Special Functions and Multiplicity Free Spaces

Singapore, January 14, 2003

http://math.rutgers.edu/~knop/talks/singapore2.pdf

Report on the paper

Knop, F.: Combinatorics and invariant differential operators on multiplicity free spaces.
To appear in: *J. Algebra* (2003), 36 pages, `math.RT/0106079`

Prerequisite:

Knop, F.: Construction of commuting difference operators for multiplicity free spaces.
Selecta Math. (New Series) **6** (2000), 443–470, `math.RT/0006004`

Multiplicity free spaces

G: connected reductive group $/\mathbb{C}$

X: a multiplicity free space for G

$\mathcal{P} = \mathbb{C}[X]$: ring of polynomial functions on X

$\mathcal{P} = \bigoplus_{\lambda \in \Lambda} \mathcal{P}^\lambda$: isotypic decomposition of $\mathcal{P}$

$\Lambda = \mathbb{N}\eta_1 + \ldots + \mathbb{N}\eta_n$.

$\operatorname{rk} X := n$: rank of X.

$\mathcal{D} := S^*(X) = \mathbb{C}[X^*]$: constant coefficient differential operators

$\mathcal{D} = \bigoplus_{\lambda \in \Lambda} \mathcal{D}_\lambda$: isotypic decomposition of $\mathcal{D}$ with $\mathcal{P}^\lambda = \mathcal{D}_\lambda^*$

$\mathcal{PD}$: polynomial coefficient differential operators

Then

$$\mathcal{PD} \cong \mathcal{P} \otimes \mathcal{D} \quad \text{as } G\text{-modules}$$

Invariant differential operators

Each $D \in \mathcal{PD}^G$ acts on $\mathcal{P}^\mu$ as a scalar:

$$D(f)|_{\mathcal{P}^\mu} = c_D(\mu)\,\mathsf{id}_{\mathcal{P}^\mu}$$

D differential operator $\Longrightarrow c_D(\mu)$ depends polynomially on μ.

Thus we get an injection

$$c_* : \quad \mathcal{PD}^G \hookrightarrow \mathbb{C}[V] : \quad D \mapsto c_D$$

where

$$V := \bigoplus_{i=1}^{n} \mathbb{C}\eta_i \subseteq \mathfrak{t}^*$$

The image of c_*

Let

$$\Delta_\Lambda^+ := \{\alpha \in \Delta(\mathfrak{g}, \mathfrak{t}) \mid \langle \Lambda | \alpha^\vee \rangle > 0\}$$

$$\Phi_X := \text{set of weights of } X$$

Lemma. The weight

$$\varrho_X := \frac{1}{2}\left(\sum \Delta_\Lambda^+ + \sum \Phi_X\right)$$

is an element of V.

Let $p_D(\lambda) := c_D(\lambda - \varrho_X)$.

Theorem. [Knop 1998]

$$p_* : \quad \mathcal{PD}^G \xrightarrow{\sim} \mathbb{C}[V]^W : \quad D \mapsto p_D$$

where $W \subseteq GL(V)$ is a finite reflection group.

Transposition

Definition: The antiautomorphism $D \mapsto {}^tD$ of $\mathcal{DP}$ with

$$ {}^t x = x, \quad {}^t\partial = -\partial \qquad \text{for all } x \in X^*,\ \partial \in X $$

is called transposition.

Theorem. For all $D \in \mathcal{PD}^G$ holds $p_{\,{}^tD}(z) = p_D(-z)$.

Example

$G = GL_p(\mathbb{C})$ acting on $X = \mathbb{C}^p$

$$\Lambda = \mathbb{N}\varepsilon_1, \quad V = \mathbb{C}\varepsilon_1$$

$$\Delta_\Lambda^+ = \{\varepsilon_1 - \varepsilon_i \mid 2 \le i \le p\}$$

$$\Phi_X = \{\varepsilon_i \mid 1 \le i \le p\}$$

$$\varrho_X = \frac{1}{2}\left((\varepsilon_1 - \varepsilon_2) + \ldots + (\varepsilon_1 - \varepsilon_p) + (\varepsilon_1 + \ldots + \varepsilon_p)\right) = \frac{p}{2}\varepsilon_1$$

$$\xi := \sum_{i=1}^{p} x_i \frac{\partial}{\partial x_i} \quad \text{(Euler operator)}$$

$$c_\xi(z) = z, \quad p_\xi(z) = z - \frac{p}{2}$$

$${}^t\xi = \sum_{i=1}^{p}(-\frac{\partial}{\partial x_i})x_i = -\sum_{i=1}^{p}(x_i\frac{\partial}{\partial x_i} + 1) = -\xi - p$$

$$p_{\,{}^t\xi}(z) = -p_\xi(z) - p = -z + \frac{p}{2} - p = -z - \frac{p}{2} = p_\xi(-z).$$

