

DRAFT

Quantum Entropy

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von Neumann (1927) defined mixed quantum state and its entropy

$$\begin{aligned} S(\rho) &= -\text{Tr } \rho \log \rho \\ &= \sum_k \lambda_k \log \lambda_k \quad \text{evals of } \rho \end{aligned}$$

Density matrix $\rho > 0$ and $\text{Tr } \rho = 1 \Rightarrow S(\rho) \geq 0$

also find ρ pure $\Leftrightarrow \rho^2 = \rho \Leftrightarrow S(\rho) = 0$

But $S(P)$ well-defined for any pos semi-def op P

$S(\rho) \geq 0$ is result of normalization and/or phys interp

Shannon (1948): classical info with entropy equiv. to diag matrix

Fundamental Properties of Quantum Entropy

Concave: $x S(\rho_1) + (1 - x)S(\rho_2) \leq S(x\rho_1 + (1 - x)\rho_2)$

Subadditive: $S(\rho_{AB}) \leq S(\rho_A) + S(\rho_B)$

with $= \Leftrightarrow \rho_{AB} = \rho_A \otimes \rho_B$

Strongly Subadditive $S(\rho_B) + S(\rho_{ABC}) \leq S(\rho_{AB}) + S(\rho_{BC})$

All indep of normalization as long as consistent, e.g.,

$$\text{Tr } \rho_B = \text{Tr } \rho_{AB} = \text{Tr } \rho_{ABC}$$

Note: $S(\mu P) = \mu S(P) - \mu \log \mu \text{Tr } P$

Concave: $x S(\rho_1) + (1 - x)S(\rho_2) \leq S(x\rho_1 + (1 - x)\rho_2)$

refers to mixture

Need Fig

Subadditive: $S(\rho_{AB}) \leq S(\rho_A) + S(\rho_B)$

refers to regions
or subsystems

Need Fig

SSA: $S(\rho_B) + S(\rho_{ABC}) \leq S(\rho_{AB}) + S(\rho_{BC})$

overlapping regions

Need Fig

Aside: Subadditive \Rightarrow Concave

$$\rho_{AB} = \begin{pmatrix} x\rho_1 & 0 \\ 0 & (1-x)\rho_2 \end{pmatrix}$$

$$\rho_A = x\rho_1 + (1-x)\rho_2, \quad \rho_B = \begin{pmatrix} x & 0 \\ 0 & 1-x \end{pmatrix}$$

$$\begin{aligned} S(\rho_{AB}) &= S(x\rho_1) + S((1-x)\rho_2) \\ &= xS(\rho_1) - x \log x + (1-x)S(\rho_2) - (1-x) \log(1-x) \end{aligned}$$

$$\begin{aligned} S(\rho_{AB}) &\leq S(\rho_A) + S(\rho_B) \\ &= S(x\rho_1 + (1-x)\rho_2) - x \log x - (1-x) \log(1-x) \end{aligned}$$

$$\Rightarrow xS(\rho_1) + (1-x)S(\rho_2) \leq S(x\rho_1 + (1-x)\rho_2)$$

Relative entropy $H(P, Q) = \text{Tr } P \log P - \text{Tr } P \log Q$

Klein's ineq: $H(P, Q) \geq \text{Tr}(P - Q) \log \frac{\text{Tr}(P - Q)}{\text{Tr } P} \geq 0$ if $\text{Tr } P = \text{Tr } Q$

assume $P, Q > 0$ strictly pos — well-def if $\ker(Q) \subset \ker(P)$

Can obtain entropy from rel ent. $H(P, \frac{1}{d}I) = -S(P) - \log d$

homogenous of degree one $H(\lambda P, \lambda Q) = \lambda H(P, Q)$

Aside: for homo fctns convex \Leftrightarrow subadd.

$$g(xA + (1-x)B) \leq g(xA) + g((1-x)B) = xg(A) + (1-x)g(B)$$

$$\frac{1}{2}g(A + B) = g\left(\frac{1}{2}(A + B)\right) \leq \frac{1}{2}g(A) + \frac{1}{2}g(B) \quad \text{cancel } \frac{1}{2}$$

