zeta^{-1}(y)$ of y contains a unique closed H orbit. Hence $X/\!/H$ may be identified with the set of all closed H orbits in X.*
(2) *Let $Z \subset X$ be an H-stable closed subvariety. Then its image $\zeta(Z)$ is closed in $X/\!/H$, and $\zeta\big|_Z : Z \to \zeta(Z)$ is an affine quotient map, i.e., $\zeta(Z) \simeq Z/\!/H$.*

1.2. **Module of covariants and multiplicities.** Let X be an irreducible affine variey and $\zeta : X \to X/\!/H$ be an affine quotient map. We write $I = \mathbb{C}[X]^H$. Denote by $\operatorname{Irr}(H)$ the set of equivalent classes of irreducible finite dimensional representations of H. For $\sigma \in \operatorname{Irr}(H)$, consider

$$L[\sigma] = \big(\sigma \otimes \mathbb{C}[X]\big)^H, \tag{1.1}$$

which is a finitely generated I-module. We shall refer to $L[\sigma]$ as a module of covariants over I. If we denote by F the quotient field of I, then we have $F \otimes_I L[\sigma] = F^d$ for a non-negative integer d. We call d the multiplicity of the I-module $L[\sigma]$, and it will be denoted by $\operatorname{rank}_I L[\sigma]$.

The following lemma is standard and so we omit its proof.

Lemma 1.2. *Let $\zeta : X \to X//H$ be an affine quotient map. Then we have*

$$\operatorname{rank}_I L[\sigma] = \dim\bigl(\sigma \otimes \mathbb{C}[\zeta^{-1}(y)]\bigr)^H < \infty,$$

where y is a general point in $X//H$.

We shall need the following result, which follows from the lemma above and the Frobenius reciprocity.

Proposition 1.3. *Let $\zeta : X \to X//H$ be an affine quotient map, and assume that a generic fiber $\zeta^{-1}(y)$ is a closed H-orbit. Then the multiplicity of the module of covariants $L[\sigma] = \bigl(\sigma \otimes \mathbb{C}[X]\bigr)^H$ over $I = \mathbb{C}[X]^H$ is given by*

$$\operatorname{rank}_I L[\sigma] = \dim \sigma^{H_x},$$

where $x \in \zeta^{-1}(y)$, and H_x denotes its stabilizer.

Remark 1.4. Since the orbit $H \cdot x \subset X$ is a closed affine variety, the stabilizer H_x is necessarily reductive.

1.3. **Lifting of coherent sheaves.** Let $K_{\mathbb{C}}$ and $K'_{\mathbb{C}}$ be reductive algebraic groups over $\mathbb{C}$, and suppose that $K_{\mathbb{C}} \times K'_{\mathbb{C}}$ acts on an irreducible affine variety Ξ. Let

$$X = \Xi//K'_{\mathbb{C}}, \quad \text{and} \quad Y = \Xi//K_{\mathbb{C}} \tag{1.2}$$

be the corresponding affine quotients. Thus we have a double fibration map by affine quotients:

$$X \xleftarrow{//K'_{\mathbb{C}}} \Xi \xrightarrow{//K_{\mathbb{C}}} Y. \tag{1.3}$$

Note that if we put $R = \mathbb{C}[\Xi]$, then we have $\mathbb{C}[X] = R^{K'_{\mathbb{C}}}$ and $\mathbb{C}[Y] = R^{K_{\mathbb{C}}}$.

Denote by $\mathcal{O}_Y$ the sheaf of regular functions on Y. For a finitely generated $\mathbb{C}[Y]$-module L, we let $\widetilde{L} = \mathcal{O}_Y \otimes_{\mathbb{C}[Y]} L$ be the corresponding coherent sheaf on Y. The functor $L \mapsto \widetilde{L}$ gives an equivalence of categories between the category of finite generated $\mathbb{C}[Y]$-modules and the category of coherent sheaves of $\mathcal{O}_Y$-modules on Y. Its inverse is the functor of taking global sections. We also write $\operatorname{rank}_Y \widetilde{L} = \operatorname{rank}_{\mathbb{C}[Y]} L$.

In the following, we shall be mainly concerned with coherent sheaves which are equivariant with respect to some group actions. For instance, since $K'_{\mathbb{C}}$ acts on Y, we may consider a $K'_{\mathbb{C}}$-equivariant coherent sheaf $\widetilde{L}$. This simply means that L is a finitely generated $(K'_{\mathbb{C}}, \mathbb{C}[Y])$-compatible module.

Definition 1.5. For a finitely generated $(K'_{\mathbb{C}}, \mathbb{C}[Y])$-compatible module L or a $K'_{\mathbb{C}}$ equivariant coherent sheaf $\widetilde{L}$ on Y, we put

$$M = (\mathbb{C}[\Xi] \otimes_{\mathbb{C}[Y]} L)^{K'_{\mathbb{C}}}, \quad \text{or} \quad \widetilde{M} = (\Xi \times_Y \widetilde{L})//K'_{\mathbb{C}}. \tag{1.4}$$

Then M is a finitely generated $(K_{\mathbb{C}}, \mathbb{C}[X])$-compatible module, and $\widetilde{M}$ is a $K_{\mathbb{C}}$-equivariant coherent sheaf on X. We call M a *lift* of L, and $\widetilde{M}$ a *lift* of $\widetilde{L}$ via the double fibration map (1.3).

We have the following commutative diagram.

$$\begin{array}{ccccc}
\widetilde{M} & \xleftarrow{//K'_{\mathbb{C}}} & \Xi \times_Y \widetilde{L} & \xrightarrow{//K_{\mathbb{C}}} & \widetilde{L} \\
\downarrow & & \downarrow & & \downarrow \\
X & \xleftarrow{//K'_{\mathbb{C}}} & \Xi & \xrightarrow{//K_{\mathbb{C}}} & Y
\end{array}$$

Lemma 1.6. *If the quotient map $\Xi \to Y$ is flat, then we have*

$$\operatorname{rank}_{\Xi}(\Xi \times_Y \widetilde{L}) = \operatorname{rank}_Y \widetilde{L}.$$

Remark 1.7. In general, we only have an inequality $\operatorname{rank}_{\Xi}(\Xi \times_Y \widetilde{L}) \leq \operatorname{rank}_Y \widetilde{L}$.

Proof. We need to prove that $\operatorname{rank}_{\mathbb{C}[Y]} L = \operatorname{rank}_{\mathbb{C}[\Xi]} \mathbb{C}[\Xi] \otimes_{\mathbb{C}[Y]} L$. Since $\mathbb{C}[\Xi]$ is flat over $\mathbb{C}[Y]$, the assertion is immediate. □

Lemma 1.8. *Assume that a generic $K'_{\mathbb{C}}$-orbit is closed in Ξ. Let $\widetilde{\mathcal{L}}$ be a $K'_{\mathbb{C}}$-equivariant coherent sheaf on Ξ and $\widetilde{\mathcal{L}}//K'_{\mathbb{C}}$ the corresponding (quotient) coherent sheaf on $X = \Xi//K'_{\mathbb{C}}$, then we have*

$$\operatorname{rank}_X(\widetilde{\mathcal{L}}//K'_{\mathbb{C}}) = \operatorname{rank}_{\Xi} \widetilde{\mathcal{L}}.$$

Proof. Put $R = \mathbb{C}[\Xi]$, then we have $R^{K'_{\mathbb{C}}} = \mathbb{C}[X]$. We are to prove that $\operatorname{rank}_R \mathcal{L} = \operatorname{rank}_{R^{K'_{\mathbb{C}}}} \mathcal{L}^{K'_{\mathbb{C}}}$ for any finitely generated R-module $\mathcal{L}$. Put $d = \operatorname{rank}_R \mathcal{L}$. Then, by definition of the multiplicity, there is an exact sequence

$$0 \longrightarrow R^d \longrightarrow \mathcal{L} \longrightarrow N \longrightarrow 0 : \text{ exact},$$

where N is an R torsion. Since taking $K'_{\mathbb{C}}$ invariants is an exact functor, we have

$$0 \longrightarrow (R^{K'_{\mathbb{C}}})^d \longrightarrow \mathcal{L}^{K'_{\mathbb{C}}} \longrightarrow N^{K'_{\mathbb{C}}} \longrightarrow 0 : \text{ exact}.$$

So it is enough to prove that $N^{K'_{\mathbb{C}}}$ is an $R^{K'_{\mathbb{C}}}$ torsion. Since N is an R torsion, $Z = \operatorname{Supp}\widetilde{N}$ (the support of $\widetilde{N}$) is a proper closed subvariety in Ξ. Clearly we have $\operatorname{Supp}\widetilde{N}^{K'_{\mathbb{C}}} \simeq Z//K'_{\mathbb{C}}$. By the assumption, there exists a closed $K'_{\mathbb{C}}$-orbit in $\Xi \backslash Z$. This implies that $\operatorname{Supp}\widetilde{N}^{K'_{\mathbb{C}}}$ cannot be the whole $X = \Xi//K'_{\mathbb{C}}$. Hence $N^{K'_{\mathbb{C}}}$ is an $R^{K'_{\mathbb{C}}}$ torsion. □

Combining the above two lemmas, we obtain the following

Proposition 1.9. *Let Ξ be an irreducible affine variety on which $K_{\mathbb{C}} \times K'_{\mathbb{C}}$ acts. Put $X = \Xi // K'_{\mathbb{C}}$ and $Y = \Xi // K_{\mathbb{C}}$. For a $K'_{\mathbb{C}}$-equivariant coherent sheaf $\widetilde{L}$ on Y, let $\widetilde{M} = (\Xi \times_Y \widetilde{L}) // K'_{\mathbb{C}}$ be the lifted sheaf, which is a $K_{\mathbb{C}}$-equivariant coherent sheaf on X. Suppose that a generic $K'_{\mathbb{C}}$-orbit is closed in Ξ and the quotient map $\Xi \to Y$ is flat, then the multiplicity is preserved by the lifting:*

$$\operatorname{rank}_X \widetilde{M} = \operatorname{rank}_Y \widetilde{L}. \tag{1.5}$$

1.4. **Lifting of covariants.** We keep the setting and notations of the previous subsection. Thus we have an irreducible affine variety Ξ on which $K_{\mathbb{C}} \times K'_{\mathbb{C}}$ acts, and $X = \Xi // K'_{\mathbb{C}}$, $Y = \Xi // K_{\mathbb{C}}$.

Now suppose that H is a reductive algebraic group over $\mathbb{C}$, and suppose that there is an irreducible affine variety Z on which $H \times K'_{\mathbb{C}}$ acts. We further assume that $Z // H \simeq Y$ as a $K'_{\mathbb{C}}$-variety. We will define the lift of the quotient map $Z \to Y$ as follows.

Let $V = (\Xi \times_Y Z) // K'_{\mathbb{C}}$. Note that since we consider $\Xi \times_Y Z$ in the scheme theoretic sense, it may not be irreducible or reduced. The same remark applies to V. We have the following commutative diagram:

$$\begin{array}{ccccc} V & \xleftarrow{//K'_{\mathbb{C}}} & \Xi \times_Y Z & \xrightarrow{//K_{\mathbb{C}}} & Z \\ \big\downarrow{\scriptstyle //H} & & \big\downarrow{\scriptstyle //H} & & \big\downarrow{\scriptstyle //H} \\ X & \xleftarrow{//K'_{\mathbb{C}}} & \Xi & \xrightarrow{//K_{\mathbb{C}}} & Y. \end{array}$$

This is clear if we show that the map $V \to X$ is an affine quotient map. To see this, we compute

$$\mathbb{C}[V]^H = (\mathbb{C}[\Xi] \otimes_{\mathbb{C}[Y]} \mathbb{C}[Z])^{K'_{\mathbb{C}} \times H} = (\mathbb{C}[\Xi] \otimes_{\mathbb{C}[Y]} \mathbb{C}[Z]^H)^{K'_{\mathbb{C}}}.$$

Since $\mathbb{C}[Y] \simeq \mathbb{C}[Z]^H$ by assumption, we get $\mathbb{C}[V]^H = \mathbb{C}[\Xi]^{K'_{\mathbb{C}}} = \mathbb{C}[X]$, which is equivalent to saying that $V \to X$ is an affine quotient map. We shall thus say that the quotient map $V \to X$ is *lifted* from the quotient map $Z \to Y$.

For $\sigma \in \operatorname{Irr}(H)$, we put

$$L[\sigma] = \big(\sigma \otimes \mathbb{C}[Z]\big)^H, \qquad M[\sigma] = \big(\sigma \otimes \mathbb{C}[V]\big)^H;$$

They are finitely generated modules of $\mathbb{C}[Z]^H \simeq \mathbb{C}[Y]$ and $\mathbb{C}[V]^H = \mathbb{C}[X]$, respectively. Let $\widetilde{L}[\sigma]$ be the associated $K'_{\mathbb{C}}$-equivariant coherent sheaf on Y, and $\widetilde{M}[\sigma]$ the associated $K_{\mathbb{C}}$-equivariant coherent sheaf on X, as before. Then we have

Proposition 1.10. *$\widetilde{M}[\sigma]$ is lifted from $\widetilde{L}[\sigma]$ in the sense of Definition 1.5.*

Proof. We put $S = \mathbb{C}[Z]$. Note that $\mathbb{C}[Y] \simeq S^H$ by assumption. We have

$$\begin{aligned} M[\sigma] &= \big(\sigma \otimes \mathbb{C}[V]\big)^H = \big(\sigma \otimes (\mathbb{C}[\Xi] \otimes_{\mathbb{C}[Y]} \mathbb{C}[Z])^{K'_{\mathbb{C}}}\big)^H \\ &\simeq \big(\sigma \otimes (\mathbb{C}[\Xi] \otimes_{S^H} S)\big)^{K'_{\mathbb{C}} \times H} \simeq \big(\mathbb{C}[\Xi] \otimes_{S^H} (\sigma \otimes S)\big)^{K'_{\mathbb{C}} \times H} \\ &\simeq \big(\mathbb{C}[\Xi] \otimes_{S^H} (\sigma \otimes S)^H\big)^{K'_{\mathbb{C}}} \simeq \big(\mathbb{C}[\Xi] \otimes_{S^H} L[\sigma]\big)^{K'_{\mathbb{C}}}. \end{aligned}$$

The assertion follows. □

2. Geometry of theta lifting

2.1. **Review of structures for dual pairs.** We review certain structural results related to our dual pairs [11].

Let $W_{\mathbb{R}} \simeq \mathbb{R}^{2N}$ be a real symplectic space which realizes $\mathbb{G} = Sp(2N, \mathbb{R})$ as a symplectic group on $W_{\mathbb{R}}$. There is a canonical complex structure on $W_{\mathbb{R}}$ and we can view $W_{\mathbb{R}}$ as (the underlying real vector space of) a complex vector space $W \simeq \mathbb{C}^N$. The symplectic form on $W_{\mathbb{R}}$ is then given by the imaginary part of a canonical positive definite Hermitian form on W. By this identification, a maximal compact subgroup $\mathbb{K}$ of $\mathbb{G}$ is realized as the unitary group $U(W) \simeq U(N)$ on $W = \mathbb{C}^N$.

We may choose maximal compact subgroups K and K' of G and G' respectively, in such a way that $K \cdot K'$ is contained in the standard maximal compact subgroup $\mathbb{K} \simeq U(N)$ of $\mathbb{G}$.

In view of Table 1 of the Introduction, we sometimes write $G = G(p, q)$, namely $G(p, q)$ will denote one of the groups $O(p, q), U(p, q)$ or $Sp(p, q)$. Note that $G(k, 0) \simeq G(0, k)$ is compact. We shall write $G(k) = G(k, 0)$ in short, if there is no possibility of misunderstanding. In the sequel, we are mainly concerned with *non-compact* cases, though results on compact cases will be used heavily. This is a basic technique in the theory of dual pairs.

From the pair (G, G'), one defines another three dual pairs, which form so-called *diamond dual pairs* (see [11, §5]). Namely, take the commutant of K in $\mathbb{G}$ and denote it by M'. The pair (K, M') is a compact dual pair, and M' is of Hermitian symmetric type and for the cases we are concerned it is isomorphic to $G' \times G'$ containing G' as the diagonal. Also, take the full commutant of K in $\mathbb{K}$, and denote it by L'. Then L' is a maximal compact subgroup of M', and it is isomorphic to $K' \times K'$ and contains K' as a diagonal subgroup. Similarly, define M as the full commutant of K' in $\mathbb{G}$, and L the full commutant in $\mathbb{K}$.

Let us summarize the somewhat complicated situation by the following diagram (Fig. 1).

An explicit description of the diamond pairs for our three cases is given in [11, (5.3)]. In the table there, L (resp. L') in our notation is written as $M^{(1,1)}$ (resp. $M'^{(1,1)}$). For convenience of readers, we reproduce the table here (Table 2 below) with additional remarks.

FIGURE 1. The diagram of a diamond pair.

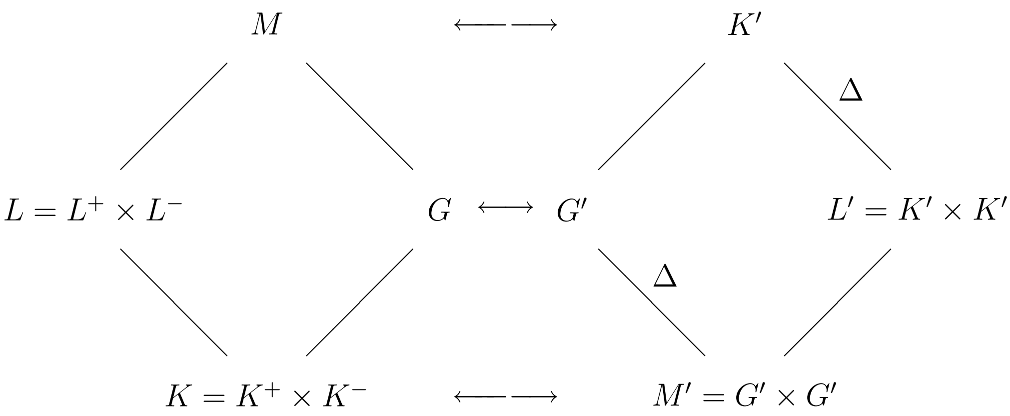

TABLE 2. Three diamond dual pairs.