More notation

Let $\ell : V \to \mathbb{C}$ with

$$f \in \mathcal{P}^\lambda \quad \Longrightarrow \quad \deg f = \ell(\lambda).$$

Recall

$$V = \bigoplus_{i=1}^{n} \mathbb{C}\eta_i.$$

Consider the dual basis $\omega_i \in V^*$: $\quad \omega_i(\eta_j) = \delta_{ij}$. Let

$$\Phi := \{w\omega_i \mid w \in W, 1 \le i \le n\} \qquad \text{Pseudoroots}$$

Consider a multiplicity function $\Phi \to \mathbb{C} : \omega \mapsto k_\omega$ with

$$k_{w\omega} = k_\omega, \quad w \in W, \omega \in \Phi; \qquad k_{-\omega} = k_\omega, \quad \omega \in \Phi \cap (-\Phi).$$

Let

$$\varrho = k_{\omega_1}\eta_1 + \ldots + k_{\omega_n}\eta_n \in V.$$

The interpolation polynomials

Definition: For any $\lambda \in \Lambda$ let $p_\lambda(z;\varrho)$ be the polynomial function on V with

1. $\deg p_\lambda(z;\varrho) = \ell(\lambda)$;
2. $p_\lambda(z;\varrho)$ is W-invariant;
3. For all $\mu \in \Lambda$ with $\ell(\mu) \le \ell(\lambda)$ holds $p_\lambda(\varrho+\mu;\varrho) = \delta_{\lambda\mu}$.

Motivation:

For each $\lambda \in \Lambda$ consider $D_\lambda \in (\mathcal{P}^\lambda \otimes \mathcal{D}_\lambda)^G$ (Capelli operator) Then

$$\mathcal{P}\mathcal{D}^G = \bigoplus_{\lambda\in\Lambda} \mathbb{C} D_\lambda .$$

Theorem. For all $\lambda, \mu \in \Lambda$ holds

$$f \in \mathcal{P}^\mu \quad \Longrightarrow \quad D_\lambda(f) = p_\lambda(\varrho_X + \mu; \varrho_X)\, f$$

In other words,

$$\begin{array}{ccc} D_\lambda & \mapsto & p_\lambda \\ \Cap & & \Cap \\ \mathcal{P}\mathcal{D}^G & \xrightarrow{\sim} & \mathbb{C}[V]^W \end{array}$$

The difference operators

Define the falling factorials:

$$[a \downarrow n] := \begin{cases} a(a-1)\dots(a-n+1) & \text{for } n \in \mathbb{N} \\ 1 & \text{otherwise} \end{cases}$$

$$\Delta = \text{ roots of } W$$

For $\eta \in V$ let

$$f_\eta(z) := \frac{\prod_{\omega \in \Phi} [\omega(z) - k_\omega \downarrow \omega(\eta)]}{\prod_{\alpha \in \Delta} [\alpha(z) \downarrow \alpha(\eta)]} \quad \text{and} \quad (T_\eta f)(z) := f(z - \eta)$$

Let

$$\Lambda_1 := \{w\eta_i \mid w \in W, \ell(\eta_i) = 1\}$$

Define the difference operator

$$L := \sum_{\eta \in \Lambda_1} f_\eta(z) T_\eta$$

$$D_h := \exp(\operatorname{ad} L)(m_h), \quad \text{where} \quad m_h(f) = hf$$

Theorem. D_h is well defined and $D_h(p_\lambda) = h(\varrho + \lambda)p_\lambda$ for all $\lambda \in \Lambda$.

The transposition formula

Let

$$q_\lambda(z) := \frac{1}{p_\lambda(-\varrho)} p_\lambda(z).$$

Theorem. $q_\lambda(-z) = \sum_{\mu \in \Lambda} (-1)^{\ell(\mu)} p_\mu(\varrho + \lambda) q_\mu(z).$

Idea of proof

By construction, f_λ has the following cut-off property

$$z \in \varrho + \Lambda, z - \eta \notin \varrho + \Lambda \quad \Longrightarrow \quad f_\eta(z) = 0$$

Key observation: f_λ has the same cut-off property for $-\varrho - \Lambda$:

$$z \in -\varrho - \Lambda, z - \eta \notin -\varrho - \Lambda \quad \Longrightarrow \quad f_\eta(z) = 0$$