Fundamental Properties of Relative Entropy

Joint Convexity $H(P_1 + P_2, Q_1 + Q_2) \leq H(P_1, Q_1) + H(P_2, Q_2)$

Monotone under Partial Trace $H(P_A, Q_A) \leq H(P_{AB}, Q_{AB})$

Monotone under CPT Map $H[\Phi(P), \Phi(Q)] \leq H(P, Q)$

Special Case of MPT $P_{AB} \rightarrow \rho_{ABC}, Q_{AB} \rightarrow I_A \otimes \rho_{BC}$

gives SSA $H(\rho_{AB}, \rho_B) \leq H(\rho_{ABC}, \rho_{BC})$

$$-S(\rho_{AB}) + S(\rho_B) \leq -S(\rho_{ABC}) + S(\rho_{BC})$$

MonoCPT with $\Phi = \text{Tr}_B \Rightarrow \text{MPT} \Rightarrow \text{SSA} \Rightarrow \text{JC}$

Will show $\text{JC} \Rightarrow \text{MPT} \Rightarrow \text{MonoCPT}$

Information Theory Expressions

Mutual Information: (always positive)

$$S(A : B) = S(\rho_A) + S(\rho_B) - S(\rho_{AB}) = H(\rho_{AB}, \rho_A \otimes \rho_B) \geq 0$$

Conditional Information: (always positive for classical systems)

$$\begin{aligned} S(A|B) &= S(\rho_{AB}) - S(\rho_B) = -H(\rho_{AB}, \rho_B) \\ &= -H(\rho_{AB}, \frac{1}{d} I_A \otimes \rho_B) + \log d \end{aligned}$$

But for max entangled quant state $S(\rho_{AB}) - S(\rho_B) = -\log d < 0$

HOW (M. Horodecki, J. Oppenheim, A. Winter) interpretation:

Nature **436**, 673–676 (2005); CMP **269**, (2007). [quant-ph/0512247](https://arxiv.org/abs/quant-ph/0512247).

Cond info measures # of bits Alice needs to transmit message to Bob when he has partial info – same in class. and quantum info

When negative, gives # of EPR pairs A and B have left for future

write density matrix $\rho = \sum_k \lambda_k |\phi_k\rangle\langle\phi_k|$ spectral decomp
 $\{g_k\}$ O.N. $|\psi\rangle = \sum_k \sqrt{\lambda_k} |\phi_k\rangle \otimes |g_k\rangle \Rightarrow \text{Tr}_B |\psi\rangle\langle\psi| = \rho$

For pure state $\rho_A = \text{Tr}_B |\psi\rangle\langle\psi|$ and $\rho_B = \text{Tr}_A |\psi\rangle\langle\psi|$
 same e-vals $\Rightarrow S(\rho_A) = S(\rho_B)$

apply to $S(\rho_{AB}) \leq S(\rho_A) + S(\rho_B)$ purified to $|\psi_{ABC}\rangle$

$$S(\rho_C) \leq S(\rho_A) + S(\rho_{AC}) \Rightarrow$$

Triangle Ineq: $|S(\rho_C) - S(\rho_A)| \leq S(\rho_{AC}) \leq S(\rho_A) + S(\rho_C)$

Purify SSA: $S(\rho_B) + S(\rho_{ABC}) \leq S(\rho_{AB}) + S(\rho_{BC})$ with $|\psi_{ABCD}\rangle$

$$S(\rho_B) + S(\rho_D) \leq S(\rho_{AB}) + S(\rho_{AD}) \quad \forall \rho_{ABD}$$

Note equality for pure ρ_{ABD}

Can have $S(\rho_B) \geq S(\rho_{AB})$ but **not for a pair from tripartite** ρ_{ABD}

Joint convexity of $H(P, Q)$ follows from:

$$(I) \quad H(P, Q) = \int_0^{\infty} \text{Tr} (Q - P) \frac{1}{L_Q + tR_P} (Q - P) \frac{1}{(1+t)^2} dt$$

$$(II) \quad (A, P, Q) \mapsto \text{Tr} A^\dagger \frac{1}{L_Q + tR_P} A \text{ is jointly convex in } A, P, Q.$$

where $L_Q(A) = QA$ and $R_P(A) = AP$

Result follows by using (II) in (I) with $A = Q - P$.

