The Combinatorics of Multiplicity Free Spaces

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http://math.rutgers.edu/~knop/talks/singapore1.pdf

Report on the paper

Knop, F.: Construction of commuting difference operators for multiplicity free spaces.
Selecta Math. (New Series) **6** (2000), 443–470, `math.RT/0006004`

Special cases are treated in:

Knop, F.; Sahi, S.: Difference equations and symmetric polynomials defined by their zeros.
Internat. Math. Res. Notices **10** (1996), 473–486, `q-alg/9610017`

Knop, F.: Semisymmetric polynomials and the invariant theory of matrix vector pairs.
Representation Theory **5** (2001), 224–266, `math.RT/9910060`

Definition of multiplicity free spaces

G: connected reductive group $/\mathbb{C}$

X: finite dimensional representation of G

$\mathcal{P} = \mathbb{C}[X]$: ring of polynomial functions on X

As a G-module, $\mathcal{P}$ has an isotypic decomposition:

$$\mathcal{P} = \bigoplus_{\lambda \in \Lambda} \mathcal{P}^\lambda$$

where $\Lambda \subseteq \mathfrak{t}^*$ is a set of dominant weights

Definition: X is a multiplicity free space if all $\mathcal{P}^\lambda$ are irreducible.

Theorem. [Howe-Umeda 1991] The set Λ is a free abelian monoid, i.e., there are linear independent weights $\eta_1, \ldots, \eta_n$ with $\Lambda = \mathbb{N}\eta_1 + \ldots + \mathbb{N}\eta_n$.

The number $n = \mathrm{rk}\, X$ is called the rank of X.

Classical examples

Rectangular matrices:

$$X = \mathbb{C}^p \otimes \mathbb{C}^q, G = GL(p, \mathbb{C}) \times GL(q, \mathbb{C})$$

$$\mathrm{rk}\, X = n := \min(p, q)$$

$$\Lambda = \{\lambda_1(\varepsilon_1 + \varepsilon_1') + \ldots + \lambda_n(\varepsilon_n + \varepsilon_n') \mid \lambda_1 \geq \ldots \geq \lambda_n \geq 0\}.$$

Symmetric matrices:

$$X = S^2\mathbb{C}^p, G = GL(p, \mathbb{C})$$

$$\mathrm{rk}\, X = n := p$$

$$\Lambda = \{\lambda_1(2\varepsilon_1) + \ldots + \lambda_n(2\varepsilon_n) \mid \lambda_1 \geq \ldots \geq \lambda_n \geq 0\}.$$

Skewsymmetric matrices:

$$X = \Lambda^2\mathbb{C}^p, G = GL(p, \mathbb{C})$$

$$\mathrm{rk}\, X = n := \left\lfloor \frac{p}{2} \right\rfloor$$

$$\Lambda = \{\lambda_1(\varepsilon_1 + \varepsilon_2) + \ldots + \lambda_n(\varepsilon_{2n-1} + \varepsilon_{2n}) \mid \lambda_1 \geq \ldots \geq \lambda_n \geq 0\}.$$

Capelli operators

$\mathcal{PD}$: differential operators with polynomial coefficients

$\mathcal{D} = \mathbb{C}[X^*]$: differential operators with constant coefficients

As G-modules:

$$\mathcal{D} = \bigoplus_{\lambda \in \Lambda} \mathcal{D}_\lambda, \text{ where } \mathcal{D}_\lambda = (\mathcal{P}^\lambda)^*$$

$$\mathcal{PD} = \mathcal{P} \otimes \mathcal{D} = \bigoplus_{\lambda, \mu \in \Lambda} \mathcal{P}^\lambda \otimes \mathcal{D}_\mu$$

Therefore:

$$\mathcal{PD}^G = \bigoplus_{\lambda \in \Lambda} \mathbb{C} D_\lambda$$

where $D_\lambda \in (\mathcal{P}^\lambda \otimes \mathcal{D}_\lambda)^G$ is called an Capelli operator.

D_λ acts on $\mathcal{P}^\mu$ as a scalar $c_\lambda(\mu)$. Normalization: $c_\lambda(\lambda) = 1$.

$$c_\lambda(\mu) = \binom{\mu}{\lambda} \quad \text{generalized Binomial coefficient}$$

[Dib 1990, Faraut-Koranyi 1990; Lassalle 1990; Yan 1992, Benson-Ratcliff 1998]

Capelli identity

Consider the case of square matrices: $X = \mathbb{C}^n \otimes \mathbb{C}^n$.

x_{ij}: coordinates of X.