Example: D_ℓ for the semiclassical case of rank 5.

$$D_\ell = z_1 + z_3 + z_5 -$$

$$-\frac{(z_1-z_2-r)(z_1-z_4-r)z_1}{(z_1-z_3)(z_1-z_5)}T_{e_1} - \frac{(z_3-z_2-r)(z_3-z_4-r)z_3}{(z_3-z_1)(z_3-z_5)}T_{e_3} - \frac{(z_5-z_2-r)(z_5-z_4-r)z_5}{(z_5-z_3)(z_5-z_1)}T_{e_5}$$

$$-\frac{(z_2-z_5-r)(z_2-z_3-r)(z_1-z_4-r)z_1}{(z_1-z_3)(z_1-z_5)(z_2-z_4)}T_{e_1+e_2} - \frac{(z_4-z_5-r)(z_4-z_3-r)(z_1-z_2-r)z_1}{(z_1-z_3)(z_1-z_5)(z_4-z_2)}T_{e_1+e_4}$$

$$-\frac{(z_2-z_5-r)(z_2-z_1-r)(z_3-z_4-r)z_3}{(z_3-z_1)(z_3-z_5)(z_2-z_4)}T_{e_3+e_2} - \frac{(z_4-z_5-r)(z_4-z_1-r)(z_3-z_2-r)z_3}{(z_3-z_1)(z_3-z_5)(z_4-z_2)}T_{e_3+e_4}$$

$$-\frac{(z_2-z_1-r)(z_2-z_3-r)(z_5-z_4-r)z_5}{(z_5-z_3)(z_5-z_1)(z_2-z_4)}T_{e_5+e_2} - \frac{(z_4-z_1-r)(z_4-z_3-r)(z_5-z_2-r)z_5}{(z_5-z_3)(z_5-z_1)(z_4-z_2)}T_{e_5+e_4}$$

For $v \in V$ consider the Dirac-measure

$$\delta_v \in \mathbb{C}[V]^* \quad \text{with} \quad f \mapsto f(v)$$

Lemma. $M := \bigoplus_{\lambda \in \Lambda} \mathbb{C}\delta_{-\varrho-\lambda} \subseteq \mathbb{C}[V]^*$ is D_h-stable.

Lemma. $\mathbb{C}[V]^W \xrightarrow{\sim} M : h \mapsto \delta_{-\varrho} D_h$

Now, for $h \in \mathbb{C}[V]^W$ consider the expansion

$$h(z) = \sum_{\mu} a_\mu(h) q_\mu(-z)$$

Then

$$a_\mu \in M, \qquad \text{hence} \qquad a_\mu(h) = (D_{g_\mu} h)(-\varrho)$$

Apply to $h = q_\lambda$ and we get

$$q_\lambda(z) = \sum_{\mu} g_\mu(\varrho + \lambda) q_\mu(-z)$$

Now check that $(-1)^{\ell(\mu)} g_\mu$ satisfies the definition of p_μ. qed

Two consequences of the transposition formula

Recall

$$q_\lambda(-z) = \sum_{\mu\in\Lambda} (-1)^{\ell(\mu)} p_\mu(\varrho+\lambda) q_\mu(z) = \sum_{\mu\in\Lambda} \frac{(-1)^{\ell(\mu)}}{p_\mu(-\varrho)} p_\mu(\varrho+\lambda) p_\mu(z).$$

Involutivity. The matrix $\big((-1)^{\ell(\mu)} p_\mu(\varrho+\lambda)\big)_{\lambda,\mu}$ is involutive.

Symmetry Theorem. $q_\lambda(-\varrho-\nu) = q_\nu(-\varrho-\lambda)$

Pieri formulas revisited

Let

$$D_h = \sum_{\eta} b^h_\eta(z) T_\eta.$$

Then

$$D_h(q_\lambda) = h(\varrho + \lambda) q_\lambda(z) \quad \text{at} \quad z = -\varrho - \nu$$

becomes

$$\sum_{\eta} b^h_\eta(-\varrho - \nu) q_\lambda(-\varrho - \nu - \eta) = h(\varrho + \lambda) q_\lambda(-\varrho - \nu).$$

Apply symmetry on both sides

$$\sum_{\eta} b^h_\eta(-\varrho - \nu) q_{\nu+\eta}(-\varrho - \lambda) = h(\varrho + \lambda) q_\nu(-\varrho - \lambda).$$