(G, G')	M	$K = K^+ \times K^-$	$L = L^+ \times L^-$	K'
$(O(p,q), Sp(2n,\mathbb{R}))$	$U(p,q)$	$O(p)\times O(q)$	$U(p)\times U(q)$	$U(n)$
$(U(p,q), U(r,s))$	$U(p,q)\times U(p,q)$	$U(p)\times U(q)$	$(U(p)\times U(p))\times(U(q)\times U(q))$	$U(r)\times U(s)$
$(Sp(p,q), O^*(2n))$	$U(2p,2q)$	$Sp(p)\times Sp(q)$	$U(2p)\times U(2q)$	$U(n)$

$$M' = G' \times G', \quad L' = K' \times K'$$

Recall $\mathbb{G} = Sp(W_\mathbb{R})$ and the complex vector space $W \simeq \mathbb{C}^N$ which is identical to $W_\mathbb{R}$ as a real vector space. Then there exist a direct sum decomposition $W_\mathbb{R} = W_\mathbb{R}^+ \oplus W_\mathbb{R}^-$ and correspondingly $W = W^+ \oplus W^-$, which are compatible with direct product decompositions $L = L^+ \times L^-$ and $K = K^+ \times K^-$ in the following way. The subgroups $L^\pm$ and $K^\pm$ are contained in the unitary group $U(W^\pm)$. Moreover the pairs $(K^\pm, G')$ and $(L^\pm, K')$ are dual pairs in $Sp(W_\mathbb{R}^\pm)$. Note that $L^\pm \supset K^\pm$ is a symmetric pair.

2.2. **Lifting of nilpotent orbits.** Let us consider the dual pair $(G, G') \subseteq \mathbb{G}$. Recall the lifting of nilpotent orbits defined in [25] (see also [26, 27, 3]):

$$G' \longrightarrow G = G(p,q),$$

$$\mathfrak{s}' \supset \mathbb{O}' \longrightarrow \mathbb{O} \subset \mathfrak{s},$$

where we assume the stable range condition. To be more specific, we have certain double fibration map

$$\begin{array}{ccc} & W & \\ {}^{\varphi}\swarrow & & \searrow^{\psi} \\ \mathfrak{s} & & \mathfrak{s}' \end{array}$$

where φ and ψ are the so-called moment maps. Then for a nilpotent $K'_{\mathbb{C}}$-orbit $\mathbb{O}'$ in $\mathfrak{s}'$, the push down of the inverse image $\varphi(\psi^{-1}(\overline{\mathbb{O}'}))$ is equal to the closure of a nilpotent $K_{\mathbb{C}}$-orbit $\overline{\mathbb{O}}$ in $\mathfrak{s}$. We often write this correspondence as $\mathbb{O} = \theta(\mathbb{O}')$, and call $\mathbb{O}$ the *theta lift* of $\mathbb{O}'$. Here $\Xi = \psi^{-1}(\overline{\mathbb{O}'}) = W \times_{\mathfrak{s}'} \overline{\mathbb{O}'}$ is the scheme theoretic fiber, and $\varphi : \Xi \to \overline{\mathbb{O}}$ is an affine quotient map by the action of $K'_{\mathbb{C}}$:

$$\mathbb{O} = \theta(\mathbb{O}') \simeq \Xi /\!/ K'_{\mathbb{C}}. \tag{2.1}$$

See [25].

Since we shall fix G' and vary $G = G(p,q)$, we often use the subscript p,q to indicate the dependence of the situation on (p,q). Thus we write $W = W_{p,q} = W_p \oplus W_q$, $\psi = \psi_{p,q}$, $\mathfrak{s} = \mathfrak{s}_{p,q}$ as well as $W_k = W_{k,0}$. We also write $\psi_{p,q} = (\psi_p^+, \psi_q^-)$ according to the vector space decomposition $W_{p,q} = W_p \oplus W_q$. Note that we may sometimes write $W_{p,q} = W_p \times W_q$ as algebraic varieties. Similar notations will be used freely.

Since G'/K' is a Hermitian symmetric space, $\mathfrak{s}'$ breaks up into irreducible pieces $\mathfrak{s}'_{\pm}$ under the action of $K'_{\mathbb{C}}$. Consider the orbit decomposition of $\mathfrak{s}'_-$ by the adjoint action of $K'_{\mathbb{C}}$. Any $K'_{\mathbb{C}}$-orbit in $\mathfrak{s}'_-$ is clearly nilpotent. We call these nilpotent orbit *holomorphic.* It is well known that $\mathfrak{s}'_-$ is a prehomogeneous vector space, and there exists a numbering of $K'_{\mathbb{C}}$-orbits $\mathbb{O}'_0, \mathbb{O}'_1, \ldots, \mathbb{O}'_l$ in such away that $\mathbb{O}'_{i-1} \subset \overline{\mathbb{O}'_i}$ for $1 \leq i \leq l$. Here l is the real rank of G', and in the cases we are concerned, $l = n, \min\{m,n\}, [\frac{n}{2}]$, respectively. As a consequence, $\mathbb{O}'_0 = \{0\}$ and $\overline{\mathbb{O}'_l} = \mathfrak{s}'_-$, i.e., $\mathbb{O}'_l$ is the open dense orbit in $\mathfrak{s}'_-$. The orbit $\mathbb{O}'_l$ is called *regular*, while the orbits $\{\mathbb{O}'_k\}_{0 \leq k < l}$ are called *singular*.

In the sequel, we shall be mainly concerned with theta lifting of a holomorphic nilpotent orbit $\mathbb{O}'^{\mathrm{hol}}_k$. Thus for $\mathbb{O}^{\mathrm{hol}}_k = \theta(\mathbb{O}'^{\mathrm{hol}}_k)$, we have

$$\overline{\mathbb{O}^{\mathrm{hol}}_k} \simeq \psi_{p,q}^{-1}(\overline{\mathbb{O}'^{\mathrm{hol}}_k}) /\!/ K'_{\mathbb{C}} \simeq \left\{ \mathfrak{N}_p^+ \times (\psi_q^-)^{-1}(\overline{\mathbb{O}'^{\mathrm{hol}}_k}) \right\} /\!/ K'_{\mathbb{C}}, \tag{2.2}$$

where $\mathfrak{N}_p^+$ is the nullcone defined by $\mathfrak{N}_p^+ = (\psi_p^+)^{-1}(0)$.

2.3. **Lifting of affine quotient map.** Let us take a holomorphic nilpotent orbit $\mathbb{O}'^{\mathrm{hol}}_k$ (of "rank" k) in $\mathfrak{s}'_-$, where $0 \leq k \leq l$, and consider its lift $\mathbb{O}^{\mathrm{hol}}_k = \theta(\mathbb{O}'^{\mathrm{hol}}_k)$.

Denote $G(k,\mathbb{C}) = G(k)_{\mathbb{C}}$ the complexification of the compact group $G(k)$. Note that $K_{\mathbb{C}} = K_{\mathbb{C}}^+ \times K_{\mathbb{C}}^- \simeq G(p,\mathbb{C}) \times G(q,\mathbb{C})$ in this notation. It is known that $\overline{\mathbb{O}'^{\mathrm{hol}}_k}$ is an affine quotient of W_k by $G(k,\mathbb{C})$:

$$W_k//G(k,\mathbb{C}) \simeq \begin{cases} \overline{\mathbb{O}'^{\mathrm{hol}}_k} & (0 \le k \le l), \\ \overline{\mathbb{O}'^{\mathrm{hol}}_l} & (k > l). \end{cases}$$

Let $\zeta' : W_k \to \overline{\mathbb{O}'^{\mathrm{hol}}_k}$ be the quotient map. Our aim is to lift the quotient map ζ' to a quotient map $\zeta : X_{p,q+k} \to \overline{\mathbb{O}^{\mathrm{hol}}_k}$. We shall specify the variety $X_{p,q+k}$ later.

Let us consider the lifting of the trivial orbit

$$G' \longrightarrow G(p, q+k),$$
$$\mathfrak{s}' \supset \mathbb{O}'^{\mathbf{1}} = \{0\} \longrightarrow \theta(\mathbb{O}'^{\mathbf{1}}) = \mathbb{O}^{\mathbf{1}} \subset \mathfrak{s} = \mathfrak{s}_{p,q+k}.$$

Here we only assume that $k \ge 0$ (in addition to the usual stable range condition for the original dual pair $(G(p,q), G')$). We put

$$X_{p,q+k} = \overline{\mathbb{O}^{\mathbf{1}}} \subset \mathfrak{s} = \mathfrak{s}_{p,q+k}, \tag{2.3}$$

the closure of the $G(p,\mathbb{C}) \times G(q+k,\mathbb{C})$-orbit lifted from the trivial one. The consideration for the variety $X_{p,q+k}$ was motivated by some result of Loke [20].

To describe the variety $X_{p,q+k}$ more explicitly, define the null cone by

$$\mathfrak{N}_{p,q+k} = \psi_{p,q+k}^{-1}(0) = \mathfrak{N}_p^+ \times \mathfrak{N}_{q+k}^- \subset W_{p,q+k}, \tag{2.4}$$

where $\mathfrak{N}_{q+k}^- = (\psi_{q+k}^-)^{-1}(0)$. By definition, $X_{p,q+k}$ is the image of the null cone $\mathfrak{N}_{p,q+k}$ by the map φ, which is an affine quotient map by the action of $K'_{\mathbb{C}}$:

$$X_{p,q+k} \simeq \mathfrak{N}_{p,q+k}//K'_{\mathbb{C}}. \tag{2.5}$$

Proposition 2.1. *We have*

$$X_{p,q+k}//G(k,\mathbb{C}) \simeq \begin{cases} \overline{\mathbb{O}^{\mathrm{hol}}_k} & (0 \le k \le l), \\ \overline{\mathbb{O}^{\mathrm{hol}}_l} & (k > l). \end{cases}$$

The quotient map

$$\zeta : X_{p,q+k} (\subset \mathfrak{s}_{p,q+k}) \longrightarrow \overline{\mathbb{O}^{\mathrm{hol}}_r} \subset \mathfrak{s}_{p,q} \qquad (r = \min\{k, l\})$$

is given by $\mathrm{proj}|_{X_{p,q+k}}$, *where* $\mathrm{proj} : \mathfrak{s}_{p,q+k} = \mathfrak{s}_{p,q} \oplus \mathfrak{s}_{p,k} \to \mathfrak{s}_{p,q}$ *is the projection map.*

To prove this, we observe the following

Lemma 2.2.

$$\mathfrak{N}_{q+k}^- \simeq W_q \times_{\mathfrak{s}'_-} W_k.$$

Proof. Clearly $(B,C) \in \mathfrak{N}^-_{q+k} \subset W_q \times W_k$ if and only if $\psi^-_{q+k}(B,C) = \psi^-_q(B) + \psi^-_k(C) = 0$, or $\psi^-_q(B) = -\psi^-_k(C)$. Namely the following diagram commutes (note the (-1) twist for the vertical map $W_k \to \mathfrak{s}'_-$):

$$\begin{array}{ccc} (B,C) \in \mathfrak{N}^-_{q+k} \subset W_q \times W_k & \xrightarrow{pr_2} & W_k \ni C \\ {\scriptstyle pr_1}\Big\downarrow & & \Big\downarrow{\scriptstyle (-1)\cdot\psi^-_k} \\ B \in W_q & \xrightarrow{\psi^-_q} & \mathfrak{s}'_- \ni \psi^-_q(B) = -\psi^-_k(C) \end{array}$$

This proves that $\mathfrak{N}^-_{q+k} \simeq W_q \times_{\mathfrak{s}'_-} W_k$ in the set theoretic sense. It is also easy to see that they are in fact isomorphic in the category of algebraic varieties. □

Proof of Proposition 2.1. By definition, we have

$$\begin{aligned} X_{p,q+k} &= \mathfrak{N}_{p,q+k}/\!/K'_{\mathbb{C}} \\ &\simeq \bigl\{\mathfrak{N}^+_p \times \bigl(W_q \times_{\mathfrak{s}'_-} W_k\bigr)\bigr\}/\!/K'_{\mathbb{C}} \quad \text{(by the above lemma);} \end{aligned} \tag{2.6}$$

therefore, if $0 \le k \le l$,

$$\begin{aligned} X_{p,q+k}/\!/G(k,\mathbb{C}) &\simeq \bigl\{\mathfrak{N}^+_p \times \bigl(W_q \times_{\mathfrak{s}'_-} (W_k/\!/G(k,\mathbb{C}))\bigr)\bigr\}/\!/K'_{\mathbb{C}} \\ &\simeq \bigl\{\mathfrak{N}^+_p \times \bigl(W_q \times_{\mathfrak{s}'_-} \overline{\mathbb{O}'^{\mathrm{hol}}_k}\bigr)\bigr\}/\!/K'_{\mathbb{C}} \\ &\simeq \bigl\{\mathfrak{N}^+_p \times (\psi^-_q)^{-1}(\overline{\mathbb{O}'^{\mathrm{hol}}_k})\bigr\}/\!/K'_{\mathbb{C}} = \psi^{-1}_{p,q}(\overline{\mathbb{O}'^{\mathrm{hol}}_k})/\!/K'_{\mathbb{C}} \simeq \overline{\mathbb{O}^{\mathrm{hol}}_k}. \end{aligned}$$

The case $k > l$ can be treated similarly.

Take $(A,B,C) \in \mathfrak{N}_{p,q+k} \subset W_{p,q+k}$. Then the quotient map

$$\mathfrak{N}_{p,q+k} \to \mathfrak{N}_{p,q+k}/\!/K'_{\mathbb{C}} \simeq X_{p,q+k}$$

is given by $\varphi_{p,q+k}$. On the other hand, the quotient map

$$W_{p,q+k} \supset \mathfrak{N}_{p,q+k} \to \mathfrak{N}_{p,q+k}/\!/G(k,\mathbb{C}) \simeq \mathfrak{N}^+_p \times (\psi^-_q)^{-1}(\overline{\mathbb{O}'^{\mathrm{hol}}_k}) = \psi^{-1}_{p,q}(\overline{\mathbb{O}'^{\mathrm{hol}}_k}) \subset W_{p,q}$$

agrees with the restriction of the projection map $W_{p,q+k} \to W_{p,q}$. Thus the quotient map $\psi^{-1}(\overline{\mathbb{O}'^{\mathrm{hol}}_k}) \to \psi^{-1}(\overline{\mathbb{O}'^{\mathrm{hol}}_k})/\!/K'_{\mathbb{C}} \simeq \overline{\mathbb{O}^{\mathrm{hol}}_k}$ is given by $\varphi_{p,q}(A,B)$. We summarize this situation by the following diagram.

$$\begin{array}{ccccc} \varphi_{p,q+k}(A,B,C) \in X_{p,q+k} & \xleftarrow{/\!/K'_{\mathbb{C}}} & (A,B,C) \in \mathfrak{N}_{p,q+k} & \xrightarrow{/\!/K_{\mathbb{C}}} & W_k \ni C \\ {\scriptstyle /\!/G(k,\mathbb{C})}\Big\downarrow & & {\scriptstyle /\!/G(k,\mathbb{C})}\Big\downarrow & & {\scriptstyle /\!/G(k,\mathbb{C})}\Big\downarrow \\ \varphi_{p,q}(A,B) \in \overline{\mathbb{O}^{\mathrm{hol}}_k} & \xleftarrow{/\!/K'_{\mathbb{C}}} & (A,B) \in \Xi_k & \xrightarrow{/\!/K_{\mathbb{C}}} & \overline{\mathbb{O}'^{\mathrm{hol}}_k} \ni \psi^-_q(B) = -\psi^-_k(C) \end{array}$$

Here $\Xi_k = \psi^{-1}_{p,q}(\overline{\mathbb{O}'^{\mathrm{hol}}_k})$. Accordingly, the quotient map $\zeta : X_{p,q+k} \to \overline{\mathbb{O}^{\mathrm{hol}}_k}$ is just the restriction of the projection proj. □

2.4. **Covariants over a lifted nilpotent orbit.** We consider the module of $H = G(k,\mathbb{C})$ covariants associated to the affine quotient map $\zeta : X_{p,q+k} \to \overline{\mathbb{O}^{\mathrm{hol}}_k}$.

Put $\Xi = \Xi_k = \psi_{p,q}^{-1}(\overline{\mathbb{O}'^{\mathrm{hol}}_k})$, and $Z = W_k$. We are in the setting of §1.4. In the notation there, we have $X = \Xi /\!/ K'_{\mathbb{C}} \simeq \overline{\mathbb{O}^{\mathrm{hol}}_k}$, and $Y = \Xi /\!/ K_{\mathbb{C}} \simeq \overline{\mathbb{O}'^{\mathrm{hol}}_k}$. Furthermore $Z /\!/ H \simeq Y$ as a $K'_{\mathbb{C}}$ variety. Note that

$$\begin{aligned} V = (\Xi \times_Y Z) /\!/ K'_{\mathbb{C}} &\simeq \bigl\{ \bigl(\mathfrak{N}_p^+ \times (W_q \times_{\mathfrak{s}'_-} \mathbb{O}'^{\mathrm{hol}}_k)\bigr) \times_{\mathbb{O}'^{\mathrm{hol}}_k} W_k \bigr\} /\!/ K'_{\mathbb{C}} \\ &\simeq \bigl\{ (\mathfrak{N}_p^+ \times (W_q \times_{\mathfrak{s}'_-} W_k) \bigr\} /\!/ K'_{\mathbb{C}} = \mathfrak{N}_{p,q+k} /\!/ K'_{\mathbb{C}} \\ &\simeq X_{p,q+k}. \end{aligned}$$

For an irreducible finite dimensional representation σ of $H = G(k,\mathbb{C})$, let $L[\sigma] = \bigl(\sigma \otimes \mathbb{C}[W_k]\bigr)^{G(k,\mathbb{C})}$ and $M[\sigma] = \bigl(\sigma \otimes \mathbb{C}[X_{p,q+k}]\bigr)^{G(k,\mathbb{C})}$ be the spaces of covariants. They are finitely generated modules over $\mathbb{C}[\overline{\mathbb{O}'^{\mathrm{hol}}_k}]$ and $\mathbb{C}[\overline{\mathbb{O}^{\mathrm{hol}}_k}]$, respectively. Let $\tilde{L}[\sigma]$ (resp. $\tilde{M}[\sigma]$) be the associated coherent sheave on $\overline{\mathbb{O}'^{\mathrm{hol}}_k}$ (resp. $\overline{\mathbb{O}^{\mathrm{hol}}_k}$). We know that the coherent sheaf $\tilde{M}[\sigma]$ on $\overline{\mathbb{O}^{\mathrm{hol}}_k}$ is *lifted* from $\tilde{L}[\sigma]$ on $\overline{\mathbb{O}'^{\mathrm{hol}}_k}$ in the sense of Definition 1.5. See Proposition 1.10.

