Effect of noise on coarsening in 1D

I. Fatkullin & E Vanden Eijnden

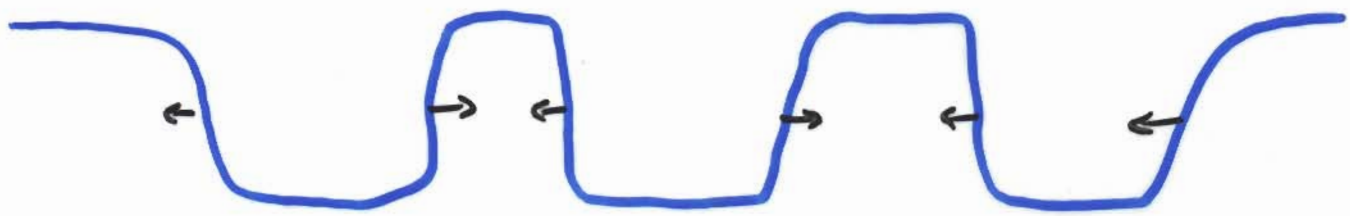
Nonconserved: $u_t = \varepsilon^2 u_{xx} + u - u^3 + \sqrt{\eta}\, \xi(x,t)$

Space-time white noise: $\langle \xi(x_1,t_1)\xi(x_2,t_2)\rangle = \delta(x_1-x_2)\delta(t_1-t_2)$

For weak noise $\eta \ll \varepsilon \ll 1$

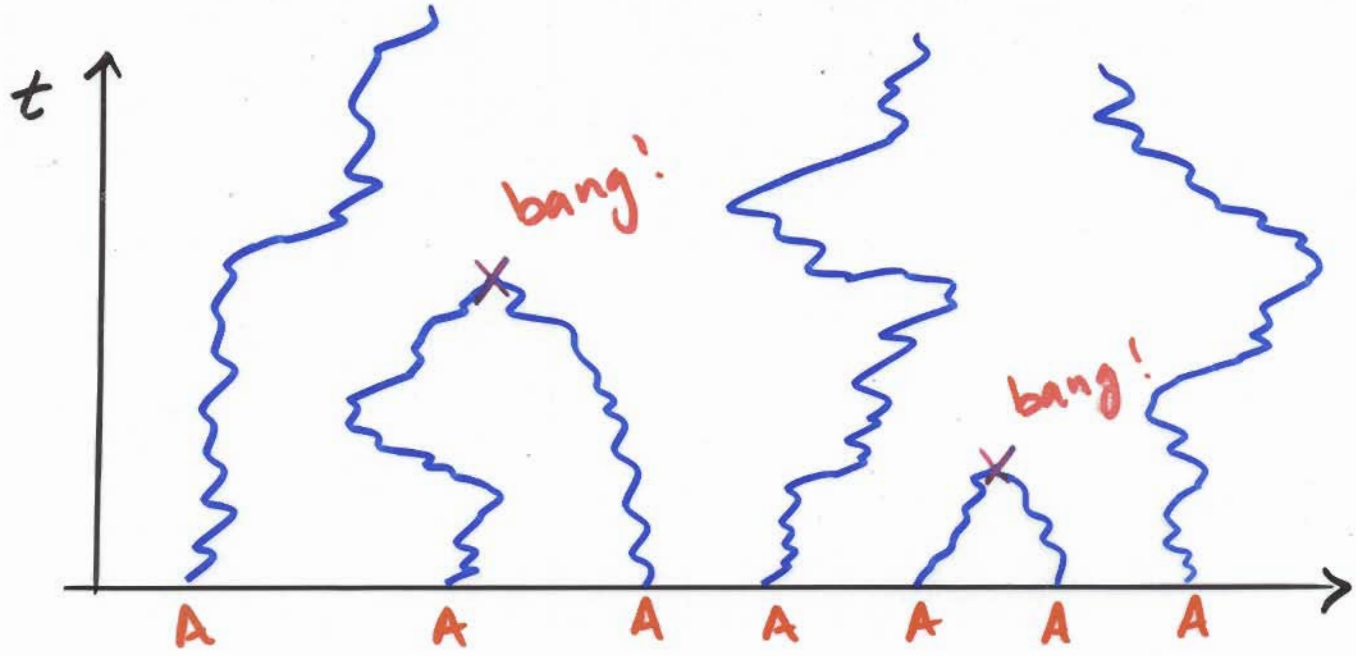
F&vE argue the effect of noise is approximately:

Kinks diffuse, annihilate upon contact



independent 1D random walkers

Vicious Walkers : a coagulation model



"Mean field" theory

$u(x,t)\,\Delta x$ = probability the distance between two walkers lies in $(x, x+\Delta x)$ at time t.

