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Entanglement Generation in Open Quantum Systems

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Outline

- **Open quantum systems:** systems in weak interaction with their environment : heat baths, external stochastic fields
- These are sources of **noise**, **dissipation** and **decoherence**
- But also of **mediated interactions** between not-directly interacting open quantum systems
- **entanglement** between two open qubits can be **generated** by the **environment**
- This **entanglement** can **persist** asymptotically

References

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Open Quantum Dynamics

- n-level quantum system (S) in interaction with an environment (E): quantum heat bath, classical noise

$$H_{S+E} = H_S^0 \otimes 1_E + 1_S \otimes H_E$$
$$+\lambda \sum_\alpha V_\alpha \otimes B_\alpha$$

$$\lambda \ll 1$$

- Coupling constant
- Bath operators
- Open system operators

n-Level Systems: n x n matrices $M_n(\mathbb{C})$

- Density matrices:

$$\rho \in M_n(\mathbb{C}) \ , \rho \geq 0 , Tr(\rho) = 1$$

Positivity

- Orthonormal basis of **traceless** matrices

$$M_n(\mathbb{C}) \ni F_\alpha \ , \quad Tr(F_\alpha) = 0 \ , \quad \alpha = 1, 2, \ldots n$$

$$Tr(F_\alpha^\dagger F_\beta) = \delta_{\alpha\beta} \ , \quad F_1 = \frac{1}{\sqrt{n}}$$

Environment: system in equilibrium

- **Environment equilibrium state**

$$\rho_E \ , \qquad \Big[H_E \, , \, \rho_E\Big] = 0$$

- **Environment 2-point correlation functions**

$$G_{ij}(t) = Tr\Big(B_i \, B_j(t)\Big) \ , \qquad B_j(t) = \mathrm{e}^{itH_E} \, B_j \, \mathrm{e}^{-itH_E}$$

Open Systems: Reduced Dynamics

- **Global initial state:**

$$\rho_{S+E} = \rho_S \otimes \rho_E$$

- **Global time-evolution:**

$$\rho_S \otimes \rho_E \mapsto \rho_{S+E}(t) = \mathrm{e}^{-itH_{S+E}}\, \rho_S \otimes \rho_E\, \mathrm{e}^{itH_{S+E}}$$

- **Reduced dynamics:**

$$\rho_S \mapsto \rho_S(t) = Tr_E\Big(\rho_{S+E}(t)\Big)$$

$\rho_S \mapsto \rho_S(t)$ is **not** a semigroup

- Memory effects due to the entanglement between S and E in the course of time
- System characteristic timescale τ_S
- Environment characteristic timescale τ_E
- Memoryless approximation when $\frac{\tau_E}{\tau_S} = \lambda^2$

$$\rho_S \mapsto \rho_S(t) = \gamma_t[\rho]$$

$$\gamma_t \circ \gamma_s = \gamma_{t+s} \; , \qquad s, t \geq 0$$

Markovian Approximations

(R.Alicki, K.Lendi: Lec. Notes Phys. 717 (2007))

- Weak-coupling limit
- Singular-coupling limit
- Low density limit

Gorini-Kossakowski-Lindblad Master equation

$$\begin{aligned} \partial_t \rho(t) &= L[\rho(t)] \\ &= -i[H_S\,,\,\rho(t)] \\ &+ \sum_{\alpha,\beta=2}^{n^2} D_{\alpha\beta}\left[F_\alpha\,\rho(t)\,F_\beta^\dagger - \frac{1}{2}\{F_\beta^\dagger\,F_\alpha\,,\,\rho(t)\}\right] \end{aligned}$$

Lamb shifted H_S^0

FRICTION

Kossakowski Matrix

- The entries $D_{\alpha\beta}$ are related to the **Fourier transforms** of the environment correlation functions

$$D_{\alpha\beta}(\omega) = \int_{-\infty}^{+\infty} \mathrm{d}t \, \mathrm{e}^{-i\omega t} \, G_{ij}(t)$$

- ω : characteristic frequency of S
- Kossakowski matrix $D = [D_{\alpha\beta}]_{\alpha,\beta=2}^{n}$