Put

$$\lambda = \varepsilon_1 + \ldots + \varepsilon_n$$

then

$$\mathcal{P}^\lambda = \mathbb{C}\,\mathsf{det}(x_{ij}).$$

Similarly,

$$\mathcal{D}_\lambda = \mathbb{C}\,\mathsf{det}(\partial_{ij}) \quad \text{where} \quad \partial_{ij} = \frac{\partial}{\partial x_{ij}}.$$

Thus

$$D_\lambda = \mathsf{det}(x_{ij})\,\mathsf{det}(\partial_{ij}) = \quad \text{right hand side of Capelli identity.}$$

The spectrum of invariant differential operators

In general, $D \in \mathcal{PD}^G$ acts on $\mathcal{P}^\mu$ as a scalar:

$$D(f) = c_D(\mu) f \text{ for all } f \in \mathcal{P}^\mu.$$

Thus we get an injection

$$\mathcal{PD}^G \hookrightarrow \mathsf{Maps}(\Lambda, \mathbb{C}) : D \mapsto c_D.$$

Let $V \subseteq \mathfrak{t}^*$ be the $\mathbb{C}$-span of Λ. Then one can show that

$$\begin{array}{ccc} \mathcal{PD}^G & \overset{c_*}{\hookrightarrow} & \mathbb{C}[V] \\ & \searrow & \downarrow \\ & & \mathsf{Maps}(\Lambda, \mathbb{C}) \end{array}$$

Let $\rho \in V$ be the orthogonal projection of $\frac{1}{2}\sum_{\alpha>0} \alpha$ to V and

$$p_D(\lambda) := c_D(\lambda - \rho).$$

Theorem. [Knop 1998] There is a finite group W acting on V such that

1. W is generated by reflections.
2. $\mathbb{C}[V]^W$ is a polynomial ring.
3. $\mathcal{PD}^G \xrightarrow{\sim} \mathbb{C}[V]^W : D \mapsto p_D$.

The characterization theorem

Thus, we obtain a distinguished basis of $\mathbb{C}[V]^W$:

$$\begin{array}{ccc} D_\lambda & \mapsto & p_\lambda \\ \Cap & & \Cap \\ \mathcal{PD}^G & \xrightarrow{\sim} & \mathbb{C}[V]^W \end{array} \quad \text{(Normalization: } p_\lambda(\rho+\lambda)=1.\text{)}$$

The polynomials p_λ can be described purely in terms of V. For this, let

$$\ell(\lambda) := \deg f \quad \text{for all} \quad f \in \mathcal{P}^\lambda, f \neq 0.$$

Characterization Theorem. [Sahi 1994, Knop 1998] The polynomial $p_\lambda \in \mathbb{C}[V]$ is uniquely determined by the following conditions:

1. Degree: $\deg p_\lambda = \ell(\lambda)$;
2. Invariance: p_λ is W-invariant;
3. Vanishing: For all $\mu \in \Lambda$ with $\ell(\mu) \leq \ell(\lambda)$ holds $p_\lambda(\rho+\mu) = \delta_{\lambda\mu}$.

Conclusion: To calculate p_λ it suffices to know the quintuple

$$[V, \Lambda, \ell, W, \rho].$$

The quintuple $[V, \Lambda, \ell, W, \rho]$ in the classical case

In the classical cases we have:

$$V = \mathbb{C}^n$$

$\Lambda = \{\lambda \in \mathbb{Z}^n \mid \lambda_1 \geq \ldots \geq \lambda_n \geq 0\}$ (Partitions)

$$\ell(\lambda) = |\lambda| := \sum_{i=1}^{n} \lambda_i$$

$$W = S_n \subseteq GL_n(\mathbb{C})$$

$\rho = r(n-1, \ldots, 1, 0)$ with $r = \begin{cases} \frac{1}{2} & \text{symmetric matrices} \\ 1 & \text{rectangular matrices} \\ 2 & \text{skewsymmetric matrices} \end{cases}$

The quintuple $[V, \Lambda, \ell, W, \rho]$ in the semiclassical case

Two not so classical cases:

$$X = (\mathbb{C}^p \otimes \mathbb{C}^q) \oplus \mathbb{C}^q, G = GL_p(\mathbb{C}) \times GL_q(\mathbb{C})$$

$$X = \Lambda^2\mathbb{C}^p \oplus \mathbb{C}^p, G = GL_p(\mathbb{C})$$

$$\mathrm{rk}\, X = n := \begin{cases} \min(2p+1, 2q) \\ p \end{cases}$$

$$V = \mathbb{C}^n$$

$$\Lambda = \{\lambda \in \mathbb{Z}^n | \lambda_1 \geq \ldots \geq \lambda_n \geq 0\}$$

$$\ell(\lambda) = \sum_{i \text{ odd}} \lambda_i$$

$$W = \{\pi \in S_n \mid \pi(i) - i \text{ even for all } i\}$$

$$\rho = r(n-1, \ldots, 1, 0),\ r = \begin{cases} \frac{1}{2} \\ 1 \end{cases}$$

The combinatorial structure of all other multiplicity free spaces has been worked out, too.