Theorem 2.3. (a) *If $k \le l$ (the real rank of G'), or if $k > l$ and G'/K' is of tube type (namely Case $\mathbb{R}$, Case $\mathbb{C}$ with $m = n$ and Case $\mathbb{H}$ with n even), then the generic fiber of the quotient map*

$$\zeta : X_{p,q+k} \longrightarrow \overline{\mathbb{O}^{\mathrm{hol}}_r} \qquad (r = \min\{k,l\})$$

is a single closed $G(k,\mathbb{C})$-orbit with a stabilizer $St = \{1\}$ in the first case and $St \simeq G(k-l,\mathbb{C})$ in the second case. Consequently for an irreducible finite dimensional representation σ of $G(k,\mathbb{C})$, the multiplicity of the space of covariants $M[\sigma] = \bigl(\sigma \otimes \mathbb{C}[X_{p,q+k}]\bigr)^{G(k,\mathbb{C})}$ is given by

$$\operatorname{rank}_{\mathbb{C}[\overline{\mathbb{O}^{\mathrm{hol}}_k}]} M[\sigma] = \begin{cases} \dim \sigma & (0 \le k \le l), \\ \dim \sigma^{G(k-l,\mathbb{C})} & (k > l \text{ and } G'/K' \text{ is of tube type}). \end{cases} \tag{2.7}$$

(b) *For the dual pair $(U(p,q), U(m,n))$ $(m \ne n)$, we have*

$$\operatorname{rank}_{\mathbb{C}[\overline{\mathbb{O}^{\mathrm{hol}}_k}]} M(\sigma) = \dim\bigl(\sigma \otimes \mathbb{C}[M_{k-n,m-n}]\bigr)^{GL_{k-n}}, \qquad (k \ge m > n),$$

and for the dual pair $(Sp(p,q), O^*(2n))$ $(n$ odd$)$, we have

$$\operatorname{rank}_{\mathbb{C}[\overline{\mathbb{O}^{\mathrm{hol}}_k}]} M(\sigma) = \dim\bigl(\sigma \otimes \mathbb{C}[\mathbb{C}^{2(k-l)}]\bigr)^{Sp(2(k-l),\mathbb{C})}, \qquad (2k > n;\ n = 2l+1).$$

Proof. (a) The proof (or the computation) for the part on the generic fiber of ζ will be carried out in the Appendix. The formula for the multiplicity $\operatorname{rank}_{\mathbb{C}[\overline{\mathbb{O}^{\mathrm{hol}}_k}]} M[\sigma]$ follows from Proposition 1.3, and Corollaries 5.1, 5.6, 5.11.

(b) We shall only prove the $(U(p,q), U(m,n))$ case. The other case is entirely similar. If $k \geq m > n$, Lemma 1.2 tells us

$$\operatorname{rank}_A M(\sigma) = \dim\big(\sigma \otimes \mathbb{C}[\zeta^{-1}((S,T))]\big)^{GL_k},$$

where $\zeta^{-1}((S,T))$ is the generic fiber as in Corollary 5.6. On the other hand, by Corollary 5.6 and Frobenius reciprocity, we have an isomorphism

$$\begin{aligned}\big(\sigma \otimes \mathbb{C}[\zeta^{-1}((S,T))]\big)^{GL_k} &\simeq \big(\sigma \otimes \mathbb{C}[GL_k *_{GL_{k-n}} M_{k-n,m-n}]\big)^{GL_k} \\ &\simeq \big(\sigma \otimes \mathbb{C}[M_{k-n,m-n}]\big)^{GL_{k-n}}.\end{aligned}$$

This proves the desired multiplicity formula. □

2.5. **$K_{\mathbb{C}}$-structures and gradings.** We consider the $K_{\mathbb{C}}$-module structures of the covariants $M[\sigma]$. First we will need some notations.

Let $\mathcal{H}^{\pm} = \mathcal{H}(K_{\mathbb{C}}^{\pm})$ denote the space of harmonic polynomials on $W^{\pm}$ (under the action of $K_{\mathbb{C}}^{\pm}$). Put

$$\operatorname{Irr}(K'_{\mathbb{C}}; \mathcal{H}^+) = \{\tau \in \operatorname{Irr}(K'_{\mathbb{C}}) \mid \operatorname{Hom}_{K'_{\mathbb{C}}}(\tau, \mathcal{H}^+) \neq 0\}$$

and

$$\operatorname{Irr}(K_{\mathbb{C}}^+; \mathcal{H}^+) = \{\sigma \in \operatorname{Irr}(K_{\mathbb{C}}^+) \mid \operatorname{Hom}_{K_{\mathbb{C}}^+}(\sigma, \mathcal{H}^+) \neq 0\}.$$

The space $\mathcal{H}(K_{\mathbb{C}}^+)$ is multiplicity-free as a representation of $K_{\mathbb{C}}^+ \times K'_{\mathbb{C}}$. Furthermore the decomposition

$$\mathcal{H}(K_{\mathbb{C}}^+)\big|_{K_{\mathbb{C}}^+ \times K'_{\mathbb{C}}} \simeq \sum\nolimits^{\oplus}_{\sigma \in \operatorname{Irr}(K_{\mathbb{C}}^+;\mathcal{H}^+), \tau \in \operatorname{Irr}(K'_{\mathbb{C}};\mathcal{H}^+)} \sigma \boxtimes \tau \tag{2.8}$$

determines a one-to-one correspondence between $\sigma \in \operatorname{Irr}(K_{\mathbb{C}}^+; \mathcal{H}^+)$ and $\tau \in \operatorname{Irr}(K'_{\mathbb{C}}; \mathcal{H}^+)$ (see [10]). Similar notations and statements apply for $\mathcal{H}^- = \mathcal{H}(K_{\mathbb{C}}^-)$.

We shall abbreviate $R^{\pm}(K'_{\mathbb{C}}) = \operatorname{Irr}(K'_{\mathbb{C}}; \mathcal{H}^{\pm})$ in the following. Under the assumption of stable range, one can explicitly check that $\tau \in R^+(K'_{\mathbb{C}})$ if and only if $\tau^* \in R^-(K'_{\mathbb{C}})$, where τ^* denotes the contragredient representation of $\tau \in \operatorname{Irr}(K')$. So we put

$$R(K'_{\mathbb{C}}) = R^+(K'_{\mathbb{C}})^* = R^-(K'_{\mathbb{C}}). \tag{2.9}$$

To summarize, for each $\tau \in R^{\pm}(K'_{\mathbb{C}})$, there is a unique $\sigma \in \operatorname{Irr}(K_{\mathbb{C}}^{\pm})$ such that

$$\operatorname{Hom}_{K_{\mathbb{C}}^{\pm} \times K'_{\mathbb{C}}}(\sigma \boxtimes \tau, \mathcal{H}(K_{\mathbb{C}}^{\pm})) \neq 0.$$

We denote this σ by $\sigma^{\pm}(\tau)$, specifying the dependency of τ and the sign $\pm$. Then, we can rewrite (2.8) as

$$\mathcal{H}(K_{\mathbb{C}}^+) \simeq \sum^{\oplus}_{\tau \in R(K'_{\mathbb{C}})} \sigma^+(\tau^*) \boxtimes \tau^*, \quad \text{and} \quad \mathcal{H}(K_{\mathbb{C}}^-) \simeq \sum^{\oplus}_{\tau \in R(K'_{\mathbb{C}})} \sigma^-(\tau) \boxtimes \tau. \tag{2.10}$$

Theorem 2.4. *There is a compatible action of $K_{\mathbb{C}}$ on $M[\sigma]$, and it decomposes as a $K_{\mathbb{C}} = K_{\mathbb{C}}^+ \times K_{\mathbb{C}}^-$-module as follows:*

$$M[\sigma] \simeq \sum_{\tau_\alpha, \tau_\beta \in R(K'_{\mathbb{C}})}^{\oplus} \mathrm{Hom}_{K'_{\mathbb{C}}}\big(\tau_\alpha \otimes \tau_\beta^*, L[\sigma]\big) \otimes \big(\sigma^+(\tau_\alpha^*) \boxtimes \sigma^-(\tau_\beta)\big), \tag{2.11}$$

where $L[\sigma] = \big(\sigma \otimes \mathbb{C}[W_k]\big)^{G(k,\mathbb{C})}$, which is a $K'_{\mathbb{C}}$-module.

Proof. Note that there is a $K_{\mathbb{C}} \times G(k,\mathbb{C})$-equivariant isomorphism

$$X_{p,q+k} \simeq \big\{\mathfrak{N}_p^+ \times \big(W_q \times_{\mathfrak{s}'_-} W_k\big)\big\}//K'_{\mathbb{C}} \qquad \text{(see (2.6))}.$$

Thus, the covariants $M[\sigma]$ is isomorphic to

$$\begin{aligned} M[\sigma] &= \big(\sigma \otimes \mathbb{C}[X_{p,q+k}]\big)^{G(k,\mathbb{C})} \\ &\simeq \big(\sigma \otimes \mathbb{C}[\mathfrak{N}_p^+ \times \big(W_q \times_{\mathfrak{s}'_-} W_k\big)]\big)^{G(k,\mathbb{C}) \times K'_{\mathbb{C}}} \\ &\simeq \big\{\mathbb{C}[\mathfrak{N}_p^+] \otimes \big(\mathbb{C}[W_q] \otimes_{\mathbb{C}[\mathfrak{s}'_-]} (\sigma \otimes \mathbb{C}[W_k])^{G(k,\mathbb{C})}\big)\big\}^{K'_{\mathbb{C}}} \\ &\simeq \big(\mathcal{H}_{p,q} \otimes L[\sigma]\big)^{K'_{\mathbb{C}}}, \end{aligned} \tag{2.12}$$

where $\mathcal{H}_{p,q} \simeq \mathcal{H}(K_{\mathbb{C}}^+) \boxtimes \mathcal{H}(K_{\mathbb{C}}^-)$ denotes the space of $K_{\mathbb{C}} = K_{\mathbb{C}}^+ \times K_{\mathbb{C}}^-$-harmonic polynomials on $W = W_{p,q}$. Note that we have used the isomorphisms $\mathbb{C}[\mathfrak{N}_p^+] \simeq \mathcal{H}(K_{\mathbb{C}}^+)$ and $\mathbb{C}[W_q] \simeq \mathcal{H}(K_{\mathbb{C}}^-) \otimes \mathbb{C}[\mathfrak{s}'_-]$ (using the stable range condition). Since

$$\mathcal{H}_{p,q} \simeq \mathcal{H}(K_{\mathbb{C}}^+) \boxtimes \mathcal{H}(K_{\mathbb{C}}^-) \simeq \sum_{\tau_\alpha, \tau_\beta \in R(K'_{\mathbb{C}})}^{\oplus} \big(\sigma^+(\tau_\alpha^*) \boxtimes \sigma^-(\tau_\beta)\big) \boxtimes \big(\tau_\alpha^* \otimes \tau_\beta\big), \tag{2.13}$$

we get the desired formula by inserting (2.13) into (2.12). □

Let us discuss certain natural grading on $L[\sigma]$ and its lifted module $M[\sigma]$. Put

$$L[\sigma]_j = \big(\sigma \otimes \mathbb{C}[W_k]_{d(\sigma^*)+2j}\big)^{G(k,\mathbb{C})} \qquad (j \geq 0), \tag{2.14}$$

where $d(\sigma^*) = \deg \sigma^*$ is the degree of σ^*, which is defined to be the lowest possible degree in $\mathbb{C}[W_k]$ in which σ^* appears. This makes $L[\sigma]$ into a $\mathbb{C}[\overline{\mathbb{O}'^{\mathrm{hol}}_k}]$-graded module. Note that $\mathbb{C}[\overline{\mathbb{O}'^{\mathrm{hol}}_k}]$ naturally inherits a structure of graded algebra because $\overline{\mathbb{O}'^{\mathrm{hol}}_k}$ is a closed cone in $\mathfrak{s}'$. The same remark applies to $\mathbb{C}[\overline{\mathbb{O}^{\mathrm{hol}}_k}]$, $\mathbb{C}[\mathfrak{N}_p^+]$ and $\mathbb{C}[\mathfrak{N}_q^-]$. Note also that the degree of $\big(\sigma^+(\tau_\alpha^*) \boxtimes \sigma^-(\tau_\beta)\big) \boxtimes \big(\tau_\alpha^* \otimes \tau_\beta\big)$ in $\mathcal{H}_{p,q}$ is $|\alpha| + |\beta|$, where $|\alpha|$ (resp. $|\beta|$) denotes the degree of τ_α^* (resp. τ_β), defined similarly.

Corollary 2.5. *We may assign a (natural) grading on the lifted module* $M[\sigma]$ *by*

$$M[\sigma]_d = \sum_{d=i+j}^{\oplus} \big((\mathcal{H}_{p,q})_{d(\sigma^*)+2i} \otimes L[\sigma]_j\big)^{K'_{\mathbb{C}}}$$
$$= \sum_{\tau_\alpha,\tau_\beta\in R(K'_{\mathbb{C}})}^{\oplus} \mathrm{Hom}_{K'_{\mathbb{C}}}\big(\tau_\alpha \otimes \tau_\beta^*, L[\sigma]_{d+\frac{1}{2}(d(\sigma^*)-|\alpha|-|\beta|)}\big) \otimes \big(\sigma^+(\tau_\alpha^*) \boxtimes \sigma^-(\tau_\beta)\big) \quad (d \geq 0).$$

With respect to this grading, $M[\sigma]$ *is a* $\mathbb{C}[\overline{\mathbb{O}_k^{\mathrm{hol}}}]$*-graded module.*

Thus we obtain the Poincaré series of $M[\sigma]$ as

$$P(M[\sigma];t) = \sum\nolimits_{d\geq 0} \dim M[\sigma]_d \cdot t^d$$
$$= t^{-d(\sigma^*)/2} \sum_{j\geq 0} \sum_{\tau_\alpha,\tau_\beta\in R(K'_{\mathbb{C}})} \dim \mathrm{Hom}_{K'_{\mathbb{C}}}\big(\tau_\alpha \otimes \tau_\beta^*, L[\sigma]_j\big) \times$$
$$\dim \sigma^+(\tau_\alpha^*)\, \dim \sigma^-(\tau_\beta) \cdot t^{j+\frac{1}{2}(|\alpha|+|\beta|)}. \tag{2.15}$$

3. Theta lifting associated to the dual pair (G, G')

3.1. **Howe's maximal quotient.** Let $(G, G') \subseteq \mathbb{G} = Sp(2N, \mathbb{R})$ be a reductive dual pair, and Ω be a fixed oscillator representation of $\widetilde{\mathbb{G}}$, the metaplectic cover of $\mathbb{G}$.

Let us denote by $\mathrm{Irr}(\mathfrak{g}', \widetilde{K}')$ the infinitesimal equivalence classes of irreducible admissible $(\mathfrak{g}', \widetilde{K}')$-modules, and $R(\mathfrak{g}', \widetilde{K}'; \Omega)$ the subset of those in $\mathrm{Irr}(\mathfrak{g}', \widetilde{K}')$ which can be realized as quotients by $(\mathfrak{g}', \widetilde{K}')$-invariant subspaces of Ω. According to [11], for each $\pi' \in R(\mathfrak{g}', \widetilde{K}'; \Omega)$ there exists a quasi-simple admissible $(\mathfrak{g}', \widetilde{K}')$-module $\Omega(\pi')$ of finite length satisfying

$$\Omega/N \simeq \Omega(\pi') \otimes \pi', \tag{3.1}$$

where

$$N = \bigcap_{\varphi\in\tilde{H}} \ker\varphi, \qquad \tilde{H} = \mathrm{Hom}_{(\mathfrak{g}', \widetilde{K}')}(\Omega, \pi').$$

Furthermore $\Omega(\pi')$ has a unique irreducible quotient, denoted by $\theta(\pi')$. The representation $\Omega(\pi')$ is called *Howe's maximal quotient* of π', and $\theta(\pi')$ (or sometimes written as $\theta_{G'\to G}(\pi')$) the *theta lift* of π'.

Denote by

$$H = \tilde{H}_{\widetilde{K}} = \mathrm{Hom}_{(\mathfrak{g}', \widetilde{K'})}(\Omega, \pi')_{\widetilde{K}\text{-finite}},$$

the subspace of $\widetilde{K}$-finite vectors of $\tilde{H}$. The following lemma is elementary.

Lemma 3.1. *Let* $\rho = \Omega(\pi')$ *be the maximal quotient. Then we have*

$$H \simeq \rho^* \qquad (\widetilde{K}\text{-finite dual}).$$

Let $(G, G') \subseteq Sp(2N, \mathbb{R})$ be one of the three dual pairs specified in the Introduction. For the moment we do not assume the stable range condition.

Recall that the maximal compact subgroup K' determines a Cartan decomposition of the complexified Lie algebra $\mathfrak{g}'$ of G' : $\mathfrak{g}' = \mathfrak{k}' \oplus \mathfrak{s}'$. Under the action of K' (via the restriction of the adjoint representation), $\mathfrak{s}'$ breaks up into irreducible pieces ${\mathfrak{s}'}_{\pm}$. Through the covering map, $\widetilde{K}'$ naturally acts on $\mathfrak{s}'$ and ${\mathfrak{s}'}_{\pm}$.