Physical Consistency: Complete Positivity

- **Positivity** of the Kossakowski matrix

$$D := [D_{\alpha\beta}] \geq 0$$

necessary for the **physical consistency** of the semigroup

$$\gamma_t = \mathrm{e}^{t\,L} \ , \ t \geq 0 \ , \ \rho \mapsto \rho(t) = \gamma_t[\rho]$$

- equivalent to **complete positivity** of γ_t

$$\gamma_t \otimes \mathrm{id}[\rho_{\mathrm{ent}}] \geq 0$$

for all **entangled** states ρ_{ent} of $S + S_n$

S_n any n-level ancilla

Single Open Qubit

- **S(ystem)+E(nvironment):**

$$H_{S+E} = \underbrace{\left(\frac{\omega_0}{2}\sum_{i=1}^{3} n_i\sigma_i\right)}_{H_S^0} \otimes 1_E + 1_S \otimes H_E + \lambda \sum_{i=1}^{3} \sigma_i \otimes B_i$$

- **Lindblad equation for 1 qubit density matrices:**

$$\partial_t \rho(t) = -i[H_S\,,\,\rho(t)] + \sum_{i,j=1}^{3} D_{ij}\left[\sigma_i\rho(t)\sigma_j - \frac{1}{2}\{\sigma_j\sigma_i\,,\,\rho(t)\}\right]$$

$$D = \begin{pmatrix} D_{11} & D_{12} & D_{13} \\ D_{21} & D_{22} & D_{13} \\ D_{31} & D_{32} & D_{33} \end{pmatrix} \geq 0$$

Two Open Qubits

- S(1)+S(2)+E:

$$H_{S+E} = \underbrace{\left(\frac{\omega_0}{2}\sum_{i=1}^{3} n_i\sigma_i\right)}_{H_1^0} \otimes (1_{S_2} \otimes 1_E)$$

$$+1_{S_1} \otimes \underbrace{\left(\frac{\omega_0}{2}\sum_{i=1}^{3} n_i\sigma_i\right)}_{H_2^0} \otimes 1_E$$

$$+1_{S_1} \otimes 1_{S_2} \otimes H_E$$

$$+\lambda\sum_{i=1}^{3}(\sigma_i \otimes 1_2) \otimes B_i + (1_1 \otimes \sigma_i) \otimes B_{i+1}$$

Lindblad equation: 2 non-directly interacting open qubits

$$\partial_t \rho(t) = \mathbb{L}[\rho(t)] = \mathbb{L}_H[\rho(t)] + \mathbb{D}[\rho(t)]$$

$$= -i[H_1 \otimes 1_2 + 1_1 \otimes H_2 \, , \, \rho(t)]$$

$A = [A_{ij}]$

$C = [C_{ij}]$

$\mathbb{D}[\rho(t)]$:

$B = [B_{ij}]$

$B^\dagger = [B^*_{ji}]$

$$+ \sum_{i,j=1}^{3} A_{ij}[(\sigma_i \otimes 1_2)\rho(t)(\sigma_j \otimes 1_2) - \frac{1}{2}\{\sigma_j\sigma_i \otimes 1_2 \, , \, \rho(t)\}]$$

$$+ \sum_{i,j=1}^{3} C_{ij}[(1_1 \otimes \sigma_i)\rho(t)(1_1 \otimes \sigma_j) - \frac{1}{2}\{1_1 \otimes \sigma_j\sigma_i \, , \, \rho(t)\}]$$

$$+ \sum_{i,j=1}^{3} B_{ij}[(\sigma_i \otimes 1_2)\rho(t)(1_1 \otimes \sigma_j) - \frac{1}{2}\{\sigma_i \otimes \sigma_j \, , \, \rho(t)\}]$$

$$+ \sum_{i,j=1}^{3} B^*_{ji}[(1_1 \otimes \sigma_i)\rho(t)(\sigma_j \otimes 1_2) - \frac{1}{2}\{\sigma_j \otimes \sigma_i \, , \, \rho(t)\}]$$

$$\mathbb{D}[\rho(t)] = \sum_{\alpha,\beta=1}^{3} D_{\alpha\beta} \left[\sigma_{(\alpha)}\, \rho(t)\, \sigma_{(\beta)} - \frac{1}{2}\{\sigma_{(\beta)}\, \sigma_{(\alpha)}\, ,\, \rho(t)\}\right]$$