Ruskai, quant-ph/0604206 *Reports on Math. Phys.* (2007).

“Another Short and Elementary Proof of Strong Subadditivity of Quantum Entropy”
based on Lesniewski-Ruskai *J. Math. Phys.* **40**, 5702–5724 (1999).

$d \times d$ matrices form Hilbert space with $\langle A, B \rangle = \text{Tr } A^\dagger B$

for lin op $\Phi : M_d \mapsto M_d$ let $\widehat{\Phi}$ denote adjoint

$$\text{Tr } A^\dagger \Phi(B) = \langle A, \Phi(B) \rangle = \langle \widehat{\Phi}(A), B \rangle = \text{Tr} [\widehat{\Phi}(A)]^\dagger B$$

Left and right mult are just linear operators on this vector space

$$L_Q(A) = QA \quad \text{and} \quad R_P(A) = AP$$

a) L_P and R_Q commute since $L_P[R_Q(A)] = PAQ = R_Q[L_P(A)]$

b) $P = P^\dagger \Rightarrow L_P$ self-adjoint in fact, $\widehat{L_P} = L_{P^\dagger}$ and $\widehat{R_P} = R_{P^\dagger}$

c) $P \geq 0 \Rightarrow L_P$ and R_P are pos semi-def

$$\langle A, R_P(A) \rangle = \text{Tr } A^\dagger R_P(A) = \text{Tr } A^\dagger AP = \text{Tr } APA^\dagger \geq 0$$

d) $(L_P)^{-1} = L_{P^{-1}}$ and $(R_Q)^{-1} = R_{Q^{-1}}$

simple form of deep idea: Araki $\Delta_{PQ} = L_{P^{-1}}R_Q$ relative modular op

Proof of II: Let $M = (\)^{-1/2}(A) - (\)^{1/2}(X)$

$$\begin{aligned}\text{Tr } M^\dagger M &= \langle M, M \rangle \\ &= \langle [(\)^{-1/2}(A) - (\)^{1/2}(X)], [(\)^{-1/2}(A) - (\)^{1/2}(X)] \rangle \\ &= \langle A, (\)^{-1}(A) \rangle - \langle A, X \rangle - \langle X, A \rangle + \langle X, (\)(X) \rangle\end{aligned}$$

Choose $M = (L_P + tR_Q)^{-1/2}(A) - (L_P + tR_Q)^{1/2}(X)$

$$\begin{aligned}\text{Tr } M^\dagger M &= \\ &\text{Tr } A^\dagger (L_P + tR_Q)^{-1}(A) - \text{Tr } A^\dagger X - \text{Tr } X^\dagger A + \text{Tr } X^\dagger (L_P + tR_Q)(X)\end{aligned}$$

Let $M_j = (L_{P_j} + tR_{Q_j})^{-1/2}(A_j) - (L_{P_j} + tR_{Q_j})^{1/2}(X)$. Then

$$0 \leq \sum_j \operatorname{Tr} M_j^\dagger M_j = \sum_j \operatorname{Tr} A_j^\dagger (L_{P_j} + tR_{Q_j})^{-1}(A_j) \\ - \operatorname{Tr} (\sum_j A_j^\dagger) X - \operatorname{Tr} X^\dagger (\sum_j A_j) + \operatorname{Tr} X^\dagger \sum_j (L_{P_j} + tR_{Q_j}) X$$

Choose $X = \frac{1}{L_{\sum_j P_j} + tR_{\sum_j Q_j}} (\sum_j A_j)$. Use $\sum_j L_{P_j} = L_{\sum_j P_j}$

$$\operatorname{Tr} X^\dagger \sum_j (L_{P_j} + tR_{Q_j}) X = \operatorname{Tr} (\sum_j A_j^\dagger) \frac{1}{L_{\sum_j P_j} + tR_{\sum_j Q_j}} (\sum_j A_j) \\ = \operatorname{Tr} (\sum_j A_j^\dagger) X = \operatorname{Tr} X^\dagger (\sum_j A_j)$$

$$0 \leq \sum_j \operatorname{Tr} A_j^\dagger \frac{1}{L_{P_j} + tR_{Q_j}} (A_j) - \operatorname{Tr} (\sum_j A_j^\dagger) \frac{1}{L_{\sum_j P_j} + tR_{\sum_j Q_j}} (\sum_j A_j)$$

compare elem. C-S ineq: $\left| \sum_k \bar{v}_k w_k \right|^2 \leq \sum_k |v_k|^2 \sum_k |w_k|^2$