The general interpolation polynomials

These examples suggest the following general

Definition: For sufficiently general $\rho \in V$ and $\lambda \in \Lambda$ let $p_\lambda(z;\rho)$ be the polynomial function on V with

1. $\deg p_\lambda(z;\rho) = \ell(\lambda)$;
2. $p_\lambda(z;\rho)$ is W-invariant;
3. For all $\mu \in \Lambda$ with $\ell(\mu) \le \ell(\lambda)$ holds $p_\lambda(\rho+\mu;\rho) = \delta_{\lambda\mu}$.

Main Theorem. [Knop 2000] Let ρ be in a certain subspace V_0 of V. Then for every $h \in \mathbb{C}[V]^W$ there is a difference operator D_h with

$$D_h(p_\lambda) = h(\rho+\lambda)p_\lambda$$

for all $\lambda \in \Lambda$.

The difference operators in the classical case

For $i = 1, \ldots, n$ let

$$T_i f(z_1, \ldots, z_n) := f(z_1, \ldots, z_{i-1}, z_i - 1, z_{i+1} \ldots, z_n)$$

With a free parameter t let

$$\mathfrak{Z}_t := \Big[(z_i + t)(z_i + r)^{n-j} - z_i^{n+1-j} T_i \Big]_{\substack{i=1\ldots n \\ j=1\ldots n}}$$

$$a(z) := \prod_{1 \le i < j \le n} (z_i - z_j) \quad \text{(Vandermonde)}$$

$$a(z)^{-1} \operatorname{det} \mathfrak{Z}_t = t^n + D_1 t^{n-1} + \ldots + D_n.$$

Theorem. [Knop-Sahi 1996] Let $[V, \Lambda, \ell, W]$ be as in the classical case and $\rho = r(n-1, \ldots, 1, 0)$. Then

$$D_i(p_\lambda) = e_i(\rho + \lambda)\, p_\lambda$$

with $e_i = i$-th elementary symmetric polynomial.

The difference operators in the semiclassical case

Notation: $x_i := z_{2i-1}, \quad y_i := z_{2i}, \quad \overline{n} := \lceil \frac{n}{2} \rceil, \quad \underline{n} := \lfloor \frac{n}{2} \rfloor.$

$$\mathfrak{X}_t := \begin{pmatrix} \left[(x_i+t)(x_i+r)^{\overline{n}-j} - x_i^{\overline{n}+1-j} T_{x,i}\right]_{\substack{i=1\ldots\overline{n} \\ j=1\ldots\overline{n}}} & \left[- x_i^{\overline{n}-j} T_{x,i}\right]_{\substack{i=1\ldots\overline{n} \\ j=1\ldots\underline{n}}} \\ \left[(y_i+r)^{\underline{n}+1-j} - y_i^{\underline{n}+1-j} T_{y,i}\right]_{\substack{i=1\ldots\underline{n} \\ j=1\ldots\overline{n}}} & \left[(y_i+r)^{\underline{n}-j}\right]_{\substack{i=1\ldots\underline{n} \\ j=1\ldots\underline{n}}} \end{pmatrix}$$

$$\mathfrak{Y}_t := \begin{pmatrix} \left[(x_i+r)^{\overline{n}-j}\right]_{\substack{i=1\ldots\overline{n} \\ j=1\ldots\overline{n}}} & \left[(x_i+r)^{\overline{n}-j} - x_i^{\overline{n}-j} T_{x,i}\right]_{\substack{i=1\ldots\overline{n} \\ j=1\ldots\underline{n}}} \\ \left[- y_i^{\underline{n}+1-j} T_{y,i}\right]_{\substack{i=1\ldots\underline{n} \\ j=1\ldots\overline{n}}} & \left[(y_i+t)(y_i+r)^{\underline{n}-j} - y_i^{\underline{n}+1-j} T_{y,i}\right]_{\substack{i=1\ldots\underline{n} \\ j=1\ldots\underline{n}}} \end{pmatrix}$$

$$a(z) := \prod_{1 \le i < j \le \overline{n}} (x_i - x_j) \prod_{1 \le i < j \le \underline{n}} (y_i - y_j).$$

$$a(z)^{-1} \det \mathfrak{X}_t = t^{\overline{n}} + X_1 t^{\overline{n}-1} + \ldots + X_{\overline{n}}, \quad a(z)^{-1} \det \mathfrak{Y}_t = t^{\underline{n}} + Y_1 t^{\underline{n}-1} + \ldots + Y_{\underline{n}}.$$

Theorem. [Knop 2001] Let $[V, \Lambda, \ell, W]$ be as in the semiclassical case and $\rho = r(n-1, \ldots, 1, 0)$. Then the operators $X_1, \ldots, X_{\overline{n}}, Y_1, \ldots, Y_{\underline{n}}$ are commuting difference operators having the p_λ's as common eigenvectors.