Recall also that an irreducible unitary representation π' of $\widetilde{G}'$ is called a lowest weight representation (or holomorphic) if there exist non-zero $\widetilde{K}'$-finite vectors v in the space of π' such that $\pi'({\mathfrak{s}'}_{-})(v) = 0$. Then the space of such vectors v is irreducible under $\widetilde{K}'$. This is the (unique) minimal $\widetilde{K}'$-type of π', and it determines the representation π' completely.

Let $\Lambda(\widetilde{K}')$ denote the set of dominant integral weights for $\widetilde{K}'$. For $\lambda \in \Lambda(\widetilde{K}')$, let $\tau^{\widetilde{K}'}(\lambda)$ denote the irreducible finite dimensional representation of $\widetilde{K}'$ with highest weight λ, and $L(\lambda)$ the unitary lowest weight representation of $\widetilde{G}'$ with the minimal $\widetilde{K}'$-type $\tau^{\widetilde{K}'}(\lambda)$. We note that if $L(\lambda)$ is a holomorphic discrete series representation, then we have

$$L(\lambda)\big|_{\widetilde{K}'} \simeq \tau^{\widetilde{K}'}(\lambda) \otimes S({\mathfrak{s}'}_{+}) \simeq \tau^{\widetilde{K}'}(\lambda) \otimes \mathbb{C}[{\mathfrak{s}'}_{-}],$$

where $S(V)$ denotes the symmetric algebra generated by a vector space V. We caution the reader that this is certainly not true for a singular unitary lowest weight representation.

We recall notations from §2.1. For a natural number k, we consider the compact dual pair $(G(k), G')$. Let Ω_k be the associated oscillator Harish-Chandra module. We have the (discrete) dual pair decomposition

$$\Omega_k\big|_{\widetilde{G(k)} \times \widetilde{G}'} \simeq \sum_{\lambda \in \Lambda_k(\widetilde{K}')}^{\oplus} \sigma_k(\lambda) \boxtimes L(\lambda), \tag{3.2}$$

where

$$\Lambda_k(\widetilde{K}') = \{\lambda \in \Lambda(\widetilde{K}') | L(\lambda) \in R(\mathfrak{g}', \widetilde{K}'; \Omega_k)\}.$$

We note that in our notation, the highest weight of $\sigma_k(\lambda) \in \operatorname{Irr}(\widetilde{G(k)})$ is not λ. The above decomposition (the description of the set $\Lambda_k(\widetilde{K}')$ and the explicit correspondence $\sigma_k(\lambda) \leftrightarrow L(\lambda)$) is well-known, and is given in [13] for the pairs $(O(k), Sp(2n, \mathbb{R}))$ and $(U(k), U(m, n))$. See [6] for the pair $(Sp(k), O^*(2n))$.

We have the compact dual pairs $(K^{\pm}, G')$ in the symplectic group $Sp(W_{\mathbb{R}}^{\pm})$. Note that $K^+ \simeq G(p, 0)$, $K^- \simeq G(0, q)$. Let $\Omega^{\pm}$ be the associated oscillator Harish-Chandra module of the metaplectic group $Mp(W_{\mathbb{R}}^{\pm})$. Writing

$$\sigma^{\widetilde{K}^+}(\mu) = \sigma_p(\mu), \qquad \mu \in \Lambda_p(\widetilde{K}'),$$
$$\sigma^{\widetilde{K}^-}(\nu) = \sigma_q(\nu), \qquad \nu \in \Lambda_q(\widetilde{K}'),$$

we have

$$\Omega^+\big|_{\widetilde{K}^+\times\widetilde{G}'} \simeq \sum_{\mu\in\Lambda_p(\widetilde{K}')}^{\oplus} \sigma^{\widetilde{K}^+}(\mu)\boxtimes L(\mu). \tag{3.3}$$

and

$$\Omega^-\big|_{\widetilde{K}^-\times\widetilde{G}'} \simeq \sum_{\nu\in\Lambda_q(\widetilde{K}')}^{\oplus} \sigma^{\widetilde{K}^-}(\nu)^*\boxtimes L(\nu)^*. \tag{3.4}$$

Remark 3.2. Assume that $(G(p,q),G')$ is in the stable range with G' the smaller member. Then for $\mu\in\Lambda_p(\widetilde{K}')$ (resp. $\nu\in\Lambda_q(\widetilde{K}')$), $L(\mu)$ (resp. $L(\nu)$) is a genuine holomorphic discrete series representation. This is also the reason that we have excluded the case $(O(p,q),Sp(2n,\mathbb{R}))$, where $2n=\min(p,q)$. See Table 1. The sets $\Lambda_p(\widetilde{K}')$ and $\Lambda_q(\widetilde{K}')$ are almost equal under the stable range condition (apart from some translation).

The following proposition is also elementary.

Proposition 3.3. *For $\pi'\in R(\mathfrak{g}',\widetilde{K}';\Omega)$, we have the $\widetilde{K}$-type decomposition*

$$H\big|_{\widetilde{K}} \simeq \sum_{\substack{\mu\in\Lambda_p(\widetilde{K}')\\ \nu\in\Lambda_q(\widetilde{K}')}}^{\oplus} \dim \operatorname{Hom}_{\mathfrak{g}',\widetilde{K}'}(L(\mu)\otimes L(\nu)^*,\pi')(\sigma^{\widetilde{K}^+}(\mu)^*\boxtimes\sigma^{\widetilde{K}^-}(\nu)).$$

Consequently we have

$$\Omega(\pi')\big|_{\widetilde{K}} \simeq \sum_{\substack{\mu\in\Lambda_p(\widetilde{K}')\\ \nu\in\Lambda_q(\widetilde{K}')}}^{\oplus} \dim \operatorname{Hom}_{\mathfrak{g}',\widetilde{K}'}(L(\mu)\otimes L(\nu)^*,\pi')(\sigma^{\widetilde{K}^+}(\mu)\boxtimes\sigma^{\widetilde{K}^-}(\nu)^*).$$

Proof. We consider the see-saw pair ([16, 9]; cf. §2.1):

$$\begin{array}{ccc} G & & M'=G'\times G' \\ \cup & & \cup \\ K=K^+\times K^- & & G' \end{array}$$

By the functoriality of the oscillator representation, we have

$$\Omega\simeq\Omega^+\otimes\Omega^-$$

as $\widetilde{K}\times\widetilde{M'}$-modules. Thus as $\widetilde{K}\times(\mathfrak{g}',\widetilde{K'})$-modules, we have

$$\begin{aligned}\Omega &\simeq \Big(\sum_{\mu\in\Lambda_p(\widetilde{K}')}^{\oplus} \sigma^{\widetilde{K}^+}(\mu)\boxtimes L(\mu)\Big)\otimes\Big(\sum_{\nu\in\Lambda_q(\widetilde{K}')}^{\oplus} \sigma^{\widetilde{K}^-}(\nu)^*\boxtimes L(\nu)^*.\Big)\\ &\simeq \sum_{\substack{\mu\in\Lambda_p(\widetilde{K}')\\ \nu\in\Lambda_q(\widetilde{K}')}}^{\oplus} \big(\sigma^{\widetilde{K}^+}(\mu)\boxtimes\sigma^{\widetilde{K}^-}(\nu)^*\big)\boxtimes\big(L(\mu)\otimes L(\nu)^*\big).\end{aligned}$$

Since $H = \mathrm{Hom}_{(\mathfrak{g}', \widetilde{K'})}(\Omega, \pi')_{\widetilde{K}\text{-finite}}$, the first assertions follows. The second assertion then follows from the isomorphism $\Omega(\pi')^* \simeq H$. □

Remark 3.4. In [24], an incorrect version of the above Proposition was given (Proposition 4.3). But it does affect the results there as long as one changes some relevant $\widetilde{K}$-types into their duals.

3.2. **Abstract K-type formula.** We now assume that the dual pair (G, G') is in the stable range with G' the smaller member. In [24], we consider the theta lift $\pi = \theta(\pi')$ for a holomorphic discrete series representation π', in particular we give the $\widetilde{K}$-type formula of π. The result in [24] is stated only for the pair $(G, G') = (U(n, n), U(p, q))$, however, the other cases can be treated by similar explicit calculations.

Here we pursue in a more sophisticated way to give a unified formula for $\widetilde{K}$-type decomposition of $\pi = \theta(\pi')$ for any unitary lowest weight representation π'. This unified description enables us to compare them with certain covariants over the associated variety of π. For this, we will need to recall Li's construction of the theta lift in the stable range [17].

We fix a genuine irreducible unitary representation ξ of $\widetilde{G}'$. Let $\mathcal{S}$ (resp. V_ξ) be the space of smooth vectors of Ω (resp. ξ). Consider the $\widetilde{G} \times \widetilde{G}'$-module $\mathcal{S} \otimes V_\xi$ where $\widetilde{G}$ acts on the first factor, and $\widetilde{G}'$ acts by $\Omega \otimes \xi$, which is in fact a representation of G'. The unitary structures on $\mathcal{S}$ and V_ξ give rise to an inner product $(,)$ on $\mathcal{S} \otimes V_\xi$. Define a sesquilinear form $(,)_\xi$ on $\mathcal{S} \otimes V_\xi$ by

$$(\Phi, \Phi')_\xi = \int_{G'} (\Phi, (\Omega \otimes \xi)(h)\Phi') dh.$$

Under the stable range condition, the above integral converges absolutely and further it defines a non-negative $\widetilde{G}$-invariant Hermitian form (see [17] for details). Let R be the radical of this form. By introducing an intertwining map to certain induced representation and making use of Mackey's Theorem, Li shows that the resulting representation on the quotient space $(\mathcal{S} \otimes V_\xi)/R$ is in fact irreducible as a representation of $\widetilde{Q}$, where Q is a subgroup of certain maximal parabolic subgroup of G. The representation of $\widetilde{G}$ on the space $(\mathcal{S} \otimes V_\xi)/R$ realizes $\theta(\xi^*)$, the theta lift of the contragradient of ξ.

The following proposition is essentially proved by Li [19]. We reproduce his argument below.

Proposition 3.5. *Let (G, G') be a reductive dual pair in the stable range with G' the smaller member. Suppose*

I: π' is a genuine unitary lowest weight representation of $\widetilde{G}'$, or

II: $\pi' = \xi^*$, the contragradient of a genuine unitary lowest weight representation ξ, *then we have the $\widetilde{K}$-type decomposition*

$$\theta(\pi')\big|_{\widetilde{K}} \simeq \begin{cases} \sum^{\oplus}_{\substack{\mu\in\Lambda_p(\widetilde{K}')\\ \nu\in\Lambda_q(\widetilde{K}')}} \dim \mathrm{Hom}_{\widetilde{G}'}(L(\mu), L(\nu)\otimes\pi')(\sigma^{\widetilde{K}^+}(\mu)\boxtimes\sigma^{\widetilde{K}^-}(\nu)^*), & \textit{Case I} \\ \sum^{\oplus}_{\substack{\mu\in\Lambda_p(\widetilde{K}')\\ \nu\in\Lambda_q(\widetilde{K}')}} \dim \mathrm{Hom}_{\widetilde{G}'}(L(\mu)\otimes\xi, L(\nu))(\sigma^{\widetilde{K}^+}(\mu)\boxtimes\sigma^{\widetilde{K}^-}(\nu)^*), & \textit{Case II.} \end{cases}$$

Here in Case I, $\mathrm{Hom}_{\widetilde{G}'}(L(\mu), L(\nu)\otimes\pi')$ *denotes the space of unitary* $\widetilde{G}'$*-intertwining maps from* $L(\mu)$ *to* $L(\nu)\otimes\pi'$*, and a similar notation in Case II.*

Proof. We shall only give the proof for Case I. Case II is similar.

Denote $\xi = (\pi')^*$. Let τ be an irreducible (unitary) representation of $\widetilde{K}$. If τ is a $\widetilde{K}$-type of $\theta(\pi')$ then there will be $\Phi, \Phi' \in \mathcal{S}\otimes V_\xi$ transforming according to τ such that $(\Phi,\Phi')_\xi \neq 0$. Write $\tau = \tau_+ \boxtimes \tau_-$, where τ_+ and τ_- are representations of $\widetilde{K}^+$ and $\widetilde{K}^-$ respectively. We have a decomposition $\mathcal{S} = \mathcal{S}_+\otimes\mathcal{S}_-$, where $\mathcal{S}_+$ and $\mathcal{S}_-$ are the spaces of smooth vectors of the oscillator representations Ω^+ and Ω^- respectively. Without loss of generality we may assume $\Phi = \phi_+\otimes\phi_-\otimes v$, where $\phi_+\in\mathcal{S}_+$, $\phi_-\in\mathcal{S}_-$, $v\in V_\xi$. Similarly write $\Phi' = \phi'_+\otimes\phi'_-\otimes v'$. Let τ'_+ (resp. $(\tau'_-)^*$) be the theta lift of τ_+ (resp. τ_-) to $\widetilde{G}'$. Both τ'_+ and τ'_- are holomorphic discrete series of $\widetilde{G}'$. Under $\widetilde{G}'$, the vectors ϕ_+, ϕ'_+ will transform according to τ'_+ and ϕ_-, ϕ'_- will transform according to $(\tau'_-)^*$. With these notations we have

$$(\Phi,\Phi')_\xi = \int_{G'} (\Omega^+(h)\phi_+, \phi'_+)(\Omega^-(h)\phi_-, \phi'_-)(\xi(h)v, v')dh.$$

Note the first factor in the integrand is a matrix coefficient of τ'_+ while the product of the last two is a matrix coefficient of the representation $(\tau'_-)^*\otimes\xi = (\tau'_-\otimes\pi')^*$. Since π' is a holomorphic representation, the tensor product $\tau'_-\otimes\pi'$ is a direct sum of irreducible holomorphic discrete series. It therefore follows from Mackey's version of the Schur's Lemma that the integral is not identically zero if and only if τ'_+ is an irreducible summand of $\tau'_-\otimes\pi'$. To summarize, this condition is necessary and sufficient for τ to be a $\widetilde{K}$-type of $\theta(\pi')$. Up to this point, everything is in [19].

Now assume that τ appears in $\theta(\pi')$. Thus $m = \dim \mathrm{Hom}_{\widetilde{G}'}(\tau'_+, \tau'_-\otimes\pi')$ is a positive integer and $m\tau'_+$ is a direct summard of $\tau'_-\otimes\pi'$. Observe that the space $\mathcal{S}\otimes V_\xi = \mathcal{S}_+\otimes\mathcal{S}_-\otimes V_\xi$ contains the following direct summard

$$(\tau'_+\otimes\tau_+)\otimes((\tau'_-)^*\otimes\tau_-)\otimes\xi = \tau\otimes(\tau'_+\otimes(\tau'_-)^*\otimes\xi)$$

as a $\widetilde{K}\times\widetilde{G}'$-module. We now apply the reasoning in the previous paragraph and take matrix coefficients of the various copies of $(\tau'_+)^*$ in $(m\tau'_+)^* \subset (\tau'_-)^*\otimes\xi = (\tau'_-\otimes\pi')^*$, and we then conclude that τ appears in $\theta(\pi')$ at least $m = \dim \mathrm{Hom}_{\widetilde{G}'}(\tau'_+, \tau'_-\otimes\pi')$ times.

On the other hand, we know from Proposition 3.3 that the multiplicity of τ in $\Omega(\pi')$ is equal to $\dim \mathrm{Hom}_{\mathfrak{g}',\widetilde{K}'}(\tau'_+\otimes(\tau'_-)^*, \pi')$. Since $\tau'_-\otimes\pi'$ is a direct sum of irreducible holomorphic discrete series representations, we have

$$\operatorname{Hom}_{\mathfrak{g}',\widetilde{K}'}(\tau'_+ \otimes (\tau'_-)^*, \pi') = \operatorname{Hom}_{\mathfrak{g}',\widetilde{K}'}(\tau'_+, \tau'_- \otimes \pi') = \operatorname{Hom}_{\widetilde{G}'}(\tau'_+, \tau'_- \otimes \pi').$$

See [24], Corollary 4.5 for the first equality.

We thus conclude that $\tau = \tau_+ \boxtimes \tau_-$ will appear in $\theta(\pi')$ exactly $\dim \operatorname{Hom}_{\widetilde{G}'}(\tau'_+, \tau'_- \otimes \pi')$ times. We note that in our previous notations, $\tau_+ = \sigma^{\tilde{K}^+}(\mu)$, $\tau_- = \sigma^{\tilde{K}^-}(\nu)^*$, and $\tau'_+ = L(\mu)$, $\tau'_- = L(\nu)$. Everything follows. □

We learned the following result first from H-Y. Loke [20].

Corollary 3.6. *Let (G, G') be a reductive dual pair in the stable range with G' the smaller member. Suppose that π' is a genuine unitary lowest weight representation of $\widetilde{G}'$ (or contragradient of such a representation), then its maximal quotient $\Omega(\pi')$ is irreducible, i.e., $\Omega(\pi') = \theta(\pi')$.*

Remark 3.7. Loke considers π' which comes from the compact dual pair correspondence of $(G(k), G')$ for some k. He obtained the above result by examining the relationship of $\Omega(\pi')$ with the maximal quotient of a character of $\widetilde{G}'$ with respect to the larger dual pair $(G(p, q + k), G')$ (cf. §2.3), and by using the result that the latter is irreducible [35] and unitary [17].

Following [24], we give a result on tensor product of a holomorphic discrete series representation with a unitary lowest weight representation.

We introduce one notation, which is not standard. For any finite dimensional unitary representation γ of $\widetilde{K}'$ with the decomposition

$$\gamma \simeq \sum_{\lambda} n(\lambda) \tau^{\tilde{K}'}(\lambda),$$

define $L(\gamma) = \sum_\lambda n(\lambda) L(\lambda)$.