For $p_k > 0$ let $v_k = p_k^{1/2}$, $w_k = p_k^{-1/2} a_k$

$$\left| \sum_k a_k \right|^2 \leq \sum_k p_k \sum_k \bar{a}_k \frac{1}{p_k} a_k$$

Rewrite $\left(\sum_k \bar{a}_k \right) \frac{1}{\sum_k p_k} \left(\sum_k a_k \right) \leq \sum_k \bar{a}_k \frac{1}{p_k} a_k$

Lieb and Ruskai (1973) proved operator version

$$\left(\sum_k A_k^\dagger \right) \frac{1}{\sum_k P_k} \left(\sum_k A_k \right) \leq \sum_k A_k^\dagger \frac{1}{P_k} A_k$$

Not suff. for SSA — need Araki rel mod op hidden in L_Q and R_P .

Compare proof: $\left| \sum_k v_k + t w_k \right|^2 \geq 0 \quad \forall t$ choose t to minimize

$$\begin{aligned}
& H\left(\sum_k P_k, \sum_k Q_k\right) \\
&= \int_0^\infty \text{Tr} \sum_k (Q_k - P_k) \frac{1}{L_{\sum_k P_k} + tR_{\sum_k Q_k}} \sum_k (Q_k - P_k) \frac{1}{(1+t)^2} dt \\
&\leq \int_0^\infty \sum_k \text{Tr} (Q_k - P_k) \frac{1}{L_{P_k} + tR_{Q_k}} (Q_k - P_k) \frac{1}{(1+t)^2} dt \\
&= \sum_k H(P_k, Q_k)
\end{aligned}$$

Only need to verify integral rep. Also works if $\frac{1}{(1+t)^2}$ replaced by $g(t) \geq 0$. Large class of gen. rel ent related to convex op functions

Functions of operators

For $A = UDU^\dagger$, define $f(A) = U f(D) U^\dagger$

$$A = U \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \dots & 0 & \lambda_m \end{pmatrix} U^\dagger \quad f(A) = U \begin{pmatrix} f(\lambda_1) & 0 & \dots & 0 \\ 0 & f(\lambda_2) & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \dots & 0 & f(\lambda_m) \end{pmatrix} U^\dagger$$

equiv. to any reasonable def using power series, integral rep., etc.

also applies to operators, e.g., L_Q acting on M_d space of matrices

In particular, $\text{Tr } L_{\log Q}(P) = \text{Tr}(\log L_Q)(P)$

Integral representation

$$\begin{aligned} -\log w &= \int_0^\infty \left[\frac{1}{w+t} - \frac{1}{1+t} \right] dt = \int_0^\infty \frac{1}{w+t} (1-w) \frac{1}{1+t} dt \\ &= (1-w) + \int_0^\infty \frac{(w-1)^2}{w+t} \frac{1}{(1+t)^2} dt \end{aligned}$$

$$\text{Tr } P \log Q = \text{Tr}(\log Q)P = \text{Tr } L_{\log Q}(P) = \text{Tr}(\log L_Q)(P)$$

$$\begin{aligned} H(P, Q) &= \text{Tr } P(\log P - \log Q) = -\text{Tr}(\log R_P^{-1})(P) - \text{Tr}(\log L_Q)(P) \\ &= -\text{Tr}[\log(L_Q R_P^{-1})](P) \\ &= \text{Tr}(I - L_Q R_P^{-1})(P) + \\ &\text{Tr} \int_0^\infty (L_Q R_P^{-1} - I) \frac{1}{L_Q R_P^{-1} + tI} (L_Q R_P^{-1} - I)(P) \frac{1}{(1+t)^2} dt \end{aligned}$$

$$(L_Q R_P^{-1} - I)(P) = Q - P \quad \text{Tr}(I - L_Q R_P^{-1})(P) = \text{Tr}(P - Q) = 0$$

$$\begin{aligned} H(P, Q) &= \int_0^\infty \text{Tr}(L_Q R_P^{-1} - I) \frac{1}{L_Q R_P^{-1} + tI} (Q - P) \frac{1}{(1+t)^2} dt \\ &= \int_0^\infty \text{Tr}(L_Q - R_P) \frac{1}{L_Q + tR_P} (Q - P) \frac{1}{(1+t)^2} dt \\ &= \int_0^\infty \text{Tr}(Q - P) \frac{1}{L_Q + tR_P} (Q - P) \frac{1}{(1+t)^2} dt \end{aligned}$$