$$\begin{array}{rclr} \sigma_{(\alpha)} & = & \sigma_\alpha \otimes 1_2 & \alpha = 1,2,3 \\ \sigma_{(\alpha)} & = & 1_1 \otimes \sigma_{\alpha-3} & \alpha = 4,5,6 \end{array}$$

$$D = \begin{pmatrix} A & B \\ B^\dagger & C \end{pmatrix} \geq 0$$

Despite noise and dissipation, can

$$\partial_t \rho(t) = -i[H_1 \otimes 1_2 + 1_1 \otimes H_2\, ,\, \rho(t)] + \mathbb{D}[\rho(t)]$$

generate entanglement ?

Sufficient Condition

(F.B., R. Floreanini, M. Piani, PRL 2003)

- an initial **2 qubit** separable state $|\phi_1\rangle\langle\phi_1| \otimes |\chi_1\rangle\langle\chi_1|$
  gets **entangled** as soon as $t > 0$ **if**

$$\langle u| A |u\rangle \langle v| C^T |v\rangle < |\langle u| Re(B) |v\rangle|^2$$

$$(Re(B))_{ij} = \frac{1}{2}(B_{ij} + B^*_{ij})$$

$$|u\rangle = \begin{pmatrix} \langle\phi_1|\sigma_1|\phi_2\rangle \\ \langle\phi_1|\sigma_2|\phi_2\rangle \\ \langle\phi_1|\sigma_3|\phi_2\rangle \end{pmatrix}, \ \phi_1 \perp \phi_2 \qquad |v\rangle = \begin{pmatrix} \langle\chi_2|\sigma_1|\chi_1\rangle \\ \langle\chi_2|\sigma_2|\chi_1\rangle \\ \langle\chi_2|\sigma_3|\chi_1\rangle \end{pmatrix}, \ \chi_1 \perp \chi_2$$

Idea for the proof: Step 1

- use **partial transposition** $\mathrm{id} \otimes T$ to check whether

$$(\mathrm{id} \circ T) \circ \gamma_t [|\phi_1\rangle\langle\phi_1| \otimes |\chi_1\rangle\langle\chi_1|] \geq 0$$

- **Partial transposition** is an **exhaustive entanglement witness** for **two** qubits

How to identify 2 qubits entanglement: partial transposition

- **transposition on 1 qubit:** $T : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & c \\ b & d \end{pmatrix}$

- **partial transposition on 2 qubits:**

$$\mathrm{id} \otimes T : M_2(\mathbb{C}) \otimes M_2(\mathbb{C}) \mapsto M_2(\mathbb{C}) \otimes M_2(\mathbb{C})$$

$$\mathrm{id} \otimes T \; |\Psi_{00}\rangle\langle\Psi_{00}| = \frac{|00\rangle + |11\rangle}{\sqrt{2}} \frac{\langle 00| + \langle 11|}{\sqrt{2}} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \frac{1}{2}\Big(|0\rangle\langle 0| \otimes |0\rangle\langle 0| + |1\rangle\langle 1| \otimes |1\rangle\langle 1| + |0\rangle\langle 1| \otimes |0\rangle\langle 1| + |1\rangle\langle 0| \otimes |1\rangle\langle 0| \Big)$$

$$= \frac{1}{2}\left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right]$$

Idea for a proof: Step 2

Since

$$(\mathrm{id}\otimes T)[|\phi_1\rangle\langle\phi_1|\otimes|\chi_1\rangle\langle\chi_1|] = |\phi_1\rangle\langle\phi_1|\otimes|\chi_1^*\rangle\langle\chi_1^*|$$

one studies the semigroup

$$g_t := (\mathrm{id}\otimes T)\circ\gamma_t\circ(\mathrm{id}\otimes T) = \mathrm{e}^{t\,\mathbb{G}}$$

with generator

$$\mathbb{G}[\rho] = -i[\widetilde{H}\,,\,\rho(t)] + \mathbb{R}[\rho(t)]$$

$$\mathbb{R}[\rho] = \sum_{\alpha,\beta=1}^{6} Q_{\alpha\beta}\Big[\sigma_{(\alpha)}\,\rho\,\sigma_{(\beta)} - \frac{1}{2}\{\sigma_{(\beta)}\,\sigma_{(\alpha)}\,,\,\rho\}\Big]$$