- Formal limit of a lattice model : A+A → inert

$$\frac{\partial u}{\partial t} = \underbrace{\frac{\partial^2 u}{\partial x^2}}_{A+A\to A} + \underbrace{\frac{\partial u}{\partial x}(0,t)\int_0^x u(y,t)\,u(x-y,t)\,dy}$$

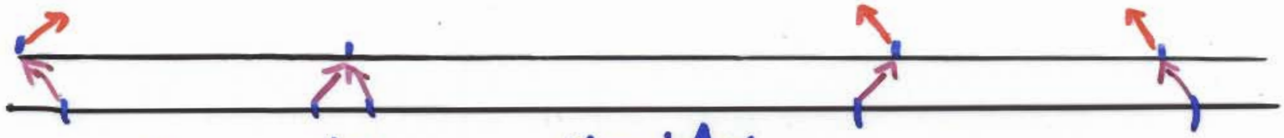
("∞ state Potts model in 1D at zero temperature"
Derrida et.al. , Mullins , ben Avraham et.al.
etc.)

Coagulation model: Formal derivation

Grid size Δx Walkers step $\pm \frac{1}{2}\Delta x$

$f_j \Delta x$ = # of intervals of size $j\Delta x$ (per unit length)

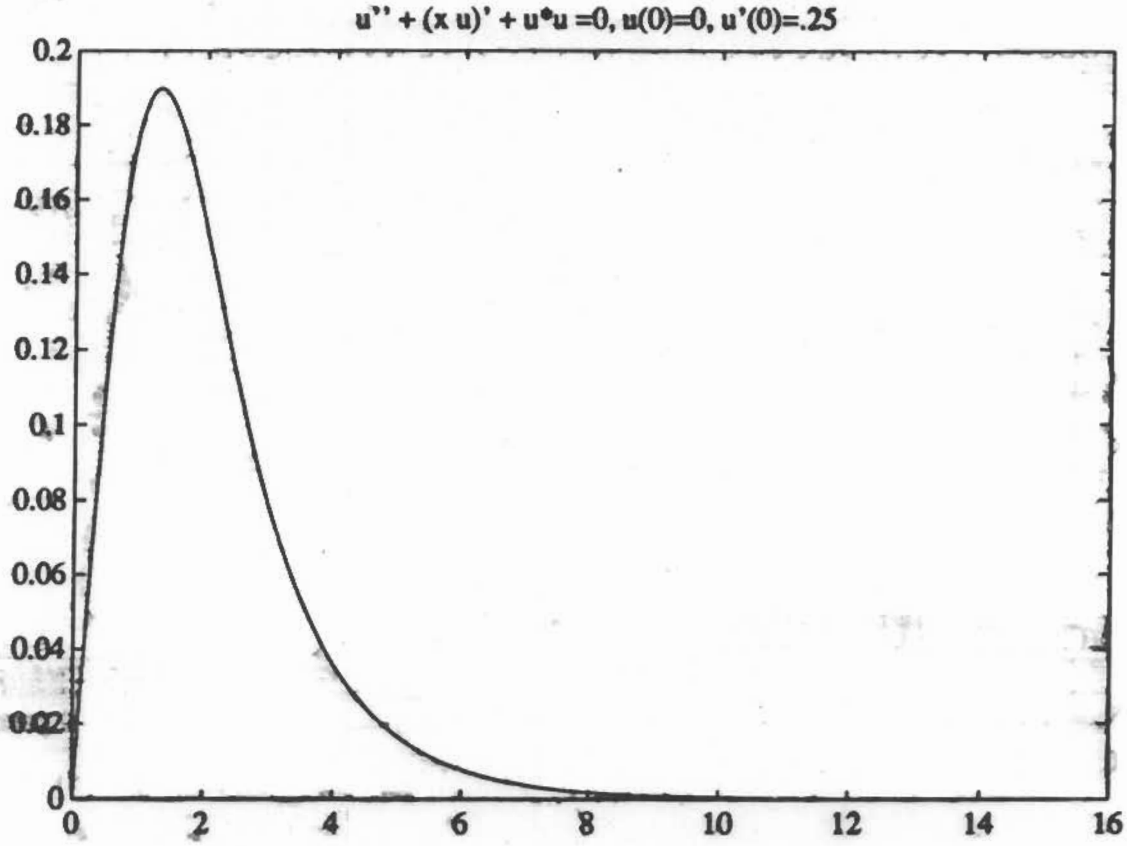
$N = \sum_j f_j \Delta x$ = total # of walkers (per unit length)