Used

$$\begin{aligned} \text{Tr}(L_Q - R_P)(B) &= \langle I, (L_Q - R_P)(B) \rangle = \langle (L_Q - R_P)(I), B \rangle \\ &= \langle (Q - P), B \rangle = \text{Tr}(Q - P)B \end{aligned}$$

Completes proof of joint convexity of $H(P, Q)$

Prove MPT: Recall gen Pauli ops,

$$Z|e_n\rangle = e^{2\pi in/d}|e_n\rangle \quad X|e_n\rangle = |e_{n+1}\rangle$$

$$\sum_j Z^j \rho Z^{-j} = d \rho_{\text{diag}} \quad \sum_j X^j \rho_{\text{diag}} X^{-j} = (\text{Tr } \rho) I$$

$$\frac{1}{d_B} \sum_j \sum_k (I_A \otimes X_B^j Z_B^k) \rho_{AB} (I_A \otimes X_B^j Z_B^k)^\dagger = \rho_A \otimes I_B$$

$$H(\rho_A, \gamma_A) = H(\rho_A \otimes \frac{1}{d_B} I, \gamma_A \otimes \frac{1}{d_B} I) =$$

$$H\left[\frac{1}{d^2} \sum_{jk} (I \otimes X^j Z^k) \rho_{AB} (I \otimes X^j Z^k)^\dagger, \frac{1}{d^2} \sum_{jk} (I \otimes X^j Z^k) \gamma_{AB} (I \otimes X^j Z^k)^\dagger\right]$$

$$\leq \frac{1}{d^2} \sum_{jk} H[(I \otimes X^j Z^k) \rho_{AB} (I \otimes X^j Z^k)^\dagger, (I \otimes X^j Z^k) \gamma_{AB} (I \otimes X^j Z^k)^\dagger]$$

$$= \frac{1}{d^2} \sum_{jk} H(\rho_{AB}, \gamma_{AB}) = H(\rho_{AB}, \gamma_{AB})$$

Proof of Mono for CPT $\Phi : \mathcal{B}(\mathcal{H}_B) \mapsto \mathcal{B}(\mathcal{H}_A)$

Lindblad/Stinespring for $A = B$ $\Phi(\rho) = \text{Tr}_E U_{AE} \rho \otimes |e\rangle\langle e| U_{AE}^\dagger$

OR $\Phi(\rho) = \text{Tr}_E V_{AE} \rho_B V_{AE}^\dagger$ $V^\dagger V = I$ $V : \mathcal{H}_B \mapsto \mathcal{H}_{AE}$

Then by cyclicity of trace

$$H(\tilde{\rho}_{AE}, \tilde{\gamma}_{AE}) \equiv H(V_{AE} \rho_B V_{AE}^\dagger, V_{AE} \gamma_B V_{AE}^\dagger) = H(\rho_B, \gamma_B)$$

$$H[\Phi(\rho), \Phi(\gamma)] = H(\tilde{\rho}_A, \tilde{\gamma}_A) \leq H(\tilde{\rho}_{AE}, \tilde{\gamma}_{AE}) = H(\rho, \gamma)$$

Notes: $\tilde{\rho}_{AE} = V_{AE} \rho V_{AE}^\dagger$ and ρ have same non-zero e-vals

$H(\rho_A, \gamma_A) \leq H(\rho_{AB}, \gamma_{AB})$ proved as cor to SSA in original paper

joint convexity of $H(\rho_{AB}, \rho_A \otimes \frac{1}{d} I_B) = -S(\rho_{AB}) + S(\rho_A) + \log d$