The Kossakowski Matrix

$$Q = \begin{pmatrix} A & Re(B) \\ Re(B^T) & C^T \end{pmatrix}$$

need not be **positive**

$$g_t = (\mathrm{id} \otimes T) \circ \gamma_t \circ (\mathrm{id} \otimes T)$$

need not preserve the positivity of

$$(\mathrm{id} \otimes T)[|\phi_1\rangle\langle\phi_1| \otimes |\chi_1\rangle\langle\chi_1|]$$

$$(\mathrm{id} \otimes T) \circ \gamma_t[|\phi_1\rangle\langle\phi_1| \otimes |\chi_1\rangle\langle\chi_1|]$$

need not be positive

$$T[|\chi_1\rangle\langle\chi_1|] = |\chi_1^*\rangle\langle\chi_1^*| \qquad |\chi_1^*\rangle = \begin{pmatrix} \chi_1^*(0) \\ \chi_1^*(1) \end{pmatrix}$$

$$\mathcal{E}_{\psi,\phi_1,\chi_1}(t) := \langle\psi|\, g_t[|\phi_1\rangle\langle\phi_1| \otimes |\chi_1^*\rangle\langle\chi_1^*|]|\psi\rangle$$

$$\mathcal{E}_{\psi,\phi_1,\chi_1}(0) = |\langle\psi|\phi_1 \otimes \chi_1^*\rangle|^2 = 0$$

$$\partial_t \mathcal{E}_{\psi,\phi_1,\chi_1}(0) = \sum_{\alpha,\beta=1}^{6} Q_{\alpha\beta}\ \langle\psi|\sigma_{(\alpha)}\left(|\phi_1\rangle\langle\phi_1| \otimes |\chi_1^*\rangle\langle\chi_1^*|\right)\sigma_{(\beta)}\,|\psi\rangle < 0$$

$$\Rightarrow \quad \mathcal{E}_{\psi,\phi,\chi_1}(t) < 0 \quad t \to 0^+$$

and $|\phi_1\rangle \otimes |\chi_1\rangle$ **get entangled**

Particular Case: $A = B = C \geq 0$

- **choose** $\phi_1 = \chi_2 \Longrightarrow |u\rangle = |v\rangle$
- **the sufficient condition becomes**

$$\langle u|A|u\rangle\langle u|A^T|u\rangle \; < \; |\langle u|Re(A)|u\rangle|^2$$

$$(\langle u|Im(A)|u\rangle)^2 \, > \, 0 \; , \; A = A^\dagger \; , \; Im(A) := \frac{1}{2}(A - A^T)$$

Example:

$$A = \begin{pmatrix} a_1 & i\,b & 0 \\ -i\,b & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix}\ ,\ a_{1,2,3} \geq 0\ ,\ a_1 a_2 \geq b^2$$

$$Im(A) = \begin{pmatrix} 0 & i\,b & 0 \\ -i\,b & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$|\phi_1\rangle = |\chi_2\rangle = |-\rangle\ ,\ \sigma_3|\pm\rangle = \pm|\pm\rangle \quad \Longrightarrow \quad |u\rangle = \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix}$$

$$\Longrightarrow \quad \langle u|Im(A)|u\rangle)^2 = 4\,b^2 > 0$$

If [illegible]) , $\quad |-\rangle\langle-| \otimes |+\rangle\langle+|$

gets entangled for small times

Two atoms in a scalar thermal field

qubit variables $\sigma_i^{(1,2)}\,,\ i=1,2,3$

linearly coupled to scalar field variables

$$F_i(x) \qquad x=(x_1,x_2,x_3,t)$$

with space-time **translation-invariant**

thermal two-point correlation functions at temperature β^{-1}

$$\langle F_i(x)\,F_j(y)\rangle = \delta_{ij} G(x-y)$$

$$= \delta_{ij}\int \frac{d^4k}{(2\pi)^3}\theta(k^0)\delta(k^2)\,\Big(\frac{\mathrm{e}^{-ik(x-y)}}{1-e^{-\beta k^0}}+\frac{\mathrm{e}^{+ik(x-y)}}{\mathrm{e}^{\beta k^0}-1}\Big)$$