Idea for proving these theorems

Let D be a difference operator. Then one shows:

1. $\deg D(f) \leq \deg f$ for all $f \in \mathbb{C}[V]^W$.
2. $wD = Dw$ for all $w \in W$.
3. If $f(\rho + \mu) = 0$ for all $\mu \in \Lambda$ with $\ell(\mu) \leq \ell(\lambda)$, $\mu \neq \lambda$ then the same holds for $D(f)$.

These conditions imply $D(p_\lambda) \in \mathbb{C} p_\lambda$.

If D has the form

$$D = \sum_{\eta \in V} b_\eta(z) T_\eta, \quad \text{where} \quad T_\eta f(z) = f(z - \eta)$$

then 3. is (roughly) equivalent to the following cut-off property of the coefficients:

3'. $\lambda \in \Lambda, \lambda - \eta \notin \Lambda \Rightarrow b_\eta(\rho + \lambda) = 0$.

Example

Take the semiclassical case of rank $n = 5$. Then

$\Lambda = \{\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4 \geq \lambda_5 \geq 0\}$

$\ell(\lambda) = \lambda_1 + \lambda_3 + \lambda_5$

$W = S(z_1, z_3, z_5) \times S(z_2, z_4)$

$D_\ell = z_1 + z_3 + z_5 -$

$$-\frac{(z_1 - z_2 - r)(z_1 - z_4 - r)z_1}{(z_1 - z_3)(z_1 - z_5)} T_{e_1} - \frac{(z_3 - z_2 - r)(z_3 - z_4 - r)z_3}{(z_3 - z_1)(z_3 - z_5)} T_{e_3} - \frac{(z_5 - z_2 - r)(z_5 - z_4 - r)z_5}{(z_5 - z_3)(z_5 - z_1)} T_{e_5}$$

$$-\frac{(z_2 - z_5 - r)(z_2 - z_3 - r)(z_1 - z_4 - r)z_1}{(z_1 - z_3)(z_1 - z_5)(z_2 - z_4)} T_{e_1+e_2} - \frac{(z_4 - z_5 - r)(z_4 - z_3 - r)(z_1 - z_2 - r)z_1}{(z_1 - z_3)(z_1 - z_5)(z_4 - z_2)} T_{e_1+e_4}$$

$$-\frac{(z_2 - z_5 - r)(z_2 - z_1 - r)(z_3 - z_4 - r)z_3}{(z_3 - z_1)(z_3 - z_5)(z_2 - z_4)} T_{e_3+e_2} - \frac{(z_4 - z_5 - r)(z_4 - z_1 - r)(z_3 - z_2 - r)z_3}{(z_3 - z_1)(z_3 - z_5)(z_4 - z_2)} T_{e_3+e_4}$$

$$-\frac{(z_2 - z_1 - r)(z_2 - z_3 - r)(z_5 - z_4 - r)z_5}{(z_5 - z_3)(z_5 - z_1)(z_2 - z_4)} T_{e_5+e_2} - \frac{(z_4 - z_1 - r)(z_4 - z_3 - r)(z_5 - z_2 - r)z_5}{(z_5 - z_3)(z_5 - z_1)(z_4 - z_2)} T_{e_5+e_4}$$

The Pseudoroots

Recall $\Lambda = \mathbb{N}\eta_1 + \ldots + \mathbb{N}\eta_n$.

Dual basis of V^*: $\omega_i(\eta_j) = \delta_{ij}$.

The elements of

$$\Phi := \{w\omega_i \mid w \in W, i = 1 \ldots n\}$$

are called pseudoroots.

The admissible ρ-shifts

Define a W-invariant multiplicity function

$$\Phi \to \mathbb{C} : \omega \mapsto k_\omega$$

with

$$k_{-\omega} = k_\omega \quad \text{for all} \quad \omega \in \Phi \cap -\Phi$$

Let

$$\rho := k_{\omega_1}\eta_1 + \ldots + k_{\omega_n}\eta_n.$$

The difference Euler operator

Define the falling factorials:

$$[a \downarrow n] := \begin{cases} a(a-1)\dots(a-n+1) & \text{for } n \in \mathbb{N} \\ 1 & \text{otherwise} \end{cases}$$

For $\eta \in V$ let

$$f_\eta(z) := \frac{\prod_{\omega \in \Phi} [\omega(z) - k_\omega \downarrow \omega(\eta)]}{\prod_{\alpha \in \Delta} [\alpha(z) \downarrow \alpha(\eta)]}$$

Let

$$\Lambda_1 := \{w\eta_i \mid w \in W, \ell(\eta_i) = 1\}$$

$\Delta =$ roots of W.

Define the difference operators

$$L := \sum_{\eta \in \Lambda_1} f_\eta(z) T_\eta \quad \text{and} \quad E := \ell(z) - L.$$

Theorem. [Knop 2000] For all $\lambda \in \Lambda$ holds $E(p_\lambda) = \ell(\rho + \lambda) p_\lambda$. In other words: $E = D_\ell$.

Construction of the other difference operators

The operator L allows to construct all other difference operators!