$\Rightarrow \rho_{AB} \mapsto S(\rho_{AB}) - S(\rho_A)$ is concave

$\Rightarrow \tilde{\rho}_{AE} \mapsto S(\tilde{\rho}_{AB}) - S(\tilde{\rho}_A)$ is concave

But $S(\rho) = S(\tilde{\rho}_{AE})$ and $S[\Phi(\rho)] = S(\tilde{\rho}_A)$

$\Rightarrow \rho \mapsto S(\rho) - S[\Phi(\rho)]$ is concave

since $\tilde{\rho}_{AE} = V_{AE} \rho V_{AE}^\dagger$ and ρ have same non-zero e-vals

alt form of SSA $S(\rho_B) - S(\rho_{AB}) + S(\rho_D) - S(\rho_{AD}) \leq 0$

= for pure ρ_{ABD} extreme; sum of convex fctns $\Rightarrow \leq \text{ext} \leq 0$

POVM $\{E_m\}$ with $E_m \geq 0$ and $\sum_M E_m = I$.

special CPT map called QC: $\Omega_{\mathcal{M}}(\rho) = \sum_m |e_m\rangle\langle e_m| \text{Tr} \rho E_m$.

Ensemble: $\{\pi_j, \rho_j\}$ with $\pi_j > 0$, $\sum_j \pi_j = 1$ and ρ_j dens. matrix

Holevo bound on accessible info or max

mutual info between ensemble and measurement outcome

$$S[\Omega_{\mathcal{M}}(\rho_{\text{av}})] - \sum_j \pi_j S[\Omega_{\mathcal{M}}(\rho_j)] \leq S(\rho_{\text{av}}) - \sum_j \pi_j S(\rho_j)$$

$$\rho_{\text{av}} = \sum_j \pi_j \rho_j \quad \text{and } = \text{ if and only if all } \rho_j \text{ commute}$$

Cor: Access info $\leq S(\rho_{\text{av}}) \leq \log d = n$ if $d = 2^n$

Holevo: direct proof in 1973 (same as SSA) **without** using SSA.

Will give 3 simple proofs of Holevo bound based on SSA

I. formal mutual info $\gamma_{AB} = \begin{pmatrix} \pi_1 \rho_1 & 0 & \dots & \dots & 0 \\ 0 & \pi_2 \rho_2 & \dots & \dots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & \dots & 0 & \pi_m \rho_m \end{pmatrix}$

Then $\gamma_A = \sum_j \pi_j \rho_j = \rho_{\text{av}}$, and $\gamma_B = \sum_j |j\rangle\langle j| \pi_j$

$$\begin{aligned} H[\gamma_{AB}, \gamma_A \otimes \gamma_B] &= -S(\gamma_{AB}) + S(\gamma_A) + S(\gamma_B) \\ &= -\sum_j S(\pi_j \rho_j) + S(\rho) + S[\pi_j] \\ &= -\sum_j \pi_j S(\rho_j) + \sum_j \pi_j \log \pi_j + S(\rho) + S[\pi_j] \\ &= S(\rho_{\text{av}}) - \sum_j \pi_j S(\rho_j) \end{aligned}$$

$$H[(\Omega_{\mathcal{M}} \otimes I)(\gamma_{AB}), (\Omega_{\mathcal{M}} \otimes I)(\gamma_A \otimes \gamma_B)] \leq H[\gamma_{AB}, \gamma_A \otimes \gamma_B]$$

LHS is mutual info between ensemble and POVM outcome

II. Yuen-Ozawa (1993): Rewrite

$$\begin{aligned} S(\rho_{\text{av}}) - \sum_j \pi_j S(\rho_j) &= \sum_j \pi_j \text{Tr} \rho_j \log \rho_j - \sum_j \text{Tr} \pi_j \rho_j \log \rho_{\text{av}} \\ &= \sum_j \pi_j \text{Tr} \rho_j (\log \rho_j - \log \rho_{\text{av}}) \\ &= \sum_j \pi_j H(\rho_j, \rho_{\text{av}}) \end{aligned}$$