Process	# per time step Δt
1) $x \longrightarrow x+y$	$2\alpha f_j \Delta x$, $\alpha = \frac{1}{4}\left(\frac{f_1 \Delta x}{N}\right)$
2) $x-\Delta x \longrightarrow x$	$\frac{1}{4}(1-2\alpha) f_{j-1} \Delta x$
$x \longrightarrow x$	$\frac{1}{2}(1-2\alpha) f_j \Delta x$
$x+\Delta x \longrightarrow x$	$\frac{1}{4}(1-2\alpha) f_{j+1} \Delta x$
3) $y + \Delta x + (x-y) \underset{\sim}{\Rightarrow} x$ (4 subcases)	$\frac{1}{4} f_1 \Delta x \sum_{k=1}^{j} \frac{f_k \Delta x}{N} \cdot \frac{f_{j-k}\Delta x}{N}$

independent?

* Sum for new $f_j \Delta x$ at next time step.



u vs. x

log10(abs(u)) vs x

Seek self similar solution: A+A → inert

$$u(x,t) = \frac{a}{\sqrt{t}}\, g\left(\frac{bx}{\sqrt{t}}\right) \qquad a^2 = 2,\ b^2 = 1/2.$$

Require:

$$0 = g''(x) + (xg)' + \int_0^x g(y)\,g(x-y)\,dy$$

$$\underline{g(0) = 0}\ , \quad \int_0^\infty x g(x) < \infty$$

Integrations ⟹

$$\int_0^\infty g = \frac{1}{2}\ , \quad \underline{g'(0) = 1/4} \quad \left(\text{IVP for } g\right)$$

Laplace transform

$$zG'(z) = z^2 G - 1/4 + G^2$$

? Is $g > 0$? Is $\int_0^\infty x g(x)\,dx < \infty$

$$z G' = z^2 G - \frac{1}{4} + G^2$$

Substitute $\quad G = -z\frac{Y'}{Y} + \frac{1}{2}(1 - z^2)$

$$Y'' + \left(1 - \frac{z^2}{4}\right) Y = 0$$

$Y(z) = U(-1, z)$ Parabolic cylinder function

$U(-1, z_0) = 0$ where $z_0 \simeq -.76495$

Identities (Abramowitz & Stegun)