Proposition 3.8. *Let $L(\mu)$ (resp. ξ) be a holomorphic discrete series representation (resp. a unitary lowest weight representation) of $\widetilde{G}'$. Suppose that ξ has the following $\widetilde{K}'$-type decomposition:*

$$\xi\big|_{\widetilde{K}'} \simeq \sum_{\tau \in \operatorname{Irr}(\widetilde{K}')} m(\tau) \tau,$$

where $m(\tau)$ is the multiplicity of τ. Then

$$L(\mu) \otimes \xi \simeq \sum_{\tau \in \operatorname{Irr}(\widetilde{K}')} m(\tau) L(\tau^{\tilde{K}'}(\mu) \otimes \tau). \tag{3.5}$$

Proof. Since $L(\mu)$ is a holomorphic discrete series with the minimal $\widetilde{K}'$-type $\tau^{\tilde{K}'}(\mu)$, and ξ is a unitary lowest weight representation, we see that for any $\widetilde{K}'$-type τ of ξ, $L(\tau^{\tilde{K}'}(\mu) \otimes \tau)$

is a direct sum of holomorphic discrete series representation. Thus we have

$$L(\mu)\big|_{\widetilde{K}'} \simeq \tau^{\widetilde{K}'}(\mu) \otimes S(\mathfrak{s}'_+),$$

and

$$L(\tau^{\widetilde{K}'}(\mu) \otimes \tau)\big|_{\widetilde{K}'} \simeq (\tau^{\widetilde{K}'}(\mu) \otimes \tau) \otimes S(\mathfrak{s}'_+).$$

Therefore the left and right hand side of (3.5) are isomorphic as $\widetilde{K}'$-modules.

On the other hand, $L(\mu) \otimes \xi$ is the direct sum of holomorphic discrete representations of $\widetilde{G}'$, and each $\widetilde{K}'$-type occurs with finite multiplicity. General theory for holomorphic representations tells us that their $\widetilde{G}'$-module decompositions (in the Grothendieck group) are determined by the weight space decompositions with respect to the compact Cartan subgroup $\widetilde{T}' \subseteq \widetilde{K}'$. We thus conclude that the isomorphism of the left and right hand side of (3.5) as $\widetilde{K}'$-modules in fact induces an isomorphism as $\widetilde{G}'$-modules. This proves the proposition. □

Theorem 3.9. *Let (G, G') be a reductive dual pair in the stable range with G' the smaller member. Suppose*

I: π' is a genuine unitary lowest weight representation of $\widetilde{G}'$, or

II: $\pi' = \xi^*$, the contragradient of a genuine unitary lowest weight representation ξ, then the $\widetilde{K}$-type formula of $\pi = \theta(\pi')$ is given by

$$\theta(\pi')\big|_{\widetilde{K}} \simeq \sum_{\substack{\mu\in\Lambda_p(\widetilde{K}')\\ \nu\in\Lambda_q(\widetilde{K}')}}^{\oplus} \dim \operatorname{Hom}_{\widetilde{K}'}(\tau^{\widetilde{K}'}(\mu) \otimes \tau^{\widetilde{K}'}(\nu)^*, \pi'\big|_{\widetilde{K}'})\big(\sigma^{\widetilde{K}^+}(\mu) \boxtimes \sigma^{\widetilde{K}^-}(\nu)^*\big)$$

in Case I, and

$$\theta(\pi')\big|_{\widetilde{K}} \simeq \sum_{\substack{\mu\in\Lambda_p(\widetilde{K}')\\ \nu\in\Lambda_q(\widetilde{K}')}}^{\oplus} \dim \operatorname{Hom}_{\widetilde{K}'}(\tau^{\widetilde{K}'}(\mu) \otimes \xi\big|_{\widetilde{K}'}, \tau^{\widetilde{K}'}(\nu))\big(\sigma^{\widetilde{K}^+}(\mu) \boxtimes \sigma^{\widetilde{K}^-}(\nu)^*\big)$$

in Case II.

Proof. This follows from Propositions 3.5 and 3.8. □

Remark 3.10. In view of the well-known fact that a unitary lowest weight (or highest weight) representation π' of $\widetilde{G}'$ is completely determined by its $\widetilde{K}'$-structure, the above result is not surprising. See [23] for an explicit decomposition formula of $\pi'\big|_{\widetilde{K}'}$.

3.3. **Good filtration and associated variety.** In this subsection, we discuss on the natural filtration on lifted modules.

Let $\pi' \in \operatorname{Irr}(\mathfrak{g}', \widetilde{K}')$ be in the domain of the theta lifting, namely π' can be realized as quotient by a $(\mathfrak{g}', \widetilde{K}')$-invariant subspace of Ω. Write $\pi = \theta(\pi')$, the theta lift of π' which is an irreducible representation of $\widetilde{G} = G(p,q)\tilde{}$. By definition of the theta correspondence, there is a (unique) surjective $(\mathfrak{g} \oplus \mathfrak{g}', K \times K')$ morphism

$$\Omega \longrightarrow \pi \otimes \pi'.$$

Denote the kernel by N, so that we have $\Omega/N \simeq \pi \otimes \pi'$. Through Fock model, we identify the representation space of Ω with the space of polynomials $\mathbb{C}[W]$. Then there is a natural filtration by degree on $\mathbb{C}[W]$. We denote it by

$$F_j(\Omega) = \sum_{i=0}^{j}{}^{\oplus} \mathbb{C}[W]_i, \qquad F_j(N) = N \cap F_j(\Omega). \tag{3.6}$$

Then, $V = \pi \otimes \pi'$ inherits an increasing filtration

$$F_j(V) = F_j(\Omega)/F_j(N) \hookrightarrow \Omega/N = V.$$

For this filtration, the Lie algebra action increases the degree by two (as opposed to one). We shall thus do some adjustment.

We adopt the following notations. For $\sigma \in \operatorname{Irr}(\widetilde{K})$, let $\deg \sigma$ be the smallest possible degree of non-zero polynomials occuring in the σ-isotypic component of $\mathbb{C}[W]$, and $\mathcal{R}(\widetilde{K}, \Omega/N)$ the set of equivalent classes of irreducible representations of $\widetilde{K}$ which occur in Ω/N. Similar notations apply to $\widetilde{K}'$. According to [11], there is an integer j_0 (which depend on N) with the following properties:

$$j_0 = \min\{\deg \sigma : \ \sigma \in \mathcal{R}(\widetilde{K}, \Omega/N)\} = \min\{\deg \sigma' : \ \sigma' \in \mathcal{R}(\widetilde{K}', \Omega/N)\}, \tag{3.7}$$

and there is a one-to-one correspondence between those $\sigma \in \mathcal{R}(\widetilde{K}, \Omega/N)$ and $\sigma' \in \mathcal{R}(\widetilde{K}', \Omega/N)$ with this minimal degree. More precisely take any $\sigma \in \mathcal{R}(\widetilde{K}, \Omega/N)$ with degree j_0, then σ occurs in $\mathcal{H}_{\widetilde{K},\widetilde{K}'} = \mathcal{H}(\widetilde{K}) \cap \mathcal{H}(\widetilde{K}')$, the space of joint harmonics. Let σ' be the $\widetilde{K}'$-type which corresponds to σ in $\mathcal{H}_{\widetilde{K},\widetilde{K}'}$, then $\sigma' \in \mathcal{R}(\widetilde{K}', \Omega/N)$ and is of the minimal degree j_0. We fix such a pair (σ, σ') of $\widetilde{K}$ and $\widetilde{K}'$-types of minimal degree j_0. Choose a new filtration of $V = \pi \otimes \pi'$ by

$$G_j(V) = F_{2j+j_0}(V) \qquad (j \geq 0). \tag{3.8}$$

Since σ' is contained in π' with multiplicity one, we have $\pi \simeq ((\sigma')^* \otimes V)^{\widetilde{K}'}$ as a $(\mathfrak{g}, \widetilde{K})$-module.

Lemma 3.11. *The filtration of $V = \pi \otimes \pi'$ induces a good filtration of π by taking*

$$G_j(\pi) \stackrel{\text{def}}{=} ((\sigma')^* \otimes G_j(V))^{\widetilde{K}'} \qquad (j \geq 0). \tag{3.9}$$

Proof. Let us denote the graded module associated with the grading $G_j(\pi)$ by

$$\operatorname{gr} \pi = \sum_{j \geq 0}{}^{\oplus} \operatorname{gr}_j \pi, \qquad \operatorname{gr}_j \pi = G_j(\pi)/G_{j-1}(\pi).$$

Since the filtration is compatible with the Lie algebra action, it is enough to verify that $S(\mathfrak{g})\operatorname{gr}_0\pi = \operatorname{gr}\pi$ (in fact, it is sufficient to prove that $\operatorname{gr}\pi$ is finitely generated over the symmetric algebra $S(\mathfrak{g})$).

Note that $G_j(\pi) = ((\sigma')^* \otimes G_j(V))^{\widetilde{K}'} = ((\sigma')^* \otimes G_j(V_{\sigma'}))^{\widetilde{K}'}$, where $V_{\sigma'}$ is σ'-isotypic component of V. Now by the standard result of Howe [11],

$$\mathbb{C}[W]_{\sigma'} = \mathcal{U}(\mathfrak{m}^{(2,0)})\mathcal{H}_{\sigma,\sigma'},$$

where $\mathfrak{m}^{(2,0)}$ consists of those operators in the complexified Lie algebra $\mathfrak{m}_{\mathbb{C}}$ (of M) which raise degrees (by 2), and $\mathcal{U}(\mathfrak{m}^{(2,0)})$ is its universal enveloping algebra. Recall that M is the full commutant of K' in $\mathbb{G}$ (the ambient symplectic group). Thus we have $V_{\sigma'} = \mathcal{U}(\mathfrak{m}^{(2,0)})\mathcal{H}_{\sigma,\sigma'} \pmod N$.

From [11], Fact 3 (on page 540), we have

$$\mathfrak{m}^{(2,0)} \oplus \mathfrak{m}^{(0,2)} = \mathfrak{s} \oplus \mathfrak{m}^{(0,2)} = \mathfrak{m}^{(2,0)} \oplus \mathfrak{s}.$$

Therefore we have

$$\operatorname{gr} V_{\sigma'} = \operatorname{gr}\big(\mathcal{U}(\mathfrak{m}^{(2,0)})\mathcal{H}_{\sigma,\sigma'}\big) = S(\mathfrak{g})\mathcal{H}_{\sigma,\sigma'},$$

where we identify $\mathcal{H}_{\sigma,\sigma'}$ with its image in Ω/N. Now it is easy to see

$$\begin{aligned} \operatorname{gr}\pi &= ((\sigma')^* \otimes \operatorname{gr} V_{\sigma'})^{\widetilde{K}'} = ((\sigma')^* \otimes S(\mathfrak{g})\mathcal{H}_{\sigma,\sigma'})^{\widetilde{K}'} \\ &= S(\mathfrak{g})((\sigma')^* \otimes \mathcal{H}_{\sigma,\sigma'})^{\widetilde{K}'} = S(\mathfrak{g})\operatorname{gr}_0\pi. \end{aligned}$$

□

Proposition 3.12. *Let $\pi = \theta(\pi')$ be the theta lift of π'. Assume that the associated variety $\mathcal{AV}(\pi')$ of π' is irreducible, hence the closure of a single nilpotent $K'_{\mathbb{C}}$-orbit $\mathbb{O}'$. Then the associated variety of π is contained in the closure of the theta lift $\mathbb{O} = \theta(\mathbb{O}')$.*

Remark 3.13. Note that in the above proposition, we do not need the assumption that π' is a unitary lowest weight module, or even π' to be unitary. We remark that Przebinda has proved a similar result about correspondence of complex associated varieties for π' unitary. See Theorem 7.9, [29].

Proof. Put $I' = \sqrt{\operatorname{Ann}\operatorname{gr}\pi'} = \mathbb{I}(\overline{\mathbb{O}'})$, and $I = \mathbb{I}(\overline{\mathbb{O}})$, where $\mathbb{I}(X)$ denotes the ideal of regular functions which vanish on X. We shall prove that $\mathbb{I}(\mathcal{AV}(\pi)) \supset I$. For this purpose, it is enough to show that $f(z) \in I$ implies that $f(z) \in \sqrt{\operatorname{Ann}\operatorname{gr}\pi}$.

Let us recall the moment maps φ and ψ (cf. §2.2). Then

$$f(z) \in I \iff f(\varphi(w)) \in \mathcal{I} \qquad (z \in \mathfrak{s}, w \in W),$$

where $\mathcal{I} = \mathbb{I}(\psi^{-1}(\overline{\mathbb{O}'})) = \mathbb{C}[W] \cdot \psi^*(I')$ is an ideal generated by $\psi^*(I')$ in $\mathbb{C}[W]$. Thus, there exist $h_i \in \mathbb{C}[W]$ and $g_i \in I'$ such that

$$f(\varphi(w)) = \varphi^*(f)(w) = \sum_i h_i(w)\psi^*(g_i)(w).$$

For $V = \Omega/N \simeq \pi \otimes \pi'$, we define the filtration $G_j(V)$ as in this section, and consider its associated graded module $\operatorname{gr} V$. Since $g_i \in I'$, a suitable power of g_i annihilates $\operatorname{gr} \pi'$. By Lemma 3.11, we can assume that $\operatorname{gr} \pi'$ comes from $\operatorname{gr} V$ taking σ-isotypic component (actually, the lemma is proved for π, but the role of π and π' is symmetric there). This means there is a sufficient large N_0 such that $\psi^*(g_i)^{N_0}$ annihilates $\operatorname{gr} V_\sigma$ for all i. Since the multiplication of $\psi^*(g_i)$ (on $\operatorname{gr} V$) is induced from the action of an element of $\mathcal{U}(\mathfrak{g}')$, it commutes with the action of $\mathfrak{g}$. This means that $\psi^*(g_i)^{N_0}$ annihilates the whole space $\operatorname{gr} \mathcal{U}(\mathfrak{g}) V_\sigma = \operatorname{gr} V$.

Thus, taking suitable power N_1, $\varphi^*(f)^{N_1}$ annihilates $\operatorname{gr} V$. By Lemma 3.11 again, taking σ'-isotypic component, we see $\varphi^*(f)^{N_1} \in \operatorname{Ann} \operatorname{gr} V_{\sigma'}$, which in turn means $f^{N_1} \in \operatorname{Ann} \operatorname{gr} \pi$ (we define $\operatorname{gr} \pi$ as in the proof of Lemma 3.11). This proves that $f(z)$ is in $\sqrt{\operatorname{Ann} \operatorname{gr} \pi}$. □

4. Poincaré series and associated cycles

4.1. **Canonical decreasing filtration.** Let π' be an admissible representation of $\widetilde{G'}$ and consider it as a $(\mathfrak{g}', \widetilde{K'})$-module. We also consider the Weil representation Ω as a Harish-Chandra module. In this subsection, we discuss the relationship between $\widetilde{K}$ types and the grading of $\pi = \theta(\pi')$ coming from the good filtration introduced in §3.3. As it turns out, this boils down to certain canonical decreasing filtration on $H = \operatorname{Hom}_{(\mathfrak{g}', \widetilde{K'})}(\Omega, \pi')_{\widetilde{K}}$.

Recall we have

$$\begin{aligned} H\big|_{\widetilde{K}} &= \operatorname{Hom}_{(\mathfrak{g}', \widetilde{K'})}(\Omega, \pi')_{\widetilde{K}} \\ &\simeq \sum^{\oplus}_{\substack{\mu \in \Lambda_p(\widetilde{K}') \\ \nu \in \Lambda_q(\widetilde{K}')}} \operatorname{Hom}_{\mathfrak{g}', \widetilde{K}'}((\sigma^{\widetilde{K}^+}(\mu) \boxtimes L(\mu)) \otimes (\sigma^{\widetilde{K}^-}(\nu)^* \boxtimes L(\nu)^*), \pi') \\ &\simeq \sum^{\oplus}_{\substack{\mu \in \Lambda_p(\widetilde{K}') \\ \nu \in \Lambda_q(\widetilde{K}')}} \operatorname{Hom}_{\mathfrak{g}', \widetilde{K}'}(L(\mu) \otimes L(\nu)^*, \pi') \otimes (\sigma^{\widetilde{K}^+}(\mu)^* \boxtimes \sigma^{\widetilde{K}^-}(\nu)). \end{aligned}$$

See §3.1 for the notations here.

The following result is already proved in §3.2, but we include the statement here as we shall need some explicit formulas later.

Lemma 4.1. *Assume that π' is the Harish-Chandra module of a unitary lowest weight representation. For $f \in \operatorname{Hom}_{(\mathfrak{g}', \widetilde{K'})}(L(\mu), L(\nu) \otimes \pi')$, define $\varphi(f)$ by*

$$\varphi(f) : L(\mu) \otimes L(\nu)^* \longrightarrow \pi', \qquad \varphi(f)(u \otimes v^*) = (v^* \otimes 1)(f(u)), \tag{4.1}$$

and $\Phi(f) = (P_\nu \otimes 1) \circ (f\big|_{\tau^{\widetilde{K}'}(\mu)})$:

$$\Phi(f) : \tau^{\widetilde{K}'}(\mu) \xrightarrow{\text{inclusion}} L(\mu) \xrightarrow{f} L(\nu) \otimes \pi' \xrightarrow{P_\nu \otimes 1} \tau^{\widetilde{K}'}(\nu) \otimes \pi', \tag{4.2}$$

where $P_\nu : L(\nu) \to \tau^{\widetilde{K}'}(\nu)$ *is the* $\widetilde{K'}$*-equivariant projection onto the lowest* $\widetilde{K}'$*-type* $\tau^{\widetilde{K}'}(\nu)$ *of* $L(\nu)$*. Then* φ *and* Φ *induce the isomorphisms:*

$$\begin{aligned}\mathrm{Hom}_{(\mathfrak{g}',\widetilde{K'})}(L(\mu)\otimes L(\nu)^*,\pi') &\simeq \mathrm{Hom}_{(\mathfrak{g}',\widetilde{K'})}(L(\mu),L(\nu)\otimes\pi')\\ &\simeq \mathrm{Hom}_{\widetilde{K'}}(\tau^{\widetilde{K}'}(\mu),\tau^{\widetilde{K}'}(\nu)\otimes\pi').\end{aligned}$$

Put

$$V=\Omega/N\simeq\rho\otimes\pi',$$

where $\rho=\Omega(\pi')$ is the maximal quotient. Recall that we have defined a natural ($\widetilde{K}$ -stable, increasing) good filtration on ρ in §3.3:

$$\rho_d=\left(V_{j_0+2d}\otimes(\sigma')^*\right)^{\widetilde{K'}}\qquad(d\ge 0),$$

where

$$\begin{aligned}&\sigma\otimes\sigma' : \text{ a joint harmonics of the lowest degree,}\\ &j_0=\deg\sigma=\deg\sigma',\\ &V_j=F_j(\Omega)/F_j(N)\quad\text{(see (3.6))}.\end{aligned}$$

Since $H=\rho^*$, this induces a canonical ($\widetilde{K}$ -stable) *decreasing* filtration $\{H^d\}$ on H given by

$$H^d=\{\tilde\varphi\in H\mid P_{\sigma'}\circ\tilde\varphi(F_{j_0+2d}(\Omega))=0\},\tag{4.3}$$

where $P_{\sigma'}:\pi'\to\sigma'$ is a $\widetilde{K'}$-equivariant projection.