Notation: For $h \in \mathbb{C}[V]^W$ let m_h be the operator "multiplication by h" on $\mathbb{C}[V]^W$.

Key Lemma. For all $h \in \mathbb{C}[V]^W$ holds

$$(\mathsf{ad}\, L)^d(m_h) = 0 \quad \text{for} \quad d > \deg h.$$

Theorem. [Knop 2000] For all $h \in \mathbb{C}[V]^W$ holds

$$D_h = \exp(\mathsf{ad}\, L)(m_h) = \sum_{d=0}^{\infty} \frac{1}{d!} (\mathsf{ad}\, L)^d(m_h).$$

Proof of the key lemma

The equality $E(p_\mu)(z) = \ell(\rho+\mu)p_\mu(z)$ evaluated at $z = \rho+\lambda$ implies the recursion formula

$$p_\mu(\rho+\lambda) = \frac{1}{\ell(\mu)} \sum_{\eta\in\Lambda_1} f_\eta(\rho+\lambda)p_\mu(\rho+\lambda-\eta)$$

and then

(A) $$p_\mu(\rho+\lambda) = \frac{1}{\ell(\mu)!} \sum_{\tau_*:\mu\to\lambda} f_{\tau_1-\tau_0}(\rho+\tau_1)f_{\tau_2-\tau_1}(\rho+\tau_2)\dots f_{\tau_d-\tau_{d-1}}(\rho+\tau_d)$$

where $\tau_* : \mu \to \lambda$ means a chain $\mu = \tau_0, \tau_1, \dots, \tau_d = \lambda$ with $\tau_i - \tau_{i-1} \in \Lambda_1$ for all i.

Consider for $h \in \mathbb{C}[V]^W$ the expansion

$$h(-z)p_\mu(z) = \sum_\tau a_\tau^h(\mu)p_{\mu+\tau}(z)$$

Observation: $a_\tau^h(\mu) = 0$ unless $\ell(\tau) \le \deg h$.

By induction on $\ell(\tau)$ and using formula (A)

(B) $$a_\tau^h(\mu) = \sum_{\tau_*:\mu\to\mu+\tau} \left[\sum_{i=0}^{d} \frac{(-1)^{d-i}}{i!(d-i)!} h(-\rho-\tau_i)\right] f_{\tau_1-\tau_0}(\rho+\tau_1)\ldots f_{\tau_d-\tau_{d-1}}(\rho+\tau_d)$$

Expand

$$\frac{1}{d!}(\mathsf{ad}\, L)^d(h) = \frac{1}{d!}\sum_{i=0}^{d}(-1)^{d-i}\binom{d}{i} L^i h L^{d-i} = \sum_{\ell(\tau)=d} b_\tau^h(z) T_\tau.$$

with

(C) $$b_\tau^h(z) := \sum_{\tau_*:0\to\tau} \left[\sum_{i=0}^{d} \frac{(-1)^{d-i}}{i!(d-i)!} h(z-\tau_i)\right] f_{\tau_1-\tau_0}(z-\tau_0)\ldots f_{\tau_d-\tau_{d-1}}(z-\tau_{d-1})$$

Comparing formulas (B) and (C) gives

$$b_\tau^h(-\rho-\mu) = (-1)^{\ell(\tau)}\frac{d_{\mu+\tau}}{d_\mu} a_\tau^h(\mu).$$

with

$$d_\lambda := (-1)^{\ell(\lambda)}\frac{f_\lambda(-\rho)}{f_\lambda(\rho+\lambda)}$$

Case I:	$GL_p(\mathbb{C})$	$S^2(\mathbb{C}^p)$
	$GL_p(\mathbb{C}) \times GL_q(\mathbb{C})$	$\mathbb{C}^p \otimes \mathbb{C}^q$
	$GL_p(\mathbb{C})$	$\Lambda^2(\mathbb{C}^p)$
	$Sp_{2p}(\mathbb{C})$	$\mathbb{C}^{2p}$
	$SO_p(\mathbb{C}) \times \mathbb{C}^*$	$\mathbb{C}^p$
	$Spin_{10}(\mathbb{C}) \times \mathbb{C}^*$	$\mathbb{C}^{16}$
	$Spin_7(\mathbb{C}) \times \mathbb{C}^*$	$\mathbb{C}^8$
	$\mathsf{G}_2 \times \mathbb{C}^*$	$\mathbb{C}^7$
	$\mathsf{E}_6 \times \mathbb{C}^*$	$\mathbb{C}^{27}$

Case II:	$GL_p(\mathbb{C}) \times GL_q(\mathbb{C})$	$(\mathbb{C}^p \otimes \mathbb{C}^q) \oplus \mathbb{C}^q$
	$GL_1(\mathbb{C}) \times GL_q(\mathbb{C})$	$(\mathbb{C} \otimes \mathbb{C}^q) \oplus (\mathbb{C}^q)^*$
	$GL_p(\mathbb{C})$	$\Lambda^2(\mathbb{C}^n) \oplus \mathbb{C}^n$