Thus we get the Pieri formula

$$\sum_{\eta} b_{\eta}^{h}(-\varrho-\nu)q_{\nu+\eta}(z)=h(-z)q_{\nu}(z).$$

Talk of January 6:

$$h(-z)p_{\nu}(z)=\sum_{\eta}(-1)^{\ell(\eta)}\frac{d_{\nu}}{d_{\nu+\eta}}b_{\eta}^{h}(-\varrho-\nu)p_{\nu+\eta}(z).$$

where d_{λ} is the virtual dimension

$$d_{\lambda}=(-1)^{\ell(\lambda)}\frac{f_{\lambda}(-\varrho)}{f_{\lambda}(\varrho+\lambda)}=\prod_{\alpha\in\Delta^{+}}\frac{\alpha(\varrho+\lambda)}{\alpha(\varrho)}\prod_{\omega\in\Phi^{+}}\frac{(\omega(\varrho)+k_{\omega})_{\omega(\lambda)}}{(\omega(\varrho)-k_{\omega}+1)_{\omega(\lambda)}}.$$

Evaluation Theorem. $p_{\lambda}(-\varrho)=(-1)^{\ell(\lambda)}d_{\lambda}$.

Corollary. Let X be a multiplicity free space. Then $\dim \mathcal{P}^\lambda = d_\lambda(\varrho_X)$

Proof: Let f_i be a basis of $\mathcal{P}^\lambda$.

Let $\partial_i \in \mathcal{D}_\lambda$ be the dual basis.

Then $D_\lambda = \sum_i f_i \partial_i$.

Moreover, ${}^tD_\lambda = (-1)^{\ell(\lambda)} \sum_i \partial_i f_i$

Therefore, ${}^tD_\lambda(1) = (-1)^{\ell(\lambda)} \sum_i \partial_i(f_i) = (-1)^{\ell(\lambda)} \dim \mathcal{P}^\lambda$

On the other hand ${}^tD_\lambda(1) = p_{{}^tD_\lambda}(\varrho) = p_{D_\lambda}(-\varrho) = (-1)^{\ell(\lambda)} d_\lambda$. qed

The scalar product

Define

$$\langle p_\lambda, p_\mu \rangle = d_\lambda \delta_{\lambda\mu}$$

With

$$g^-(z) = g(-z)$$

the transposition formula can be reformulated as

$$\langle q_\lambda^-, h \rangle = h(\varrho + \lambda)$$

For an operator Ξ put

$$\Xi^-(g) = \Xi(g^-)^-$$

Then we have the following adjoints

$$D_h^* = D_h$$

$$m_h^* = D_h^-$$

$$L^* = L - 2\ell - L^-$$

$$(L^-)^* = L^-$$

The sl_2-triple

An sl_2-triple (e, f, h) satisfies

$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h.$$

Theorem. $(-L, 2E, L^*)$ is an sl_2-triple.

Theorem. The adjoint action of this triple can be integrated to a $PGL_2(\mathbb{C})$-action on the algebra B generated by all m_h, L, and L^-.

Example: The automorphism $X \mapsto X^-$ corresponds to the matrix

$$\begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Thus

$$q_\lambda^- = (-1)^{\ell(\lambda)} \operatorname{exp}(L^*) q_\lambda.$$

The differential limit

The algebra B is filtered by degree:

$$B = \bigcup_{d=-\infty}^{\infty} B_{\leq d}$$

Associated graded is

$$\overline{B} = \bigoplus_{d=-\infty}^{\infty} B_{\leq d}/B_{<d}$$

Theorem. The algebra $\mathbb{C}[V]^W$ is a module for $\overline{B}$. Each element of $\overline{B}$ acts on $\mathbb{C}[V]^W$ by differential operators.

Theorem. Consider a multiplicity free space X. Then $\mathbb{C}[V]^W = \mathbb{C}[X \oplus X^*]^G$. Moreover,

$$\{\overline{D_h}\} \quad \leftrightarrow \quad \{\text{invariant differential operators on } X \text{ extended to } X \oplus X^*\}$$

$$\overline{L} \quad \leftrightarrow \quad \text{evaluation function } X \oplus X^* \to \mathbb{C}$$

$$\overline{L^*} \quad \leftrightarrow \quad \text{Laplacian on } X \oplus X^*$$

Theorem. The algebra B and $\overline{B}$ are isomorphic. More precisely, let

$$B_d := \text{ } d\text{-eigenspace of } \mathrm{ad}\, E \text{ in } B.$$

Then $\Phi : B_d \xrightarrow{\sim} \overline{B}_d$

Explanation: There are two different ways to embed $\mathbb{C}[V]^W$ into B

$$\begin{array}{ccc} \mathbb{C}[V]^W & \hookrightarrow & \overline{B} \\ & \searrow_j & \downarrow \Phi^{-1} \\ \mathbb{C}[V]^W & \underset{i}{\hookrightarrow} & B \end{array}$$