$$U'(-1, z) + \frac{z}{2} U(-1, z) - \frac{1}{2} U(0, z) = 0$$

$$z\, U(0, z) - U(-1, z) + \frac{1}{2} U(1, z) = 0$$

$$\Rightarrow \boxed{G(z) = \frac{1}{4} \frac{U(1, z)}{U(-1, z)}}$$

Theorem $G(z)$ is completely monotone

Hence $g(x) > 0$ and decays exponentially as $x \to \infty$

Proof $4\,G(z) = U(1,x) \cdot \frac{1}{U(-1,x)}$

$$\phi = U(a,z)\,e^{z^2/4} =$$

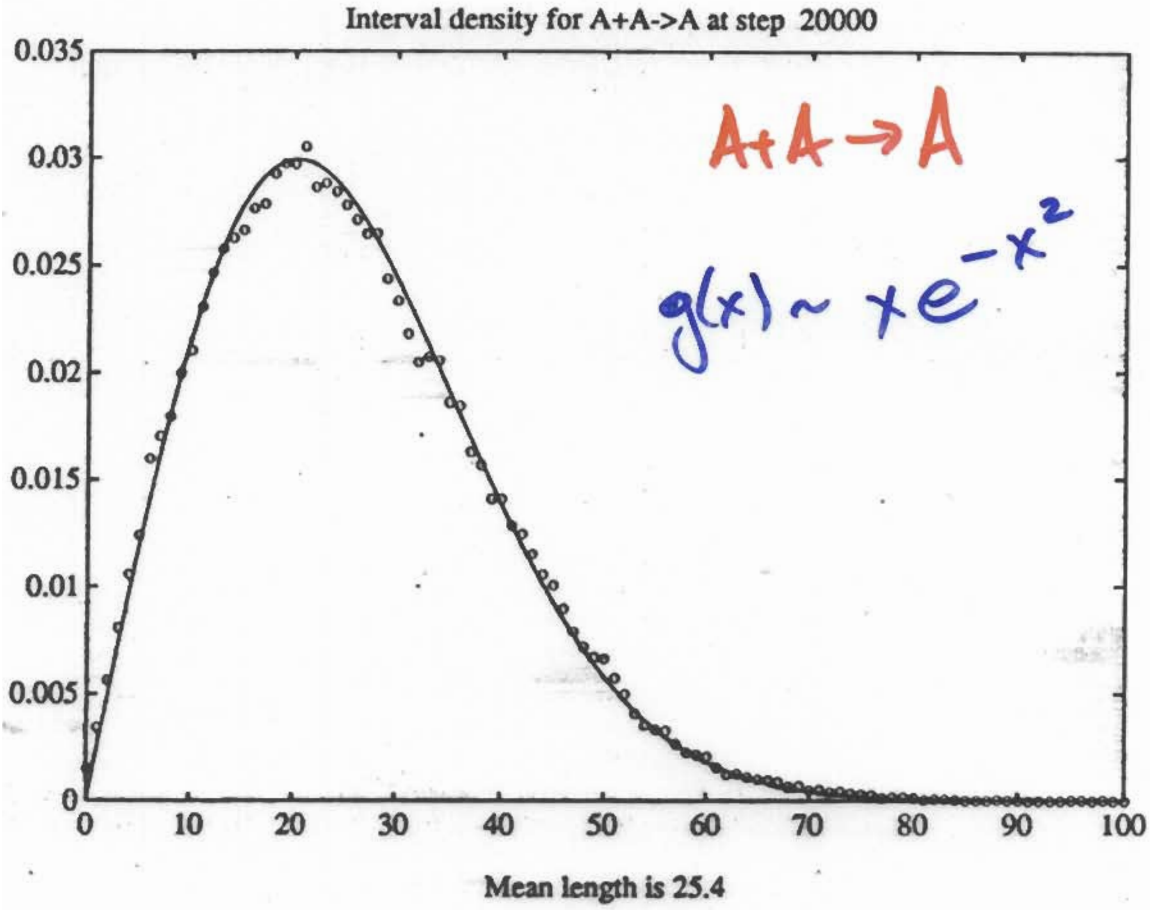
$$\frac{1}{\Gamma(a+\frac{1}{2})} \int_0^\infty e^{-zx}\, e^{-x^2/2}\, x^{a-\frac{1}{2}}\, dx \quad (a > -\tfrac{1}{2})$$

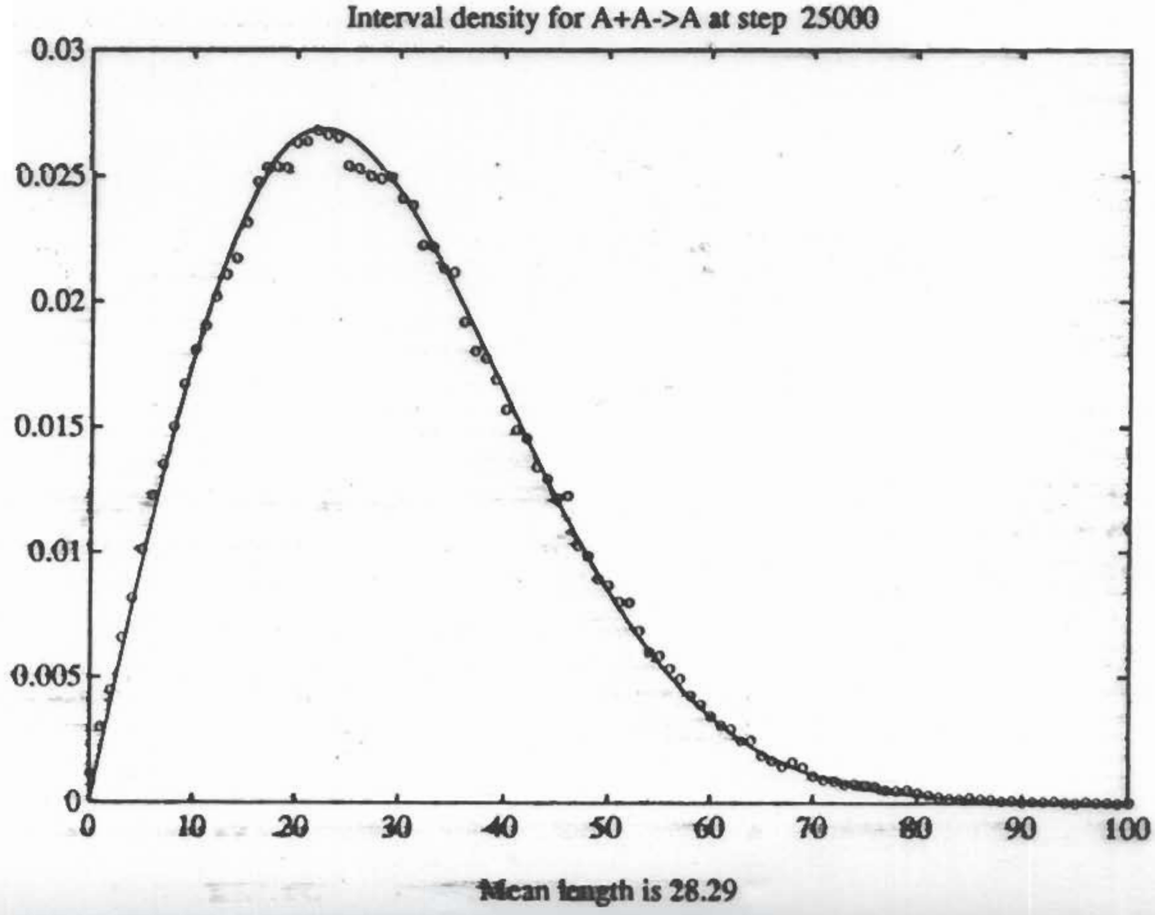
ϕ is completely monotone (esp. for $a = 1$)