Then by monotonicity of relative entropy

$$\begin{aligned} S[\Omega_{\mathcal{M}}(\rho_{\text{av}})] - \sum_j \pi_j S[\Omega_{\mathcal{M}}(\rho_j)] &= \sum_j \pi_j H[\Omega_{\mathcal{M}}(\rho_j), \Omega_{\mathcal{M}}(\rho_{\text{av}})] \\ &\leq \sum_j \pi_j H(\rho_j, \rho_{\text{av}}) \\ &= S(\rho_{\text{av}}) - \sum_j \pi_j S(\rho_j) \end{aligned}$$

III. Lieb-Seiringer

observed: $\rho \mapsto S(\rho) - S[\Omega_{\mathcal{M}}(\rho)]$ concave means

$$S(\rho_{\text{av}}) - S[\Omega_{\mathcal{M}}(\rho_{\text{av}})] \geq \sum_j \pi_j (S(\rho_j) - S[\Omega_{\mathcal{M}}(\rho_j)])$$

equiv to

$$S(\rho_{\text{av}}) - \sum_j \pi_j S(\rho_j) \geq S[\Omega_{\mathcal{M}}(\rho_{\text{av}})] - S[\Omega_{\mathcal{M}}(\rho_j)]$$

All three proofs extend to partial measurement

$$\Omega_{\mathcal{M}_B} : \gamma_{AB} \mapsto \sum_j |j\rangle\langle j| \text{Tr}_B \gamma_{AB} I_A \otimes F_j \quad \sum_j F_j = I_B$$

“no transparent proof of SSA is known”

p. 645 of *Quantum Computation and Quantum Information*

Michael A. Nielsen and Isaac L. Chuang (Cambridge Press, 2000)

based on B. Simon's version adapted from Uhlmann (1977)

of “elementary” proof using only Schwarz inequality

of Lieb's Thm: $(A, B) \mapsto \text{Tr} A^{1-t} K^* B^t K$ jointly concave

similar argument in Wehrl *Rev. Mod. Phys* (198?).

MBR, “Lieb's simple proof of concavity of $\text{Tr} A^p K^* B^{1-p} K \dots$ ”

Int. J. Quant Info. **3**, 579–590 (2005); arXiv:quant-ph/0404126

Need only Schwarz inequality and maximum modulus principle

Petz – another argument in book and arXiv:quant-ph/0408130

MBR – proof presented here for SSA works for $\text{Tr} A^p K^* B^{1-p} K$

Wigner-Yanase-Dyson revisited

WYD skew entropy $\frac{1}{2} \text{Tr} [K, \gamma^p][K, \gamma^{1-p}]$ for $0 < p < 1$, $K = K^\dagger$

WY Proved concave for $p = \frac{1}{2}$; Dyson suggested $p \in (0, 1)$

led to **Conjecture** $\text{Tr} K \gamma^p K \gamma^{1-p} - \text{Tr} K \gamma K$ concave in γ

Lieb threw away linear term — seems reasonable but

proved $(A, B) \mapsto \text{Tr} K^\dagger A^p K B^{1-p}$ concave for $A, B \geq 0$, $p \in (0, 1)$.

Ando (1978) proved joint convexity $p \in (1, 2)$ ignored

until recent Lieb-Carlen work on $(\text{Tr}_1(\text{Tr}_2 A_{12}^p)^{q/p})^{1/q}$

Jencova-Ruskai: Unified treatment – retain linear term in WYD

$$J_p(K, P, Q) = \frac{1}{p(1-p)} (\text{Tr } K^\dagger P K - \text{Tr } K^\dagger P^p K Q^{1-p})$$

well-def for $p \in (0, 2)$; by cont at $p = 1$, $J_1(I, P, Q) = H(P, Q)$

Thm: $(P, Q) \mapsto J_p(K, P, Q)$ is jointly convex for $p \in (0, 2)$

Cor: Lieb concavity for $p \in (0, 1)$ because of sign change at $p = 1$.

