Mathematical models of interface dynamics and coarsening

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Abstracts of the lectures:

I. Sharp and diffuse models of interface dynamics

Diffusion and adherence can drive the migration of material interfaces by flux deposition, by diffusion along the interface, or by fluctuation-induced Brownian motion. Classically the dynamics of interfaces is described by evolving surfaces, but on a finer scale interfaces can be modeled as diffuse zones of rapid transition of an order parameter. In this talk I'll focus on two problems recently treated by such models: (i) On vicinal surfaces of crystals, the step edges of atomically flat terraces can evolve by such mechanisms, and I'll describe work of Otto et al that recovers the BCF (Burton-Cabrera-Frank) sharp-interface model of step migration from a viscous Cahn-Hilliard equation with degenerate mobility. (ii) For Hele-Shaw flow between parallel plates, Glasner has adapted Otto's description of Hele-Shaw dynamics as gradient flow with respect to the Wasserstein transport metric, in order to derive a degenerate Cahn-Hilliard equation as a suitable diffuse interface approximation.

II. Paradigms for coarsening dynamics: metastability, geometric models, mean-field self-similarity

Entropy increases, and at constant temperature, free energy decreases. Morphologically complex systems dominated by interfacial energy exhibit coarsening behavior associated with dynamic scaling laws which are not so well understood. I'll describe some simple scaling concepts, and hierarchies of models in 1D, 2D and 3D that describe coarsening. Especially I'll discuss recent mathematical progress in analyzing the universality of self-similar behavior in idealized mean-field models such as the classic LSW (Lifshitz-Slyozov-Wagner) model of Ostwald ripening that serves as a paradigm for dynamic scaling in materials science.

III. Rigorous bounds on coarsening rates

I'll discuss recent progress deriving from work of Kohn and Otto, toward justifying dynamic scaling laws through rigorous upper bounds derived from underlying diffuse-interface models. I'll also discuss how this work has been extended in a number of directions by Kohn and Yan, by B. Li and J.-G. Liu, and by S. Dai and myself. Work on rigorous bounds is particularly intriguing in light of recent experimental evidence of Voorhees, suggesting that power-law coarsening rates can occur in the absence of statistical self-similarity at the microscopic level.

IV. Mean-field agglomeration models, stochastic effects.

Models of interfacial growth and coarsening can involve stochastic effects. Understanding the statistical properties of solutions of nonlinear field equations is a fundamental scientific problem in general. I'll describe several mean-field models of coarsening and discuss the effects of noise. Also I'll describe 2-D models of epitaxial growth by ballistic deposition, including the KPZ (Kadar-Parisi-Zhang) and the Lai-dasSarma models. An important mean-field model of agglomeration or clustering is Smoluchowski's coagulation equation. Surprisingly it has a rigorous connection to Burgers' classic "turbulence" model.

Outline of the lectures

I. Sharp & diffuse models of interface dynamics

- Step edge dynamics on vicinal surfaces
  - Phenomena. Yellow diamonds by chemical vapor deposition [19]
  - BCF model of step edge motion [12, 4]
  - Diffuse-interface models — background [41, 13, 40]
  - Diffuse-interface BCF model by Otto et al [39]
  - Limiting regimes: Diffusion-limited, attachment-limited kinetics
- Hele-Shaw flow between parallel plates
  - Gradient flow and the Wasserstein transport metric [38, 6]
  - A Cahn-Hilliard-type diffuse-interface model [27]

II. Paradigms for coarsening dynamics: metastability, geometric models, mean-field self-similarity

- Phase-space geometry of relaxation in 1D – FHCP picture [25, 16, 14]
- Remarkable solution of a 1D mean field model [15, 26]
- LSW mean-field models, analysis, and computation [5, 2, 3, 36, 37, 35, 43, 44]

III. Rigorous bounds on coarsening rates

- Experimental evidence for statistical non-self-similarity [33]
- Kohn-Otto rigorous upper bounds on coarsening rates [28, 29, 30, 32, 18]
- Rigorous upper bounds for mean-field models [17]

IV. Mean-field agglomeration models, stochastic effects.

- Effect of noise on coarsening in 1D [23, 20]
- KPZ equation, Lai-das Sarma model [46, 31]
- Burgers turbulence [11, 22, 24, 7, 8, 9]
- Smoluchowski's coagulation equation [42, 45, 21, 1, 34, 10]

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