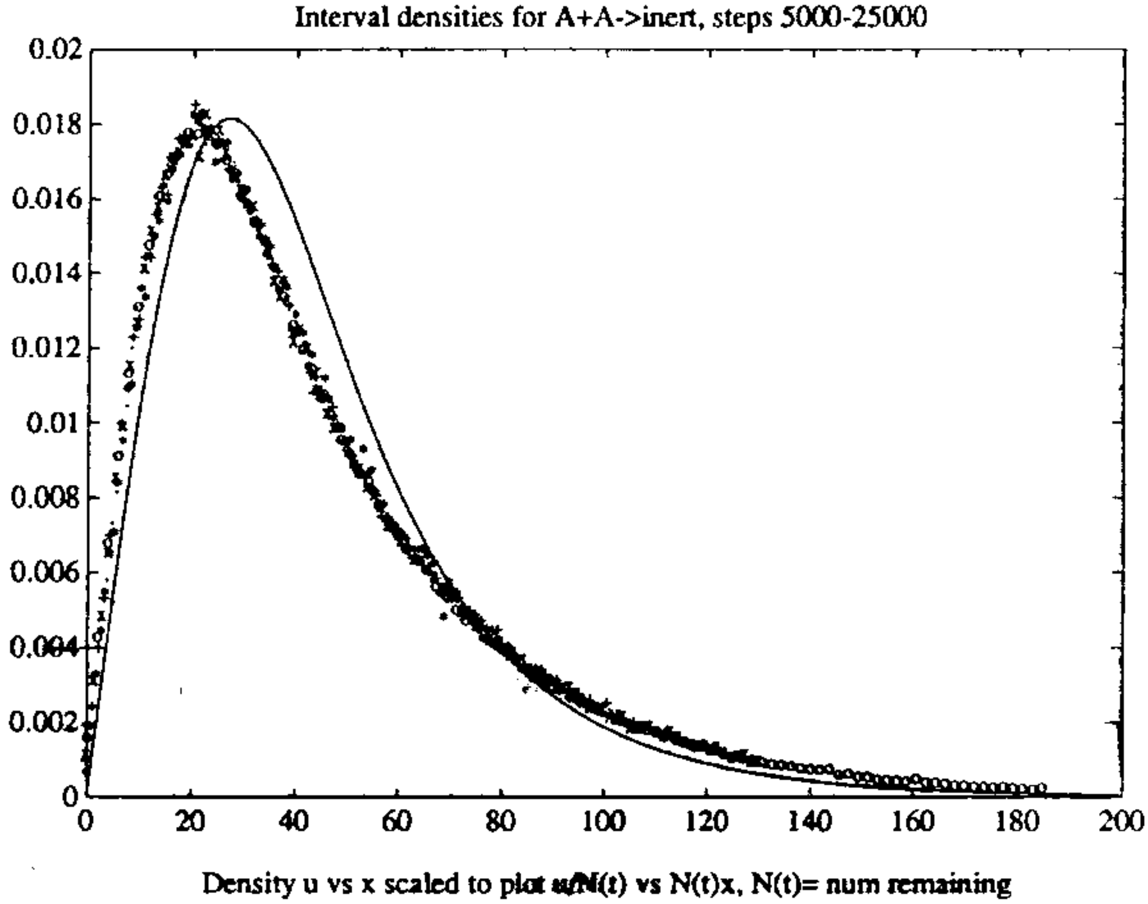
$$\psi = U(-1,z)\,e^{z^2/4} \Rightarrow \psi'(z) = \tfrac{1}{2}\, U(0,z)\, e^{z^2/4}$$