Qubits + Scalar Field: Full Hamiltonian

- use smeared field variables

$$F_i(f) = \int_{\mathbf{R}^3} \mathrm{dx}\, f(x)\, F_i(x)$$

- with localized smearing functions

$$f(x) = \frac{1}{\pi^2} \frac{\varepsilon/2}{x^2 + (\varepsilon/2)^2}$$

Localized qubit-field interactions

- **Total Hamiltonian**

$$H_{S+E} = H_1^0 + H_2^0 + H_E$$
$$+\lambda \sum_{i=1}^{3} ((\sigma_i \otimes 1_2) + (1_1 \otimes \sigma_i)) \otimes F_i(f)$$

- **Qubit Hamiltonian**

$$H_1^0 = H_2^0 = \frac{\omega_0}{2} \sum_{i=1}^{3} n_i \, \sigma_i$$

Kossakowski matrix D explicitly calculable

$$D = \begin{pmatrix} A & A \\ A & A \end{pmatrix}$$

$$A = \begin{pmatrix} a + c\,n_1^2 & c\,n_1 n_2 - i\,b\,n_3 & c\,n_1 n_3 + i\,b\,n_2 \\ c\,n_1 n_2 + i\,b\,n_3 & a + c\,n_2^2 & c\,n_2 n_3 - i\,b\,n_1 \\ c\,n_1 n_3 - i\,b\,n_2 & c\,n_2 n_3 + i\,b\,n_1 & a + c\,n_3^2 \end{pmatrix}$$

$$a = \frac{\omega}{4\pi}\,\frac{1 + \mathrm{e}^{-\beta\omega}}{1 - \mathrm{e}^{-\beta\omega}} \qquad b = \frac{\omega}{4\pi} \qquad c = \frac{1}{2\pi\beta} - \frac{\omega}{4\pi}\,\frac{1 + \mathrm{e}^{-\beta\omega}}{1 - \mathrm{e}^{-\beta\omega}}$$

Can entanglement created irreversibly survive decoherence?

YES

Moreover, a state initially entangled can remain entangled asymptotically and even become more entangled

F.B., R. Floreanini (Int. J. Quant. Inf. 2006)

Quantifying Entanglement: Concurrence

- **2 qubits entanglement content :**

$$\rho \mapsto \hat{\rho} := (\sigma_2 \otimes \sigma_2)\, \rho^* \,(\sigma_2 \otimes \sigma_2) \;\mapsto\; R = \rho\, \hat{\rho}$$

- **spectrum(R) = spectrum($\sqrt{\rho}\hat{\rho}\sqrt{\rho}$)=** $\lambda_1^2 \geq \lambda_2^2 \geq \lambda_3^2 \geq \lambda_4^2$

- **concurrence:**

$$C(\rho) := \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}$$

Any initial state

$$\rho = \frac{1}{4}\left(1_1 \otimes 1_2 + \sum_{i=1}^{3} \rho_{0i}\, 1_1 \otimes \sigma_i + \sum_{i=1}^{3} \rho_{i0}\, \sigma_i \otimes 1_2 + \sum_{i,j=1}^{3} \rho_{ij}\, \sigma_i \otimes \sigma_j\right)$$

goes into

$$\rho_\infty = \frac{1}{4}\left(1_1 \otimes 1_2 \quad - \sum_{i=1}^{3} \frac{R\, n_i(\tau+3)}{3+R^2}(1_1 \otimes \sigma_i + \sigma_i \otimes 1_2)\right.$$

$$\left. + \sum_{i=1}^{3} \frac{\tau - R^2(1-(\tau+3)n_i^2)}{2(3+R^2)}\, \sigma_i \otimes \sigma_i \; + \sum_{i \neq j} \frac{R^2(\tau+3)\, n_i\, n_j}{2(3+R^2)}\, \sigma_i \otimes \sigma_j\right)$$

$$\tau = \sum_{i=1}^{3} \mathrm{Tr}(\rho\, \sigma_i \otimes \sigma_i)$$

$$0 \leq R = \frac{b}{a} = \frac{1-\mathrm{e}^{-\beta\omega}}{1+\mathrm{e}^{-\beta\omega}} \leq 1$$