$$g_p(x) = \begin{cases} \frac{x-x^p}{p(1-p)} & p \neq 1 \\ x \log x & p = 1 \end{cases} \quad J_p(K, P, Q) = \text{Tr } K^* g_p(L_P R_Q^{-1})(KQ)$$

operator convexity of $g_p(x) \Rightarrow$ int rep as for $H(P, Q) \Rightarrow$ Thm

apparent symmetry $p \leftrightarrow 1 - p$ at [more subtle](#)

$$\tilde{g}_p(x) = w g_{1-p}(w^{-1}) = \begin{cases} \frac{1-x^p}{p(1-p)} & p \neq 0 \\ -\log x & p = 0 \end{cases}$$

$$J_p(K, P, Q) = \tilde{J}_{1-p}(K^\dagger, Q, P) \text{ with } \tilde{J}_p \text{ well-def on } (-1, 1)$$

$$J_p(K, P, Q) = \frac{1}{p(1-p)} (\text{Tr } K^\dagger P K - \text{Tr } K^\dagger P^p K Q^{1-p})$$

also has same monotonicity props as $H(P, Q)$

$$J_p(K_A, P_A, Q_A) \leq J_p(K_{AB}, P_{AB}, Q_{AB})$$

analogue of SSA $J_p(I, \rho_{AB}, \rho_B) \leq J_p(I, \rho_{ABC}, \rho_{BC})$

$$\frac{1}{p(1-p)} \text{Tr } \rho_{AB}^p \rho_B^{1-p} \leq \frac{1}{p(1-p)} \text{Tr } \rho_{ABC}^p \rho_{BC}^{1-p}$$

very different, but more natural than Renyi or Tsallis entropy

log enters only in weight in integral rep – doesn't affect props

No obvious connection to additivity conj – maybe why so hard

Equality conds for Carlen-Lieb or any J_p are exactly same as SSA

- For SSA go back to Klein's inequality in original proof
quantum Markov cond $\log \rho_{ABC} - \log \rho_{AB} = \log \rho_{BC} - \log \rho_B$

- Petz (1986) used Connes co-cycle –

- Jencova-Ruskai (2008) equal from C-S argument $\text{Tr } M_j^\dagger M_j \geq 0$

$$M_j = (L_{P_j} + tR_{Q_j})^{-1/2}(A_j) - (L_{P_j} + tR_{Q_j})^{-1/2}(X) = 0 \quad \forall j \forall t \geq 0$$

Insert X modified to include K and let $P = \sum_j P_j$ etc.

$$\frac{1}{I + tL_{P_j}^{-1}R_{Q_j}}(K) = \frac{1}{I + tL_P^{-1}R_Q}(K) \quad \forall j \forall t \geq 0$$

can anal cont except $(-\infty, 0]$ and extend to $g(L_P^{-1}R_Q)(K)$

$\rho_{ABC} = \rho_A \otimes \rho_{BC}$ or $\rho_{AB} \otimes \rho_C$ but not necessary

$\rho_{ABC} = \rho_{AB_1} \otimes \rho_{B_2C}$ where $\mathcal{H}_B = \mathcal{H}_{B_1} \otimes \mathcal{H}_{B_2}$

Most general nasc (HJPW)

$$\rho_{ABC} = \bigoplus_k \rho_{A_k B'_k} \otimes \rho_{B''_k C_k}$$

Proof of Klein's inequality: $\text{Tr } A \log A - \text{Tr } A \log B \geq \text{Tr}(A - B)$

g convex means diff quotients increase

$$\Rightarrow \frac{g(b) - g(a)}{b - a} \leq g'(b) \text{ for } a < b$$

$$\Rightarrow g(b) - g(a) \leq (b - a)g'(b) \text{ for all } a, b$$

$$\text{Tr} [g'(B)(B - A) - g(B) + g(A)]$$

$$= \sum_k \left[g'(b_k)(b_k - \langle \beta_k, A \beta_k \rangle) - g(b_k) + \langle \beta_k, g(A) \beta_k \rangle \right]$$

$$\text{Jensen } g(\langle \beta_k, A \beta_k \rangle) \leq \langle \beta_k, g(A) \beta_k \rangle$$

$$\geq \sum_k \left[g'(b_k)(b_k - \langle \beta_k, A \beta_k \rangle) - g(b_k) + g(\langle \beta_k, A \beta_k \rangle) \right] \geq 0$$

For $g(x) = x \log x$, $g'(x) = 1 + \log x$

$$\text{Tr} [(B - A)(I + \log B) - B \log B + A \log A] \geq 0$$

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