Consider the B-module $M = \mathbb{C}[V]^W$. Then there are two different identifications of M with $\mathbb{C}[V]^W$

$$h \mapsto i(h)(1) \qquad \text{and} \qquad h \mapsto j(h)(1).$$

Then

$i \rightsquigarrow$ difference operators $\qquad j \rightsquigarrow$ differential operators

More commutative subalgebras

Let $S := \langle L, E, L^* \rangle_{\mathbb{C}}$.

So far, we have the following commutative subalgebras

Commutative algebra	Intersection with S
$\{m_h\}$	$\mathbb{C}(L+E) = \mathbb{C}\ell$
$\{D_h\}$	$\mathbb{C}E$
$\{j(h)\}$	$\mathbb{C}L$
$\{j(h)^*\}$	$\mathbb{C}L^*$

Using the $PGL_2(\mathbb{C})$-action on B one proves:

Theorem. For every line $u \in \mathbf{P}(S) \cong \mathbf{P}^2$ there is a commutative subalgebra A_u of B with $A_u \cong \mathbb{C}[V]^W$ and $A_u \cap S = u$.

Remark: $\{m_h\}$ and $\{D_h\}$ are in the same $PGL_2(\mathbb{C})$-orbit while $\{j(h)\}$ is not.

M as an A_u-module

$u = \mathbb{C}L^*$	$\iff$	A_u^+ acts on M locally nilpotently
$u \subset \langle E, L^* \rangle_{\mathbb{C}} - \mathbb{C}L^*$	$\iff$	the action of A_u on M is diagonalizable
$u \not\subset \langle E, L^* \rangle_{\mathbb{C}}$	$\iff$	$A_u \to M : \Xi \mapsto \Xi(1)$ is an isomorphism

We can play the following game:
Choose $u \subset \langle E, L^* \rangle_{\mathbb{C}} - \mathbb{C}L^*$ and $v \not\subset \langle E, L^* \rangle_{\mathbb{C}}$. Then A_u acts diagonalizable on $A_v = M$.

The action of $\langle E, L^* \rangle_{\mathbb{C}}$ integrates to $M \Longrightarrow$ may assume $u = \mathbb{C}E$.

$u = \mathbb{C}L$	$\rightsquigarrow$	generalized monomials $\overline{p}_\lambda$
u nilpotent, $u \neq \mathbb{C}L, \mathbb{C}L^*$	$\rightsquigarrow$	generalized Laguerre polynomials
u semisimple, $u \subset \langle L, E \rangle_{\mathbb{C}}$	$\rightsquigarrow$	generalized binomials p_λ
u semisimple, $u \not\subseteq \langle L, E \rangle_{\mathbb{C}} \cup \langle E, L^* \rangle_{\mathbb{C}}$	$\rightsquigarrow$	generalized Meixner polynomials

New scalar products

Theorem. The generalized Meixner polynomials can be expressed as

$$\mathcal{M}_\lambda(z;\varrho,c) := \sum_{\mu\in\Lambda} \alpha^{\ell(\mu)} p_\mu(\varrho+\lambda) q_\mu(z) = \alpha^{\ell(\lambda)} \operatorname{exp}(-L^*/\alpha) q_\lambda \qquad \text{with } \alpha = c^{-1}-1$$

They are orthogonal with respect to the scalar product

$$\langle f,g\rangle_c := \sum_{\mu\in\Lambda} c^{\ell(\mu)} d_\mu\, f(\varrho+\mu) g(\varrho+\mu)$$

Theorem. The generalized Laguerre polynomials can be expressed as

$$\mathcal{L}_\lambda(z;\varrho,c) := \sum_{\mu\in\Lambda} p_\mu(\varrho+\lambda)\overline{q}_\mu(z) = \operatorname{exp}(-\overline{L}^*)\overline{q}_\lambda$$

They are orthogonal with respect to the scalar product

$$\langle f,g\rangle_c' := \int_{\mu\in\Lambda_{\mathbb{R}}} e^{-\ell(\mu)} \overline{d}_\mu f(\mu) g(\mu)\, d\mu$$

where

$$\overline{d}_\mu := \lim_{N\to\infty} N^{-\ell_0(\lambda)} d_{N\lambda}$$