Definition 4.2. For a space V with a decreasing filtration, we say that $\deg v\ge d$ if $v\in V$ belongs to the d-th filtered subspace.

Through the isomorphisms $\varphi(f)\leftrightarrow f\leftrightarrow\Phi(f)$ given in Lemma 4.1, we have the corresponding decreasing filtrations in appropriate spaces. We will determine the various filtrations.

Denote by $||\mu||=\deg\sigma^{\widetilde{K}^+}(\mu)$, $||\nu||=\deg\sigma^{\widetilde{K}^-}(\nu)^*$. Note that as usual, the degree of $\sigma^{\widetilde{K}^+}(\mu)$ is defined to be the lowest possible degree in (a Fock model of) Ω^+ in which $\sigma^{\widetilde{K}^+}(\mu)$ occurs, and likewise for the degree of $\sigma^{\widetilde{K}^-}(\nu)^*$.

Lemma 4.3. *Let* $f\in\mathrm{Hom}_{(\mathfrak{g}',\widetilde{K'})}(L(\mu),L(\nu)\otimes\pi')$*. Then* $\deg f\ge d$ *if and only if*

$$(1\otimes P_{\sigma'})\circ f(L(\mu)_i)\subset L(\nu)^{d+\frac12(j_0-||\mu||-||\nu||)-i}\otimes\sigma'\qquad(\forall i\ge 0),\tag{4.4}$$

where $\{L(\mu)_i\}$ *is the natural increasing good filtration of a holomorphic discrete series, and* $\{L(\nu)^j\}$ *is the decreasing filtration determined by the natural increasing good filtration* $\{(L(\nu)^*)_j\}$ *of an anti-holomorphic discrete series.*

Proof. Let $\tilde{\varphi} \in \operatorname{Hom}_{(\mathfrak{g}', \widetilde{K'})}\big((\sigma^{\tilde{K}^+}(\mu) \boxtimes L(\mu)) \otimes (\sigma^{\tilde{K}^-}(\nu)^* \boxtimes L(\nu)^*), \pi'\big) \subset H\big|_{\tilde{K}}$. Then $\deg \tilde{\varphi} \geq d$ if and only if

$$P_{\sigma'} \circ \tilde{\varphi}(F_i(\sigma^{\tilde{K}^+}(\mu) \boxtimes L(\mu)) \otimes F_j(\sigma^{\tilde{K}^-}(\nu)^* \boxtimes L(\nu)^*)) = 0, \qquad (\forall i + j \leq j_0 + 2d).$$

If in addition $\tilde{\varphi}$ is of the form

$$\tilde{\varphi} = \varphi \otimes (u_\mu \otimes v_\nu), \qquad \text{where } \begin{cases} u_\mu \in \sigma^{\tilde{K}^+}(\mu)^*, v_\nu \in \sigma^{\tilde{K}^-}(\nu), \\ \varphi \in \operatorname{Hom}_{(\mathfrak{g}', \widetilde{K'})}(L(\mu) \otimes L(\nu)^*, \pi'), \end{cases}$$

this is equivalent to saying that

$$P_{\sigma'} \circ \varphi(L(\mu)_i \otimes (L(\nu)^*)_j) = 0, \qquad ((||\mu|| + 2i) + (||\nu|| + 2j) \leq j_0 + 2d). \tag{4.5}$$

Note that

$$\sigma^{\tilde{K}^+}(\mu) \boxtimes L(\mu)_i = F_{||\mu||+2i}(\sigma^{\tilde{K}^+}(\mu) \boxtimes L(\mu)),$$
$$\sigma^{\tilde{K}^-}(\nu)^* \boxtimes (L(\nu)^*)_j = F_{||\nu||+2j}(\sigma^{\tilde{K}^-}(\nu)^* \boxtimes L(\nu)^*).$$

Let $\varphi = \varphi(f) \leftrightarrow f \in \operatorname{Hom}_{(\mathfrak{g}', \widetilde{K'})}(L(\mu), L(\nu) \otimes \pi')$ be the correspondence given in Lemma 4.1. From (4.5), we see that $\deg f \geq d$ if and only if

$$(1 \otimes P_{\sigma'}) \circ f(L(\mu)_i) \subset L(\nu)^j \otimes \sigma', \tag{4.6}$$

for any i, j satisfying $j \leq d + \frac{1}{2}(j_0 - ||\mu|| - ||\nu||) - i$. Since the filtration $\{L(\nu)^j\}$ is decreasing, the inclusion in (4.6) is equivalent to

$$(1 \otimes P_{\sigma'}) \circ f(L(\mu)_i) \subset L(\nu)^{d + \frac{1}{2}(j_0 - ||\mu|| - ||\nu||) - i} \otimes \sigma' \qquad (\forall i \geq 0).$$

□

Finally, let $f \leftrightarrow \Phi = \Phi(f)$ be the second isomorphic correspondence in Lemma 4.1. We want to determine the degree of Φ (in the decreasing filtration). Recall that

$$\begin{array}{ccc} \operatorname{Hom}_{(\mathfrak{g}', \widetilde{K'})}(L(\mu), L(\nu) \otimes \pi') & \simeq & \operatorname{Hom}_{\widetilde{K'}}(\tau^{\tilde{K}'}(\mu), \tau^{\tilde{K}'}(\nu) \otimes \pi') \\ \Cup & & \Cup \\ f & \leftrightarrow & \Phi \end{array}.$$

Lemma 4.4. $\deg \Phi \geq d$ *if and only if*

$$\Phi(\tau^{\tilde{K}'}(\mu)) \subset \tau^{\tilde{K}'}(\nu) \otimes (\pi')^{d + \frac{1}{2}(j_0 - ||\mu|| - ||\nu||)},$$

where $\{(\pi')^j\}$ *is the decreasing filtration determined by the natural* (*increasing*) *good filtration of the unitary highest weight module* $(\pi')^*$.

Proof. By applying the Lie algebra action of $\mathfrak{s}'$, we see that the inclusion in (4.4) is in fact equivilent to

$$(1 \otimes P_{\sigma'}) \circ f(\tau^{\tilde{K}'}(\mu)) \subset L(\nu)^{d + \frac{1}{2}(j_0 - ||\mu|| - ||\nu||)} \otimes \sigma'. \tag{4.7}$$

We claim that the containment

$$(1 \otimes P_{\sigma'}) \circ f(\tau^{\widetilde{K}'}(\mu)) \subset L(\nu)^j \otimes \sigma'$$

holds if and only if

$$(P_\nu \otimes 1) \circ f(\tau^{\widetilde{K}'}(\mu)) \subset \tau^{\widetilde{K}'}(\nu) \otimes (\pi')^j.$$

To see this, we note that the first containment implies a linear inequality of the exponents of the central character of $\widetilde{K}'$ on the two sides. The same inequality ensures the second containment, since the central character encodes the degree of the natural filtration in π'.

In view of (4.7), the lemma follows. □

4.2. **Associated cycles of theta lifts.** From the decreasing filtration on $H = \rho^*$, we get an increasing filtration for $\rho = \Omega(\pi')$, and hence obtain the graded module as follows:

$$\begin{aligned} \mathrm{gr}(\rho) &= \sum_{d\geq 0}^{\oplus} \mathrm{gr}_d(\rho), \\ \mathrm{gr}_d(\rho) &= \sum_{\substack{\mu\in\Lambda_p(\widetilde{K}') \\ \nu\in\Lambda_q(\widetilde{K}')}}^{\oplus} \mathrm{Hom}_{\widetilde{K'}}\big(\tau^{\widetilde{K}'}(\mu), \tau^{\widetilde{K}'}(\nu) \otimes \mathrm{gr}_{d+\frac{1}{2}(j_0-||\mu||-||\nu||)}(\pi')\big) \otimes (\sigma^{\widetilde{K}^+}(\mu) \boxtimes \sigma^{\widetilde{K}^-}(\nu)^*). \end{aligned} \tag{4.8}$$

This in fact defines the Poincaré series for $\pi = \theta(\pi')$, as the maximal quotient $\rho = \Omega(\pi')$ is already irreducible in the present case. See Corollary 3.6.

Remark 4.5. By comparing central characters (of $\widetilde{K}'$), it is easy to see that summation in $\mathrm{gr}_d(\rho)$ is actually finite.

Let σ be an irreducible (finite-dimensional) representation of $G(k, \mathbb{C})$ which appears in $\mathbb{C}[W_k]$. We also take π' to be a unitary lowest weight representation of $\widetilde{G}'$ which comes from the compact dual pair correspondence of $(G(k), G')$. To fix notations, we assume π' and σ are related by the formula:

$$\pi'\big|_{\widetilde{K'}} \simeq \chi_k \otimes L[\sigma], \qquad L[\sigma] = \big(\sigma \otimes \mathbb{C}[W_k]\big)^{G(k,\mathbb{C})}, \tag{4.9}$$

where χ_k is certain character of $\widetilde{K'}$.

Recall from §2.4 the lifted module $M[\sigma]$ of $L[\sigma]$:

$$M[\sigma] = \big(\sigma \otimes \mathbb{C}[X_{p,q+k}]\big)^{G(k,\mathbb{C})}.$$

Its Poincaré series is given in (2.15).

Lemma 4.6. *Let π' be a unitary lowest weight module, and let σ and k be related to π' through the formula* (4.9). *Assume that π' is genuine with respect to the dual pair $(G(p, q), G')$, and let $\pi = \theta(\pi')$ be the theta lift. Let j_0 be the lowest degree of the joint* *harmonics for* $\pi \otimes \pi'$ *(cf.* (3.7)*), and* $j_1 = \deg \sigma^*$ *(cf.* (2.14)*). Then the following inequalities hold for the Poincaré series:*

Case $\mathbb{R}$*: denote* $\kappa_0 = (p-q-k)/2 \in \mathbb{Z}$*. Then we have*

$$t^{(-n|\kappa_0|+j_0-j_1)/2} P(\pi;t) \geq P(M[\sigma];t) \geq t^{(n|\kappa_0|+j_0-j_1)/2} P(\pi;t), \qquad (t>0),$$

Case $\mathbb{C}$*: denote* $\kappa_0 = (p-q-k)/2 \in \mathbb{Z}$*. Then we have*

$$t^{(-(n+m)|\kappa_0|+j_0-j_1)/2} P(\pi;t) \geq P(M[\sigma];t) \geq t^{((m+n)|\kappa_0|+j_0-j_1)/2} P(\pi;t), \qquad (t>0).$$

Case $\mathbb{H}$*: denote* $\kappa_0 = p-q-k$*. Then we have*

$$t^{(-n|\kappa_0|+j_0-j_1)/2} P(\pi;t) \geq P(M[\sigma];t) \geq t^{(n|\kappa_0|+j_0-j_1)/2} P(\pi;t), \qquad (t>0).$$

Proof. We shall only give the proof for Case $\mathbb{R}$. The other two cases are entirely similar. Also we assume that $\kappa_0 \geq 0$ since the case $\kappa_0 < 0$ can be treated in a similar way.

Now $\mu \in \Lambda_p(\widetilde{K'})$ and $\nu \in \Lambda_q(\widetilde{K'})$ if and only if $\mu = \alpha + \frac{p}{2}\mathbb{I}_n$ and $\nu = \beta + \frac{q}{2}\mathbb{I}_n$, where $\alpha, \beta \in \mathcal{P}_n$ (the set of partitions of n), and $\mathbb{I}_n = (1, \ldots, 1) \in \mathcal{P}_n$. Also we have the degrees $||\mu|| = |\alpha| = \alpha_1 + \cdots + \alpha_n$, and $||\nu|| = |\beta| = \beta_1 + \cdots + \beta_n$. For $\alpha \in \mathcal{P}_n$, denote by $\tau_\alpha^{(n)}$ (resp. $\sigma_\alpha^{(p)}$) the irreducible finite dimensional representation of GL_n (resp. $O(p,\mathbb{C})$) with highest weight α. Then we have $\sigma^{\tilde{\kappa}^+}(\mu) = \sigma_\alpha^{(p)}$, $\sigma^{\tilde{\kappa}^-}(\nu)^* = \sigma_\beta^{(q)}$.

Note that

$$\pi'\big|_{\widetilde{K'}} \simeq \det_n^{k/2} \otimes L[\sigma], \qquad L[\sigma] = \big(\sigma \otimes \mathbb{C}[M_{k,n}]\big)^{O(k,\mathbb{C})},$$

where $K' = U(n)$. Thus we can rewrite

$$\mathrm{Hom}_{\widetilde{K'}}\big(\tau^{(n)}_{\alpha+\frac{p}{2}\mathbb{I}_n}, \tau^{(n)}_{\beta+\frac{q}{2}\mathbb{I}_n} \otimes \mathrm{gr}_j(\pi')\big) \simeq \mathrm{Hom}_{\widetilde{K'}}\big(\det_n^{(p-q-k)/2} \otimes \tau_\alpha^{(n)}, \tau_\beta^{(n)} \otimes \mathrm{gr}_j L[\sigma]\big).$$

Therefore the Poincaré series of $\mathrm{gr}\,\pi = \mathrm{gr}\,\rho$ becomes

$$\begin{aligned} P(\pi;t) &= \sum_{d\geq 0} \dim \mathrm{gr}_d(\pi) \cdot t^d \\ &= \sum_{d\geq 0} \sum_{\alpha,\beta\in\mathcal{P}_n} \dim \mathrm{Hom}_{\widetilde{K'}}\big(\det_n^{(p-q-k)/2} \otimes \tau_\alpha^{(n)}, \tau_\beta^{(n)} \otimes \mathrm{gr}_{d+\frac{1}{2}(j_0-|\alpha|-|\beta|)} L[\sigma]\big) \times \\ &\qquad\qquad \dim\big(\sigma_\alpha^{(p)} \boxtimes \sigma_\beta^{(q)}\big) \cdot t^d \\ &= t^{-j_0/2} \cdot \sum_{j\geq 0} \sum_{\alpha,\beta\in\mathcal{P}_n} \dim \mathrm{Hom}_{\widetilde{K'}}\big(\det_n^{(p-q-k)/2} \otimes \tau_\alpha^{(n)}, \tau_\beta^{(n)} \otimes \mathrm{gr}_j L[\sigma]\big) \times \\ &\qquad\qquad \dim\big(\sigma_\alpha^{(p)} \boxtimes \sigma_\beta^{(q)}\big) \cdot t^{j+\frac{1}{2}(|\alpha|+|\beta|)}. \end{aligned}$$

We put $B(\alpha, \beta; j) = \dim \operatorname{Hom}_{GL_n}\big(\tau_\alpha^{(n)} \otimes \tau_\beta^{(n)*}, \operatorname{gr}_j L[\sigma]\big)$. Then, we can express

$$\begin{aligned}
t^{j_0/2} P(\pi; t) &= \sum_{j \geq 0} \sum_{\alpha, \beta \in \mathcal{P}_n} B(\alpha + \kappa_0 \mathbb{I}_n, \beta; j) \dim\big(\sigma_\alpha^{(p)} \boxtimes \sigma_\beta^{(q)}\big) \cdot t^{j + (|\alpha| + |\beta|)/2} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\text{put } \alpha' = \alpha + \kappa_0 \mathbb{I}_n) \\
&= \sum_{j \geq 0} \sum_{\substack{\alpha' \in \mathcal{P}_n + \kappa_0 \mathbb{I}_n \\ \beta \in \mathcal{P}_n}} B(\alpha', \beta; j) \dim\big(\sigma_{\alpha' - \kappa_0 \mathbb{I}_n}^{(p)} \boxtimes \sigma_\beta^{(q)}\big) \cdot t^{j + (|\alpha'| + |\beta| - n\kappa_0)/2} \\
&\leq \sum_{j \geq 0} \sum_{\alpha', \beta \in \mathcal{P}_n} B(\alpha', \beta; j) \dim\big(\sigma_{\alpha'}^{(p)} \boxtimes \sigma_\beta^{(q)}\big) \cdot t^{j + (|\alpha'| + |\beta| - n\kappa_0)/2} \\
&= P(M[\sigma]; t) \cdot t^{(j_1 - n\kappa_0)/2}.
\end{aligned}$$

Here we have used the inequality $\dim \sigma_{\alpha' - \kappa_0 \mathbb{I}_n}^{(p)} \leq \dim \sigma_{\alpha'}^{(p)}$. On the other hand, we have

$$\begin{aligned}
t^{j_0/2} P(\pi; t) &= \sum_{j \geq 0} \sum_{\alpha, \beta \in \mathcal{P}_n} B(\alpha, \beta - \kappa_0 \mathbb{I}_n; j) \dim\big(\sigma_\alpha^{(p)} \boxtimes \sigma_\beta^{(q)}\big) \cdot t^{j + (|\alpha| + |\beta|)/2} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\text{put } \beta' = \beta - \kappa_0 \mathbb{I}_n) \\
&= \sum_{j \geq 0} \sum_{\substack{\alpha \in \mathcal{P}_n \\ \beta' \in \mathcal{P}_n - \kappa_0 \mathbb{I}_n}} B(\alpha, \beta'; j) \dim\big(\sigma_\alpha^{(p)} \boxtimes \sigma_{\beta' + \kappa_0 \mathbb{I}_n}^{(q)}\big) \cdot t^{j + (|\alpha| + |\beta'| + n\kappa_0)/2} \\
&\geq \sum_{j \geq 0} \sum_{\alpha, \beta' \in \mathcal{P}_n} B(\alpha, \beta'; j) \dim\big(\sigma_\alpha^{(p)} \boxtimes \sigma_{\beta'}^{(q)}\big) \cdot t^{j + (|\alpha| + |\beta'| + n\kappa_0)/2} \\
&= P(M[\sigma]; t) \cdot t^{(j_1 + n\kappa_0)/2}.
\end{aligned}$$

These two formulas prove the lemma. □

Theorem 4.7. *Let π' be a unitary lowest weight representation of $\widetilde{G}'$ arising from the compact dual pair correspondence of $(G(k), G')$, for some k. Assume that π is genuine with respect to the dual pair $(G(p,q), G')$, and let $\pi = \theta(\pi')$ be the theta lift of π'. If $\mathcal{AC}(\pi') = m_{\pi'}[\overline{\mathbb{O}'^{\mathrm{hol}}}]$, then we have*

$$\mathcal{AC}(\pi) = m_{\pi'}[\overline{\mathbb{O}^{\mathrm{hol}}}], \qquad \textit{where} \qquad \mathbb{O}^{\mathrm{hol}} = \theta(\mathbb{O}'^{\mathrm{hol}}).$$

Moreover if σ is related to π' through the formula (4.9)*, this multiplicity is given by the following formulas:*

Case $\mathbb{R}$:

$$m_{\pi'} = \begin{cases} \dim \sigma & (k \leq n), \\ \dim \sigma^{O(k-n, \mathbb{C})} & (k > n). \end{cases}$$

Case $\mathbb{C}$*:*

$$m_{\pi'} = \begin{cases} \dim \sigma & (0 \le k \le \min\{m, n\}), \\ \dim \sigma^{GL_{k-n}} & (k > m = n), \\ \dim\big(\sigma \otimes \mathbb{C}[M_{k-n,m-n}]\big)^{GL_{k-n}} & (k \ge m > n). \end{cases}$$

Case $\mathbb{H}$*:*

$$m_{\pi'} = \begin{cases} \dim \sigma & (2k \le n), \\ \dim \sigma^{Sp(2(k-l),\mathbb{C})} & (2k > n;\ n = 2l), \\ \dim\big(\sigma \otimes \mathbb{C}[\mathbb{C}^{2(k-l)}]\big)^{Sp(2(k-l),\mathbb{C})} & (2k > n;\ n = 2l+1). \end{cases}$$

Proof. Since $\pi'\big|_{\widetilde{K'}} \simeq \chi_k \otimes L[\sigma]$, we see that $m_{\pi'}$ is equal to the multiplicity $\operatorname{rank}_{\overline{\mathbb{O}'^{\text{hol}}}} L[\sigma]$. This is computed in [23] (see also [34]), and it is as given in the current theorem.