Case III:	$GL_p(\mathbb{C}) \times GL_q(\mathbb{C})$	$(\mathbb{C}^p \otimes \mathbb{C}^q) \oplus (\mathbb{C}^q)^*$
	$GL_p(\mathbb{C}) \times \mathbb{C}^*$	$\Lambda^2(\mathbb{C}^p) \oplus (\mathbb{C}^p)^*$

$\longrightarrow$back

Case IVa:	$Sp_{2p}(\mathbb{C}) \times GL_3(\mathbb{C})$	$\mathbb{C}^{2p} \otimes \mathbb{C}^3$
Case IVb:	$Sp_4(\mathbb{C}) \times GL_p(\mathbb{C})$	$\mathbb{C}^4 \otimes \mathbb{C}^p$
Case IVc:	$Sp_4(\mathbb{C}) \times GL_3(\mathbb{C})$	$\mathbb{C}^4 \otimes \mathbb{C}^3$
Case V:	$(Sp_{2p}(\mathbb{C}) \times \mathbb{C}^*) \times GL_2(\mathbb{C})$	$(\mathbb{C}^{2p} \otimes \mathbb{C}^2) \oplus \mathbb{C}^2$
	$GL_p(\mathbb{C}) \times SL_2(\mathbb{C}) \times GL_q(\mathbb{C})$	$(\mathbb{C}^p \otimes \mathbb{C}^2) \oplus (\mathbb{C}^2 \otimes \mathbb{C}^q)$
	$(Sp_{2p}(\mathbb{C}) \times \mathbb{C}^*) \times SL_2(\mathbb{C}) \times GL_q(\mathbb{C})$	$(\mathbb{C}^{2p} \otimes \mathbb{C}^2) \oplus (\mathbb{C}^2 \otimes \mathbb{C}^q)$
	$(Sp_{2p}(\mathbb{C}) \times \mathbb{C}^*) \times SL_2(\mathbb{C}) \times (Sp_{2q}(\mathbb{C}) \times \mathbb{C}^*)$	$(\mathbb{C}^{2p} \otimes \mathbb{C}^2) \oplus (\mathbb{C}^2 \otimes \mathbb{C}^{2q})$
	$Spin_8(\mathbb{C}) \times \mathbb{C}^* \times \mathbb{C}^*$	$\mathbb{C}^8_+ \oplus \mathbb{C}^8_-$
Case VIa:	$Sp_{2p}(\mathbb{C}) \times GL_2(\mathbb{C})$	$\mathbb{C}^{2p} \otimes \mathbb{C}^2$
	$Spin_9(\mathbb{C}) \times \mathbb{C}^*$	$\mathbb{C}^{16}$
Case VIb:	$Sp_{2p}(\mathbb{C}) \times \mathbb{C}^* \times \mathbb{C}^*$	$\mathbb{C}^{2p} \oplus \mathbb{C}^{2p}$

→**back**

Case I: The classical cases: $(1 \le n)$

$$\{\omega_i\} := \{z_1 - z_2, z_2 - z_3, \dots, z_{n-1} - z_n, z_n\}.$$

$$\{\eta_i\} = \{e_1, e_1 + e_2, e_1 + e_2 + e_3, \dots, e_1 + e_2 + \dots + e_n\}$$

$$\ell := z_1 + z_2 + \dots + z_n$$

$$W := S_n = \langle s_{12}, s_{23}, \dots, s_{n-1\,n} \rangle$$

$$\Delta^+ = \{z_i - z_j \mid 1 \le i < j \le n\}$$

$$\Phi^+ = \{z_i - z_j \mid 1 \le i < j \le n\} \cup \{z_i \mid 1 \le i \le n\}$$

$$\rho = ((n-1)r + s, (n-2)r + s, \dots, r + s, s)$$

→**back**

Case II: The semiclassical cases: $(3 \leq n)$

$$\{\omega_i\} := \{z_1 - z_2, z_2 - z_3, \ldots, z_{n-1} - z_n, z_n\}$$

$$\{\eta_i\} = \{e_1, e_1 + e_2, e_1 + e_2 + e_3, \ldots, e_1 + e_2 + \ldots + e_n\}$$

$$\ell := z_1 + z_3 + z_5 + \ldots$$

$$W := \{\pi \in S_n \mid \forall i : \pi(i) - i \text{ even}\} = \langle s_{13}, s_{24}, \ldots, s_{n-2\,n}\rangle$$

$$\Delta^+ = \{z_i - z_j \mid 1 \leq i < j \leq n, i - j \text{ even}\}$$

$$\Phi^+ = \{z_i - z_j \mid 1 \leq i < j \leq n, i - j \text{ odd}\} \cup \{z_i \mid 1 \leq i \leq n, n - i \text{ even}\}$$

$$\rho = ((n-1)r + s, (n-2)r + s, \ldots, r + s, s)$$

→**back**

Case III: The quasiclassical cases: $(3 \leq n)$

n **odd**

$V := \{z \in \mathbb{C}^{n+1} \mid z_n = 0\}$

$\{\omega_i\} := \{z_1 - z_2, z_2 - z_3, \ldots, z_n - z_{n+1}\}$

$\{\eta_i\} = \{e_1, e_1 + e_2, \ldots, e_1 + e_2 + \ldots + e_{n-1}, -e_{n+1}\}$