Asymptotic Concurrence:

$$C(\rho_\infty) = \frac{3-R^2}{2(3+R^2)}\left[\frac{5R^2-3}{3-R^2} - \tau\right]$$

- **initial state:** $\rho = \frac{s}{4}\, 1_1 \otimes 1_1 + (1-s)\,|\Psi_{01}\rangle\langle\Psi_{01}|$
- **concurrence:** $C(\rho) = 1 - \frac{3s}{2}\ , \quad s < \frac{2}{3}$
- **asymptotic gain:** $C(\rho_\infty) - C(\rho) = \frac{3R^2\, s}{3+R^2}$

Two atoms separated by a distance L

- **interaction Hamiltonian:**

$$H_{int} = \sum_{i=1}^{3} \left(\sigma_i^{(1)} \otimes F_i(f_1) + \sigma_i^{(2)} \otimes F(f_2) \right)$$

- **smearing functions:**

$$f_1(x) = \frac{1}{\pi^2} \frac{\varepsilon/2}{x^2 + (\varepsilon/2)^2}$$

$$f_2(x) = f(x + L)$$

Kossakowski Matrix $D = \begin{pmatrix} A & A' \\ A' & A \end{pmatrix}$

$$A = \begin{pmatrix} a + c\,n_1^2 & c\,n_1 n_2 - i\,b\,n_3 & c\,n_1 n_3 + i\,b\,n_2 \\ c\,n_1 n_2 + i\,b\,n_3 & a + c\,n_2^2 & c\,n_2 n_3 - i\,b\,n_1 \\ c\,n_1 n_3 - i\,b\,n_2 & c\,n_2 n_3 + i\,b\,n_1 & a + c\,n_3^2 \end{pmatrix}$$

$$a = \frac{\omega}{4\pi}\frac{1 + e^{-\beta\omega}}{1 - e^{-\beta\omega}} \qquad b = \frac{\omega}{4\pi} \qquad c = \frac{1}{2\pi\beta} - \frac{\omega}{4\pi}\frac{1 + e^{-\beta\omega}}{1 - e^{-\beta\omega}}$$

$$A' = \begin{pmatrix} a' + c\,n_1^2 & c'\,n_1 n_2 - i\,b'\,n_3 & c'\,n_1 n_3 + i\,b'\,n_2 \\ c'\,n_1 n_2 + i\,b'\,n_3 & a' + c\,n_2^2 & c'\,n_2 n_3 - i\,b'\,n_1 \\ c'\,n_1 n_3 - i\,b'\,n_2 & c'\,n_2 n_3 + i\,b'\,n_1 & a' + c'\,n_3^2 \end{pmatrix}$$

$$a' = \frac{\omega}{4\pi}\frac{1 + e^{-\beta\omega}}{1 - e^{-\beta\omega}}\frac{\sin\omega L}{\omega L} \qquad b' = \frac{\omega}{4\pi}\frac{\sin(\omega L)}{\omega L} \qquad c' = \frac{1}{2\pi\beta} - \frac{\omega}{4\pi}\frac{1 + e^{-\beta\omega}}{1 - e^{-\beta\omega}}\frac{\sin(L)}{\omega L}$$

Entanglement Creation

- **separable initial state:** $\rho = |-\rangle - | \otimes |+\rangle\langle+|$
- **sufficient condition:**

$$\langle u|A|u\rangle\langle v|A^T|v\rangle \ < \ |\langle u|Re(A')|v\rangle|^2$$

- **with** $|u\rangle = |v\rangle = (1, -i, 0)$ **it becomes**

$$R^2 + S^2 > 1$$

$$R = \frac{b}{a} = \frac{1 - \mathrm{e}^{-\beta\omega}}{1 + \mathrm{e}^{-\beta\omega}}$$

$$S = \frac{\sin(\omega L)}{\omega L}$$

- **no entanglement asymptotically if L>0**