By Lemma 4.6 we have

$$\lim_{t \uparrow 1} \frac{P(\pi; t)}{P(M[\sigma]; t)} = 1. \tag{4.10}$$

This means that the dimension of the support of a finitely generated $\mathbb{C}[\mathfrak{s}]$ module $\operatorname{gr} \pi$ is equal to the dimension of $\overline{\mathbb{O}^{\text{hol}}}$, which is the support of $M[\sigma]$. We know that the associated variety $\mathcal{AV}(\pi)$ of π is contained in $\overline{\mathbb{O}^{\text{hol}}}$ by Proposition 3.12. Since they have the same dimension, we conclude that $\mathcal{AV}(\pi) = \overline{\mathbb{O}^{\text{hol}}}$. Now, the same formula (4.10) implies that $\operatorname{gr} \pi$ and $M[\sigma]$ have the same multiplicity m_π along their common support $\overline{\mathbb{O}^{\text{hol}}}$. This multiplicity is given in Theorem 2.3, and it is the same as $m_{\pi'}$. □

Remark 4.8. Note that $M[\sigma]$ is lifted from $L[\sigma]$ (Proposition 1.10). It is easily checked that the hypothesis in Proposition 1.9 is satisfied when G'/K' is of tube type. Thus in this case the equality $\operatorname{rank}_{\overline{\mathbb{O}^{\text{hol}}}} M[\sigma] = \operatorname{rank}_{\overline{\mathbb{O}'^{\text{hol}}}} L[\sigma]$ follows directly from Proposition 1.9. For the non-tube case, one may also prove the preservation of multiplicty along this line, with a bit more work.

Corollary 4.9. *The Gelfand-Kirillov dimension of* π *is* $\operatorname{GKdim} \pi = \dim \overline{\mathbb{O}_k^{\text{hol}}}$*, which is equal to the following numbers for Cases* $\mathbb{R}$,$\mathbb{C}$,$\mathbb{H}$*, respectively:*

$$n(p + q - 2n + r - 1) - \frac{1}{2} r(r - 1), \qquad r = \min\{k, n\}.$$

$$(m + n)(p + q - m - n + r) - r^2, \qquad r = \min\{k, m, n\}.$$

$$n(2p + 2q - 2n + 2r + 1) - r(2r + 1), \qquad r = \min\{k, [n/2]\}.$$

As remarked in the Introduction, if π' is a holomorphic discrete series representation, the multiplicity $m_{\pi'}$ may also be given by the dimension of its lowest $\widetilde{K}'$-type. This of course implies an equality of dimension of certain space of fixed vectors of σ with the dimension of the lowest $\widetilde{K}'$-type of π'. We prove a proposition in that direction.

Recall the complex vector space W_k canonically associated to the compact dual pair $(G(k), G')$. As usual, let $\mathcal{H}_k$ denote the space of $G(k,\mathbb{C})$-harmonic polynomials on W_k, which has a dual pair correspondence as a $G(k,\mathbb{C}) \times K'_{\mathbb{C}}$-module (see §2.5). Write

$$\mathcal{H}_k\big|_{G(k,\mathbb{C})\times K'_{\mathbb{C}}} \simeq \sum_{\tau}^{\oplus} \sigma_k(\tau) \boxtimes \tau. \tag{4.11}$$

Recall also the see-saw pair (cf. §2.1):

$$\begin{array}{cc} L(k) & G' \\ \cup & \cup \\ G(k) & K' \end{array}$$

Let

$$\mathbb{C}[W_k]\big|_{L(k,\mathbb{C})\times K'_{\mathbb{C}}} \simeq \sum_{\tau}^{\oplus} \rho_k(\tau) \boxtimes \tau \tag{4.12}$$

be the corresponding decomposition. Note that $L(k,\mathbb{C}) \supset G(k,\mathbb{C})$ is symmetric pair.

Proposition 4.10. *Assume that the following conditions hold:*

$$\begin{cases} k \geq 2n, & \textit{if } G' = Sp(2n,\mathbb{R}), \\ k \geq m+n, & \textit{if } G' = U(m,n), \\ k \geq n, & \textit{if } G' = O^*(2n). \end{cases} \tag{4.13}$$

Denote l the real rank of G', as before. Then there is an isomorphism of $G(l,\mathbb{C})$-modules

$$\sigma_k(\tau)^{G(k-l,\mathbb{C})}\big|_{G(l,\mathbb{C})} \simeq \rho_l(\tau)\big|_{G(l,\mathbb{C})},$$

for any τ occurring in $\mathcal{H}_k$ (or equivalently $\mathbb{C}[W_k]$).

Proof. Under the given assumption on k, we have

$$\mathbb{C}[W_k] \simeq \mathcal{H}_k \otimes \mathbb{C}[W_k]^{G(k,\mathbb{C})} \simeq \mathcal{H}_k \otimes \mathbb{C}[\mathfrak{s}'] \qquad \text{(as } G(k,\mathbb{C}) \times K'_{\mathbb{C}}\text{-modules)}.$$

Taking the $G(k-l,\mathbb{C})$ invariants, we get

$$\mathbb{C}[W_k]^{G(k-l,\mathbb{C})} \simeq (\mathcal{H}_k)^{G(k-l,\mathbb{C})} \otimes \mathbb{C}[\mathfrak{s}'].$$

On the other hand, we have the decomposition $W_k = W_l \oplus W_{k-l}$ and so $\mathbb{C}[W_k] = \mathbb{C}[W_l] \otimes \mathbb{C}[W_{k-l}]$ as $G(l,\mathbb{C}) \times G(k-l,\mathbb{C}) \times K'_{\mathbb{C}}$-modules. Thus the space of $G(k-l,\mathbb{C})$ invariants of $\mathbb{C}[W_k]$ becomes

$$\mathbb{C}[W_k]^{G(k-l)} \simeq \mathbb{C}[W_l] \otimes \mathbb{C}[W_{k-l}]^{G(k-l)} \simeq \mathbb{C}[W_l] \otimes \mathbb{C}[\mathfrak{s}'].$$

Here we have used the isomorphism $\mathbb{C}[W_{k-l}]^{G(k-l)} \simeq \mathbb{C}[\mathfrak{s}']$, again by our assumption on k.

Thus we obtain the following isomorphism of $G(l,\mathbb{C}) \times K'_{\mathbb{C}}$-modules:

$$(\mathcal{H}_k)^{G(k-l,\mathbb{C})} \simeq \mathbb{C}[W_l]. \tag{4.14}$$

The desired isomorphism follows immediately. □

Remark 4.11. When G'/K' is of tube type, the above isomorphism yields especially pleasant formulas, which we list below. Note that they were proved by Knapp [14].

$$\big(\sigma_\eta^{(k)}\big)^{O(k-n,\mathbb{C})}\big|_{O(n,\mathbb{C})} \simeq \tau_\eta^{(n)}\big|_{O(n,\mathbb{C})} \qquad (k \geq 2n; \eta \in \mathcal{P}_n).$$

$$\big(\tau_{\alpha\odot\beta}^{(k)}\big)^{GL_{k-n}}\big|_{GL_n} \simeq \tau_\alpha^{(n)} \otimes \tau_\beta^{(n)*} \qquad (m = n, k \geq 2n; \alpha, \beta \in \mathcal{P}_n).$$

$$\big(\sigma_\eta^{(k)}\big)^{Sp(2(k-l),\mathbb{C})}\big|_{Sp(2l,\mathbb{C})} \simeq \tau_\eta^{(n)}\big|_{Sp(2l,\mathbb{C})} \qquad (n = 2l, k \geq n; \eta \in \mathcal{P}_n).$$

Here for $\eta \in \mathcal{P}_n$ (the set of partitions of n), $\tau_\eta^{(n)}$ denotes the irreducible finite dimensional representation of GL_n of highest weight η. We use the same notation $\sigma_\eta^{(k)}$ to denote a copy of irreducible finite dimensional representation of either $O(k,\mathbb{C})$ or $Sp(2k,\mathbb{C})$ with η as the highest weight. Finally $\alpha \odot \beta$ is defined to be $(\alpha_1, ..., \alpha_n, ..., -\beta_n, ..., -\beta_1)$.

5. Appendix: Computation of generic fibers

5.1. $O(p,q) \times Sp(2n,\mathbb{R})$ $(2n < p, q)$. We have the double fibration map

$$\begin{array}{ccc} & W = W_{p,q} = M_{p,n} \oplus M_{q,n} & \\ & {\scriptstyle\varphi} \swarrow \qquad \searrow {\scriptstyle\psi} & \\ \mathfrak{s} = M_{p,q} & & \operatorname{Sym}_n \oplus \operatorname{Sym}_n = \mathfrak{s}' \end{array}$$

where the moment maps φ and ψ are given by

$$(A,B) \in M_{p,n} \oplus M_{q,n}$$
$$\begin{cases} \varphi(A,B) = A\,{}^tB \in M_{p,q} = \mathfrak{s}, \\ \psi(A,B) = ({}^tAA, {}^tBB) \in \operatorname{Sym}_n \oplus \operatorname{Sym}_n = \mathfrak{s}'. \end{cases}$$

Take a holomorphic nilpotent orbit $\mathbb{O}'^{\mathrm{hol}}_k$ of rank k in $\mathfrak{s}'_- = \operatorname{Sym}_n$, where $0 \leq k \leq n$, and consider its lift $\mathbb{O}^{\mathrm{hol}}_k = \theta(\mathbb{O}'^{\mathrm{hol}}_k)$. The closure $\overline{\mathbb{O}'^{\mathrm{hol}}_k}$ is an affine quotient of $W_k = M_{k,n}$ by $O(k,\mathbb{C})$:

$$M_{k,n}/\!/O(k,\mathbb{C}) \simeq \begin{cases} \overline{\mathbb{O}'^{\mathrm{hol}}_k} & (0 \leq k \leq n), \\ \overline{\mathbb{O}'^{\mathrm{hol}}_n} & (k > n). \end{cases}$$

Here $O(k,\mathbb{C})$ acts on $M_{k,n}$ via the left multiplication of matrices. The quotient map $\zeta' : M_{k,n} \to \overline{\mathbb{O}'^{\text{hol}}_k} \subset \operatorname{Sym}_n$ is given by $\zeta'(C) = {}^tCC \quad (C \in M_{k,n})$.

The null cone is

$$\begin{aligned}\mathfrak{N}_{p,q+k} =& \psi^{-1}_{p,q+k}(0) = \mathfrak{N}^+_p \times \mathfrak{N}^-_{q+k} \\ =& \{(A,B,C) \mid {}^tAA = 0,\ {}^tBB + {}^tCC = 0\} \subset M_{p+(q+k),n}\end{aligned}$$

We have the following diagram:

$$\begin{array}{ccccc} (A\,{}^tB, A\,{}^tC) \in X_{p,q+k} & \xleftarrow{//GL_n} & (A,B,C) \in \mathfrak{N}_{p,q+k} & \xrightarrow{//O(p,\mathbb{C})\times O(q,\mathbb{C})} & M_{k,n} \ni C \\ \big\downarrow{\scriptstyle //O(k,\mathbb{C})} & & \big\downarrow{\scriptstyle //O(k,\mathbb{C})} & & \big\downarrow{\scriptstyle //O(k,\mathbb{C})} \\ A\,{}^tB \in \overline{\mathbb{O}^{\text{hol}}_r} & \xleftarrow{//GL_n} & (A,B) \in \Xi_r & \xrightarrow{//O(p,\mathbb{C})\times O(q,\mathbb{C})} & \overline{\mathbb{O}'^{\text{hol}}_r} \ni {}^tBB = -\,{}^tCC \end{array}$$

Here $\Xi_r = \psi^{-1}(\overline{\mathbb{O}'^{\text{hol}}_r})$, and $r = \min\{k,n\}$.

We consider the generic fiber of $\zeta : X_{p,q+k} \to \overline{\mathbb{O}^{\text{hol}}_k} \subset M_{p,q}$. The fiber of $Z \in \overline{\mathbb{O}^{\text{hol}}_k}$ in $\mathfrak{N}_{p,q+k}$ is

$$\{(A,B,C) \in \mathfrak{N}_{p,q+k} \mid Z = A\,{}^tB\}. \tag{5.1}$$

If $\operatorname{rank} Z = n$, it implies $\operatorname{rank} A = \operatorname{rank} B = n$ and one can easily deduce that

$$\{(A,B) \in \mathfrak{N}^+_p \times M_{q,n} \mid Z = A\,{}^tB, \operatorname{rank} A = \operatorname{rank} B = n\}$$

is a single GL_n-orbit. Now assume that $0 \le k \le n$. Note that $\operatorname{rank} Z\,{}^tZ \le \operatorname{rank} {}^tBB = \operatorname{rank}(-\,{}^tCC) \le k$. We assume that Z satisfies the generic condition $\operatorname{rank} Z\,{}^tZ = k$. Then we must have $\operatorname{rank} {}^tBB = k$. Thus for each fixed B, the set $\{C \in M_{k,n} \mid {}^tBB = -\,{}^tCC\}$ is a single $O(k,\mathbb{C})$-orbit. We have thus proved that the set (5.1) is a single $GL_n \times O(k,\mathbb{C})$-orbit, which is closed (since the map $\mathfrak{N}_{p,q+k} \to \overline{\mathbb{O}^{\text{hol}}_k}$ is an affine quotient map by $GL_n \times O(k,\mathbb{C})$).

The fiber $\zeta^{-1}(Z)$ is an $O(k,\mathbb{C})$-equivariant quotient of the closed subvariety (5.1) by the action of GL_n. Therefore, $\zeta^{-1}(Z)$ is a single $O(k,\mathbb{C})$-orbit, which is closed.

The case $k > n$ can be treated similarly.

We have in fact proved the following

Corollary 5.1. *Put $r = \min\{k,n\}$. If $Z \in \overline{\mathbb{O}^{\text{hol}}_k}$ satisfies $\operatorname{rank} Z = n, \operatorname{rank} Z\,{}^tZ = r$, then the fiber $\zeta^{-1}(Z)$ is a single $O(k,\mathbb{C})$-orbit. The stabilizer at $x \in \zeta^{-1}(Z)$ is conjugate to*

$$\begin{cases} \{1_k\} & (0 \le k \le n) \\ O(k-n,\mathbb{C}) & (k > n) \end{cases}$$

Remark 5.2. Clearly, $Z \in \overline{\mathbb{O}^{\text{hol}}_k}$ satisfies $\operatorname{rank} Z \le n$ and $\operatorname{rank} Z\,{}^tZ \le r$. Thus, the condition $\operatorname{rank} Z = n, \operatorname{rank} Z\,{}^tZ = r$ is generic (it is easy to see that the set satisfying the condition is non-empty).