$\ell := \sum_{i=1}^{n+1} \frac{1}{2}(1 - 3(-1)^i) z_i = 2z_1 - z_2 + 2z_3 - + \ldots - z_{n+1}$

$W := \{\pi \in S_{n+1} \mid \forall i : \pi(i) - i \text{ even}, \pi(n) = n\} = \langle s_{13}, s_{24}, \ldots, s_{n-3\, n-1}, s_{n-1\, n+1}\rangle$

$\Delta^+ = \{z_i - z_j \mid 1 \leq i < j \leq n+1, i - j \text{ even}, j \neq n\}$

$\Phi^+ = \{z_i - z_j \mid 1 \leq i < j \leq n+1, i - j \text{ odd}\}$ (with $z_n = 0$)

$\rho = ((n-2)r + s, (n-3)r + s, \ldots, r + s, s, 0, -s)$

n **even**

$\{\omega_i\} := \{z_1 - z_2, z_2 - z_3, \ldots, z_{n-1} - z_n, z_{n-1}\}$

$\{\eta_i\} = \{e_1, e_1 + e_2, \ldots, e_1 + e_2 + \ldots + e_{n-2}, -e_n, e_1 + e_2 + \ldots + e_n\}$

$\ell := \sum_{i=1}^{n} \frac{1}{2}(1 - 3(-1)^i) z_i = 2z_1 - z_2 + 2z_3 - + \ldots - z_n$

$W := \{\pi \in S_n \mid \forall i : \pi(i) - i \text{ even}\} = \langle s_{13}, s_{24}, \ldots, s_{n-2\, n}\rangle$

$\Delta^+ = \{z_i - z_j \mid 1 \leq i < j \leq n, i - j \text{ even}\}$

$\Phi^+ = \{z_i - z_j \mid 1 \leq i < j \leq n, i - j \text{ odd}\} \cup \{z_i \mid 1 \leq i \leq n, i \text{ odd}\}$

$\rho = ((n-2)r + s, (n-3)r + s, \ldots, r + s, s, -r + s)$

→**back**

Case IVa:

$$V := \{z \in \mathbb{C}^7 \mid z_2 + z_4 + z_6 + z_7 = 0\}$$

$$\{\omega_i\} := \{z_1 - z_2, z_2 - z_3, z_3 - z_4, z_4 - z_5, z_5 - z_6, z_4 + z_6\}$$

$$\begin{aligned}\{\eta_i\} = \{&(1,0,0,0,0,0,0), (1,1,0,0,0,0,-1), (1,1,1,0,0,0,-1),\\ &(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2}), (\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2}), (\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},-\tfrac{3}{2})\}\end{aligned}$$

$$\ell := 2(z_1 + z_3 + z_5)$$

$$W := S_3(z_1, z_3, z_5) \times S_4(z_2, z_4, z_6, z_7)$$

$$\Delta^+ = \{z_i - z_j \mid (i,j) = (1,3), (1,5), (3,5), (2,4), (2,6), (2,7), (4,6), (4,7), (6,7)\}$$

$$\begin{aligned}\Phi^+ = &\{\mathsf{sign}(j-i)(z_i - z_j) \mid i = 1,3,5; j = 2,4,6,7\} \cup\\ &\{(z_i + z_j) \mid (i,j) = (2,4), (2,6), (4,6)\}\end{aligned}$$

$$\rho = (\tfrac{s}{2} + 4r, \tfrac{s}{2} + 3r, \tfrac{s}{2} + 2r, \tfrac{s}{2} + r, \tfrac{s}{2}, \tfrac{s}{2} - r, -\tfrac{3}{2}s - 3r)$$

→**back**

Case IVb:

$$\{\omega_i\} := \{\tfrac{1}{2}(z_1 - z_2 - z_4 - z_6), z_2 - z_3, z_3 - z_4, z_4 - z_5, z_5 - z_6, z_5 + z_6\}$$

$$\{\eta_i\} = \{(2,0,0,0,0,0), (1,1,0,0,0,0), (1,1,1,0,0,0), (2,1,1,1,0,0), (\tfrac{3}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}), (\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2})\}$$

$$\ell := 2z_1$$

$$W := \mathsf{D}_3(z_2, z_4, z_6) \times \mathsf{C}_2(z_3, z_5)$$

$$\Delta^+ = \{z_i \pm z_j \mid (i,j) = (2,4), (2,6), (4,6)\} \cup \{z_3 \pm z_5, 2z_3, 2z_5\}$$

$$\Phi^+ = \{z_i \pm z_j \mid 2 \le i < j \le 6, i - j \text{ odd}\} \cup \{\tfrac{1}{2}(z_1 \pm z_2 \pm z_4 \pm z_6) \mid \text{1 or 3 minus signs}\}$$

$$\rho = (2s + 6r, 4r, 3r, 2r, r, 0)$$

→**back**

Case IVc:

$\{\omega_i\} := \{z_1 - z_2, z_2 - z_3, z_3 - z_4, z_4 - z_5, z_4 + z_5\}$

$\{\eta_i\} = \langle(1,0,0,0,0),(1,1,0,0,0),(1,1,1,0,0),(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}),(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2})\rangle$

$\ell := 2(z_1 + z_3 + z_5)$

$W := S_3(z_1, z_3, z_5) \times \mathsf{C}_2(z_2, z_4)$

$\Delta^+ = \{z_i - z_j \mid (i,j) = (1,3),(1,5),(3,5)\} \cup \{z_2 \pm z_4, 2z_2, 2z_4\}$

$\Phi^+ = \{z_i \pm z_j \mid 1 \leq i < j \leq 5, i - j \text{ odd}\}$

$\rho = (4r, 3r, 2r, r, 0)$

→**back**

Case V: ($1 \leq b \leq a \leq 3$)

$$\{\omega_i\} = \{z_1 - z_2, \ldots, z_{a-1} - z_a, z_1' - z_2', \ldots, z_{b-1}' - z_b', \\ z_a + z_b' - z'', z_a - z_b' + z'', -z_a + z_b' + z''\}$$

$$\{\eta_i\} = \{e_1, e_1 + e_2, \ldots, e_1 + e_2 + \ldots + e_{a-1}, e_1', e_1' + e_2', \ldots, e_1' + e_2' + \ldots + e_{b-1}', \\ \tfrac{1}{2}\sum e_i + \tfrac{1}{2}\sum e_i', \tfrac{1}{2}\sum e_i + \tfrac{1}{2}e'', \tfrac{1}{2}\sum e_i' + \tfrac{1}{2}e''\}$$

$$\ell := 2(z_1 + z_1')$$

$$W := (\mathbb{Z}/2\mathbb{Z})^{a+b-1} = \{(z_1, \pm z_2, \ldots, \pm z_a, z_1', \pm z_2', \ldots, \pm z_b', \pm z'')\}$$

$$\Delta^+ = \{2z_2, \ldots, 2z_a, 2z_2', \ldots, 2z_b', 2z''\}$$

$$\Phi^+ = \{z_i \pm z_{i+1} \mid 1 \leq i < a\} \cup \{z_i' \pm z_{i+1}' \mid 1 \leq i < b\} \cup \\ \{\pm z_a \pm z_b' \pm z'' \mid \text{at most one minus sign}\}$$

$$\rho = (r_1 + \ldots + r_{a-1} + s, r_2 + \ldots + r_{a-1} + s, \ldots, r_{a-1} + s, s; \\ r_1' + \ldots + r_{b-1}' + s, r_2' + \ldots + r_{b-1}' + s, \ldots, r_{b-1}' + s, s; s)$$

→**back**

Case VIa:

$$\{\omega_i\} := \{z_1 - z_2, z_2 - z_3, 2z_3\}$$

$$\{\eta_i\} = \{(1,0,0),(1,1,0),(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2})\}$$

$$\ell := 2z_1$$

$$W := (\mathbb{Z}/2\mathbb{Z})^2 = \{(z_1,z_2,z_3) \mapsto (z_1, \pm z_2, \pm z_3)\}$$

$$\Delta^+ = \{2z_2, 2z_3\}$$

$$\Phi^+ = \{z_1 \pm z_2, z_2 \pm z_3, 2z_3\}$$

$$\rho = (r + s + \tfrac{t}{2}, s + \tfrac{t}{2}, \tfrac{t}{2})$$

→back

Case VIb:

$$\{\omega_i\} := \{z_1 - z_2, z_2 - z_3, z_3 - z_4, z_3 + z_4\}$$

$$\{\eta_i\} = \{(1,0,0,0), (1,1,0,0), (\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}), (\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, -\tfrac{1}{2})\}$$

$$\ell := 2z_1$$

$$W := (\mathbb{Z}/2\mathbb{Z})^2 = \{(z_1, z_2, z_3, z_4) \mapsto (z_1, \pm z_2, \pm z_3, z_4)\}$$

$$\Delta^+ = \{2z_2, 2z_3\}$$

$$\Phi^+ = \{z_1 \pm z_2, z_2 \pm z_3, z_3 \pm z_4\}$$

$$\rho = (r+s+t, s+t, t, 0)$$

→back