5.2. $U(p,q) \times U(m,n) \quad (m+n \leq p,q)$**.** We have the double fibration map

$$\begin{array}{ccc} & W = M_{p+q,m+n} & \\ \varphi \swarrow & & \searrow \psi \\ \mathfrak{s} = M_{p,q} \oplus M_{q,p} & & M_{m,n} \oplus M_{n,m} = \mathfrak{s}' \end{array}$$

where the moment maps φ and ψ are given by

$$Z = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in M_{p+q,m+n}$$

$$\begin{cases} \varphi(Z) = (A\,{}^tC, D\,{}^tB) \in M_{p,q} \oplus M_{q,p} = \mathfrak{s}, \\ \psi(Z) = ({}^tAB, {}^tDC) \in M_{m,n} \oplus M_{n,m} = \mathfrak{s}'. \end{cases}$$

Take a holomorphic nilpotent orbit $\mathbb{O}'^{\mathrm{hol}}_k$ of rank k in $\mathfrak{s}'_- = M_{n,m}$, where $0 \leq k \leq \min\{m,n\}$, and consider its lift $\mathbb{O}^{\mathrm{hol}}_k = \theta(\mathbb{O}'^{\mathrm{hol}}_k)$. The closure $\overline{\mathbb{O}'^{\mathrm{hol}}_k}$ is an affine quotient of $W_k = M_{k,m+n}$ by GL_k:

$$M_{k,m+n}/\!/GL_k \simeq \begin{cases} \overline{\mathbb{O}'^{\mathrm{hol}}_k} & (0 \leq k \leq \min\{m,n\}), \\ \overline{\mathbb{O}'^{\mathrm{hol}}_n} & (n = \min\{m,n\} < k). \end{cases}$$

Here GL_k acts on $M_{k,m+n}$ via

$$g \cdot (E,F) = (gE, {}^tg^{-1}F), \qquad (g \in GL_k, E \in M_{k,m}, F \in M_{k,n}).$$

The quotient map $\zeta' : M_{k,m+n} \to \overline{\mathbb{O}'^{\mathrm{hol}}_k} \subset M_{n,m}$ is given by

$$\zeta'((E,F)) = {}^tFE \qquad ((E,F) \in M_{k,m+n}).$$

The null cone is

$$\begin{aligned} \mathfrak{N}_{p,q+k} =& \psi^{-1}_{p,q+k}(0) = \mathfrak{N}^+_p \times \mathfrak{N}^-_{q+k} \\ =& \left\{ Z = \begin{pmatrix} A & B \\ C & D \\ E & F \end{pmatrix} \mid {}^tAB = 0, {}^tDC + {}^tFE = 0 \right\} \subset M_{p+(q+k),m+n}. \end{aligned}$$

We have the following diagram:

$$\begin{array}{ccccc}
X_{p,q+k} & \xleftarrow{//GL_m\times GL_n} & \mathfrak{N}_{p,q+k} & \xrightarrow{//GL_p\times GL_q} & M_{k,m+n} \\
{\scriptstyle //GL_k}\big\downarrow & & {\scriptstyle //GL_k}\big\downarrow & & {\scriptstyle //GL_k}\big\downarrow \\
\overline{\mathbb{O}_r^{\mathrm{hol}}} & \xleftarrow{//GL_m\times GL_n} & \Xi_r & \xrightarrow{//GL_p\times GL_q} & \overline{\mathbb{O}'^{\mathrm{hol}}_r}
\end{array}$$

$$\begin{array}{ccccc}
\left((A\,{}^tC, A\,{}^tE), \begin{pmatrix} D\,{}^tB \\ F\,{}^tB \end{pmatrix}\right) & \xleftarrow{\varphi_{p,q+k}} & \begin{pmatrix} A & B \\ C & D \\ E & F \end{pmatrix} & \xrightarrow{\mathrm{proj}} & (E, F) \\
\zeta\big\downarrow & & \mathrm{proj}\big\downarrow & & \zeta'\big\downarrow \\
(A\,{}^tC, D\,{}^tB) & \xleftarrow{\varphi} & \begin{pmatrix} A & B \\ C & D \end{pmatrix} & \xrightarrow{\psi} & {}^tDC = -\,{}^tFE
\end{array}$$

Here $\Xi_r = \psi^{-1}(\overline{\mathbb{O}'^{\mathrm{hol}}_r})$, and $r = \min\{k, m, n\}$.

Let us consider a "contraction map"

$$f : M_{m+n,k} \ni \begin{pmatrix} A \\ B \end{pmatrix} \longmapsto A\,{}^tB \in M_{m,n}. \tag{5.2}$$

Then the image of f is a closed subvariety

$$V_r = \{Y \in M_{m,n} \mid \operatorname{rank} Y \le r\} \subset M_{m,n} \qquad (r = \min\{k, m, n\}),$$

which is called the determinantal variety of rank r. If we define an action of $g \in GL_k$ on $\begin{pmatrix} A \\ B \end{pmatrix} \in M_{m+n,k}$ as $g \cdot \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} A\,{}^tg \\ Bg^{-1} \end{pmatrix}$, then V_r is isomorphic to the affine quotient of $M_{m+n,k}$ by GL_k, and f is a quotient map:

$$f : M_{m+n,k} \longrightarrow V_r \simeq M_{m+n,k}//GL_k.$$

The following results can be proved by elementary calculations. We shall be contented to just state them. Denote $G *_H V$ the G-principal bundle over G/H, where V is an H-module.

Lemma 5.3. *We keep the notation above. Let us take $Y \in V_r$ with* $\operatorname{rank} Y = r$.

(1) *If $k \le \min\{m, n\}$, then the fiber $f^{-1}(Y)$ is a closed GL_k-orbit isomorphic to GL_k.*
(2) *If $k > m = n$, the fiber $f^{-1}(Y)$ is a closed GL_k-orbit isomorphic to GL_k/GL_{k-n}.*
(3) *If $k \ge m > n$, there is a GL_k-equivariant isomorphism*

$$f^{-1}(Y) \simeq GL_k *_{GL_{k-n}} M_{k-n,m-n},$$

where GL_{k-n} acts on $M_{k-n,m-n}$ by the matrix multiplication on the left. Thus the fiber $f^{-1}(Y)$ contains exactly two GL_k-orbits if $m = n + 1$; otherwise, there exist infinitely many orbits in $f^{-1}(Y)$. Among the GL_k-orbits in $f^{-1}(Y)$, there exists a unique closed orbit isomorphic to GL_k/GL_{k-n}.

Corollary 5.4. *The generic fiber of the quotient map* $f : M_{m+n,k} \to V_r$ *is a single closed* GL_k*-orbit if and only if* $k \leq \min\{m, n\}$ *or* $m = n$ *holds.*

Now we consider the generic fiber of $\zeta : X_{p,q+k} \to \overline{\mathbb{O}_r^{\mathrm{hol}}} \subset M_{p,q} \oplus M_{q,p}$. Take $(S, T) \in \overline{\mathbb{O}_r^{\mathrm{hol}}}$. Instead of investigating the fiber $\zeta^{-1}((S, T)) \subset X_{p,q+k}$, we consider the fiber in $\mathfrak{N}_{p,q+k}$:

$$\Upsilon(S, T) = \left\{ \begin{pmatrix} A & B \\ C & D \\ E & F \end{pmatrix} \in \mathfrak{N}_{p,q+k} \,\middle|\, S = A\,{}^tC, T = D\,{}^tB \right\}. \tag{5.3}$$

We use the notations of Lemma 5.3 in the following

Lemma 5.5. *If* $\operatorname{rank} S = m, \operatorname{rank} T = n$ *and* $\operatorname{rank} ST = r$*, then there exists a* $Y \in M_{m,n}$ *with* $\operatorname{rank} Y = r$ *such that*

$$\Upsilon(S, T) \simeq GL_m \times GL_n \times f^{-1}(Y) \qquad (GL_m \times GL_n \times GL_k\text{-equivariant isomorphism}).$$

In particular, we have

$$\Upsilon(S, T) \simeq \begin{cases} GL_m \times GL_n \times GL_k & (k \leq \min\{m, n\}), \\ GL_m \times GL_n \times (GL_k/GL_{k-n}) & (k > m = n), \\ GL_m \times GL_n \times (GL_k *_{GL_{k-n}} M_{k-n,m-n}) & (k \geq m > n). \end{cases}$$

Corollary 5.6. *Let us assume that* $\operatorname{rank} S = m, \operatorname{rank} T = n$ *and* $\operatorname{rank} ST = r$*. Then, we have* $\zeta^{-1}((S, T)) \simeq f^{-1}(Y)$ *for* $Y \in M_{m,n}$ *as in the lemma above. More precisely,*

$$\zeta^{-1}((S, T)) \simeq \begin{cases} GL_k & (k \leq \min\{m, n\}), \\ GL_k/GL_{k-n} & (k > m = n), \\ GL_k *_{GL_{k-n}} M_{k-n,m-n} & (k \geq m > n). \end{cases}$$

Thus, the generic fiber $\zeta^{-1}((S, T))$ *is a closed* GL_k*-orbit if and only if* $k \leq \min\{m, n\}$ *or* $m = n$*.*

5.3. $Sp(p, q) \times O^*(2n) \quad (n \leq p, q)$**.** We have the following double fibration map

$$\begin{array}{ccc} & W = M_{2p+2q,n} & \\ {}^{\varphi}\swarrow & & \searrow^{\psi} \\ \mathfrak{s} = M_{2p,2q} & & \operatorname{Alt}_n \oplus \operatorname{Alt}_n = \mathfrak{s}' \end{array}$$

where the moment maps φ and ψ are given by

$$Z = \begin{pmatrix} A \\ B \end{pmatrix} \in M_{2p+2q,n}$$

$$\begin{cases} \varphi(Z) = A\,{}^tB \in M_{2p,2q} = \mathfrak{s} \\ \psi(Z) = ({}^tAJ_pA,\ {}^tBJ_qB) \in \mathrm{Alt}_n \oplus \mathrm{Alt}_n = \mathfrak{s}' \end{cases} \qquad J_p = \begin{pmatrix} 0 & 1_p \\ -1_p & 0 \end{pmatrix}$$

Take a holomorphic nilpotent orbit $\mathbb{O}'^{\mathrm{hol}}_k$ of rank $2k$ in $\mathfrak{s}'_- = \mathrm{Alt}_n$, where $0 \leq 2k \leq n$, and consider its lift $\mathbb{O}^{\mathrm{hol}}_k = \theta(\mathbb{O}'^{\mathrm{hol}}_k)$. Note that the rank is necessarily an even integer. The closure $\overline{\mathbb{O}'^{\mathrm{hol}}_k}$ is an affine quotient of $W_k = M_{2k,n}$ by $Sp(2k,\mathbb{C})$:

$$M_{2k,n}//Sp(2k,\mathbb{C}) \simeq \begin{cases} \overline{\mathbb{O}'^{\mathrm{hol}}_k} & (0 \leq 2k \leq n) \\ \overline{\mathbb{O}'^{\mathrm{hol}}_l} & (n < 2k) \end{cases} \qquad l = [n/2]$$

Here $Sp(2k,\mathbb{C})$ acts on $M_{2k,n}$ via the left multiplication of matrices. The quotient map $\zeta' : M_{2k,n} \to \overline{\mathbb{O}'^{\mathrm{hol}}_k} \subset \mathrm{Alt}_n$ is given by $\zeta'(A) = {}^tAJ_kA \quad (A \in M_{2k,n})$.

The null cone is

$$\begin{aligned} \mathfrak{N}_{p,q+k} &= \psi^{-1}_{p,q+k}(0) = \mathfrak{N}^+_p \times \mathfrak{N}^-_{q+k} \\ &= \{(A,B,C) \mid {}^tAJ_pA = 0,\ {}^tBJ_qB + {}^tCJ_kC = 0\} \subset M_{2p+(2q+2k),n}. \end{aligned}$$

We have the following diagram:

$$\begin{array}{ccccc} X_{p,q+k} & \xleftarrow{//GL_n} & \mathfrak{N}_{p,q+k} & \xrightarrow{//Sp(2p,\mathbb{C})\times Sp(2q,\mathbb{C})} & M_{2k,n} \\ {\scriptstyle //Sp(2k,\mathbb{C})}\downarrow & & {\scriptstyle //Sp(2k,\mathbb{C})}\downarrow & & {\scriptstyle //Sp(2k,\mathbb{C})}\downarrow \\ \overline{\mathbb{O}^{\mathrm{hol}}_r} & \xleftarrow{//GL_n} & \Xi_r & \xrightarrow{//Sp(2p,\mathbb{C})\times Sp(2q,\mathbb{C})} & \overline{\mathbb{O}'^{\mathrm{hol}}_r} \end{array}$$

$$\begin{array}{ccccc} (A\,{}^tB, A\,{}^tC) & \xleftarrow{\varphi_{p,q+k}} & (A,B,C) & \xrightarrow{\mathrm{proj}} & C \\ {\scriptstyle \zeta}\downarrow & & {\scriptstyle \mathrm{proj}}\downarrow & & {\scriptstyle \zeta'}\downarrow \\ A\,{}^tB & \xleftarrow{\varphi} & (A,B) & \xrightarrow{\psi} & {}^tBJ_qB = -\,{}^tCJ_kC \end{array}$$

Here $\Xi_r = \psi^{-1}(\overline{\mathbb{O}'^{\mathrm{hol}}_r})$, and $r = \min\{k,l\}$.

Let us consider a "contraction map"

$$f : M_{2k,n} \ni A \longmapsto {}^tAJ_kA \in \mathrm{Alt}_n. \tag{5.4}$$

Then the image of f is a closed subvariety

$$V_r = \{Y \in \mathrm{Alt}_n \mid \mathrm{rank}\,Y \leq 2r\} \subset \mathrm{Alt}_n \qquad (r = \min\{k,l\}),$$

which is a closed subvariety of the determinantal variety of rank $2r$. If we define an action of $g \in Sp(2k, \mathbb{C})$ on $M_{2k,n}$ by the left multiplication of matrices, then V_r is isomorphic to the affine quotient of $M_{2k,n}$ by $Sp(2k, \mathbb{C})$, and f is a quotient map:

$$f : M_{2k,n} \longrightarrow V_r \simeq M_{2k,n} /\!/ Sp(2k, \mathbb{C}).$$

Again we state the following results without proof.

Lemma 5.7. *We keep the notation above. Let us take* $Y \in V_r$ *with* $\operatorname{rank} Y = 2r$ *and* $r = \min\{k, l = [n/2]\}$*.*

(1) *If* $2k \leq n$*, then the fiber* $f^{-1}(Y)$ *is a closed* $Sp(2k, \mathbb{C})$*-orbit isomorphic to* $Sp(2k, \mathbb{C})$*.*

(2) *If* $2k > n$ *and* n *is even, the fiber* $f^{-1}(Y)$ *is a closed* $Sp(2k, \mathbb{C})$*-orbit isomorphic to* $Sp(2k, \mathbb{C})/Sp(2(k-r), \mathbb{C})$*.*

(3) *If* $2k > n$ *and* n *is odd, the fiber* $f^{-1}(Y)$ *consists of exactly two orbits. The closed orbit which is smaller is isomorphic to* $Sp(2k, \mathbb{C})/Sp(2(k-r), \mathbb{C})$*, and the larger orbit is isomorphic to* $Sp(2k, \mathbb{C})/Q_v$*, where* $Q_v \subset Sp(2(k-r), \mathbb{C})$ *is the stabilizer of a vector* $v \neq 0$ *in* $\mathbb{C}^{2(k-r)}$*:*

$$\begin{aligned} f^{-1}(Y) &\simeq Sp(2k, \mathbb{C})/Sp(2(k-r), \mathbb{C}) \sqcup Sp(2k, \mathbb{C})/Q_v \\ &\simeq Sp(2k, \mathbb{C}) *_{Sp(2(k-r), \mathbb{C})} \mathbb{C}^{2(k-r)}. \end{aligned}$$

Remark 5.8. The maximal parabolic subgroup which leaves $\mathbb{C}v$ stable is isomorphic to $Q_v \times \mathbb{C}^{\times}$.

Corollary 5.9. *The generic fiber of the quotient map* $f : M_{2k,n} \to V_r$ *is a single closed* $Sp(2k, \mathbb{C})$*-orbit if and only if* $2k \leq n$ *or* n *is even.*

Now we consider the generic fiber of $\zeta : X_{p,q+k} \to \overline{\mathbb{O}_r^{\mathrm{hol}}} \subset M_{2p,2q}$. Take $S \in \overline{\mathbb{O}_r^{\mathrm{hol}}}$. Instead of investigating the fiber $\zeta^{-1}(S) \subset X_{p,q+k}$, we consider the fiber in $\mathfrak{N}_{p,q+k}$:

$$\Upsilon(S) = \Big\{ (A, B, C) \in \mathfrak{N}_{p,q+k} \Big| \, S = A\, {}^t B \Big\}. \tag{5.5}$$

We use the notations of Lemma 5.7 in the following

Lemma 5.10. *Put* $r = \min\{k, l\}$ *as above. If* $\operatorname{rank} S = n$ *and* $\operatorname{rank} S J_q {}^t S = 2r$*, then there exists a* $Y \in \operatorname{Alt}_n$ *with* $\operatorname{rank} Y = 2r$ *such that*

$$\Upsilon(S) \simeq GL_n \times f^{-1}(Y) \qquad (GL_n \times Sp(2k, \mathbb{C})\text{-equivariant isomorphism}).$$

In particular, we have

$$\Upsilon(S) \simeq \begin{cases} GL_n \times Sp(2k, \mathbb{C}) & (2k \leq n), \\ GL_n \times (Sp(2k, \mathbb{C})/Sp(2(k-r), \mathbb{C}) & (2k > n;\ n \text{ is even}), \\ GL_n \times (Sp(2k, \mathbb{C}) *_{Sp(2(k-r), \mathbb{C})} \mathbb{C}^{2(k-r)}) & (2k > n;\ n \text{ is odd}). \end{cases}$$

Corollary 5.11. *Let us assume that* $\operatorname{rank} S = n$ *and* $\operatorname{rank} S J_q {}^t S = 2r$. *Then, we have* $\zeta^{-1}(S) \simeq f^{-1}(Y)$ *for* $Y \in \operatorname{Alt}_n$ *in the above lemma. More precisely,*

$$\zeta^{-1}(S) \simeq \begin{cases} Sp(2k,\mathbb{C}) & (2k \le n), \\ Sp(2k,\mathbb{C})/Sp(2(k-r),\mathbb{C}) & (2k > n;\ n \text{ is even}), \\ Sp(2k,\mathbb{C}) *_{Sp(2(k-r),\mathbb{C})} \mathbb{C}^{2(k-r)} & (2k > n;\ n \text{ is odd}). \end{cases}$$

Thus, the generic fiber $\zeta^{-1}(S)$ *is a closed* $Sp(2k,\mathbb{C})$*-orbit if and only if* $2k \le n$ *or* n *is even.*

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