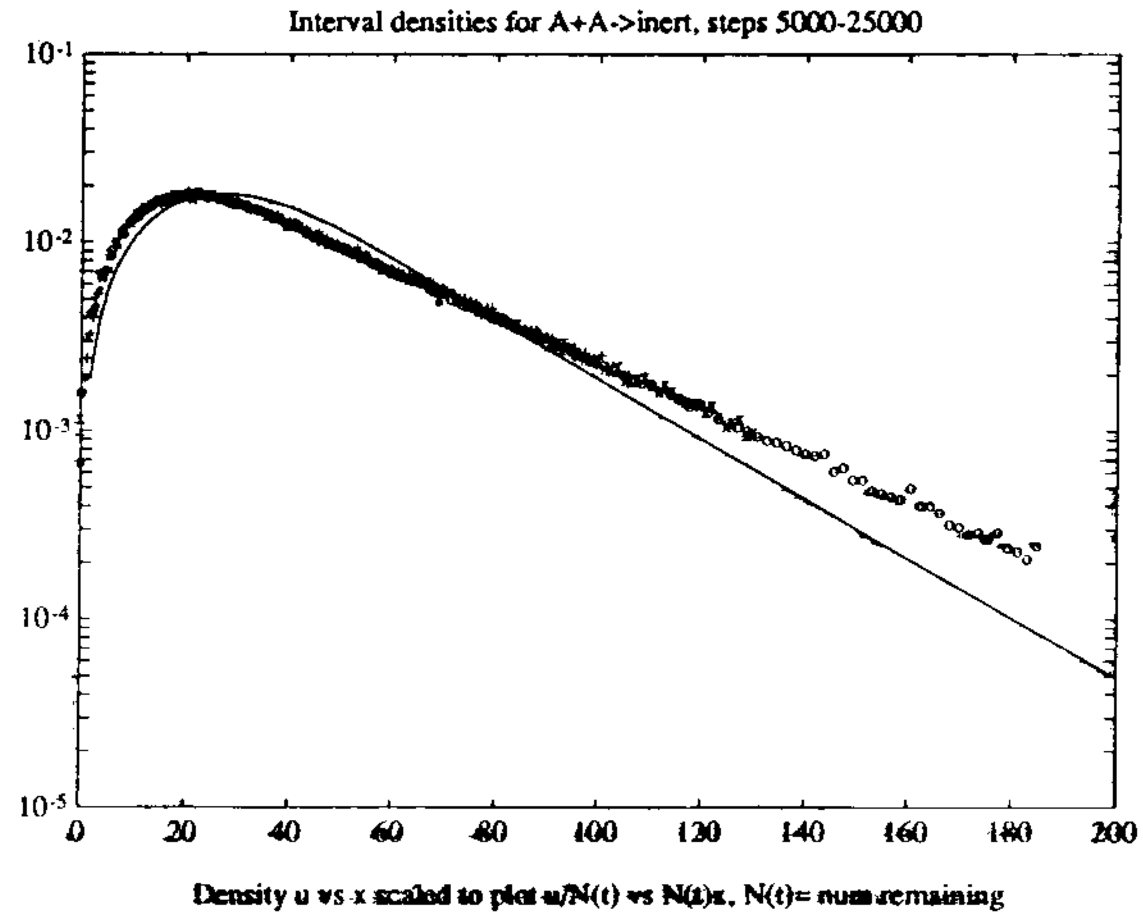
Now, ψ' is CM. Hence $\frac{1}{\psi}$ is CM.

Hence $G = \frac{1}{4}\phi \cdot \frac{1}{\psi}$ is CM. ∎

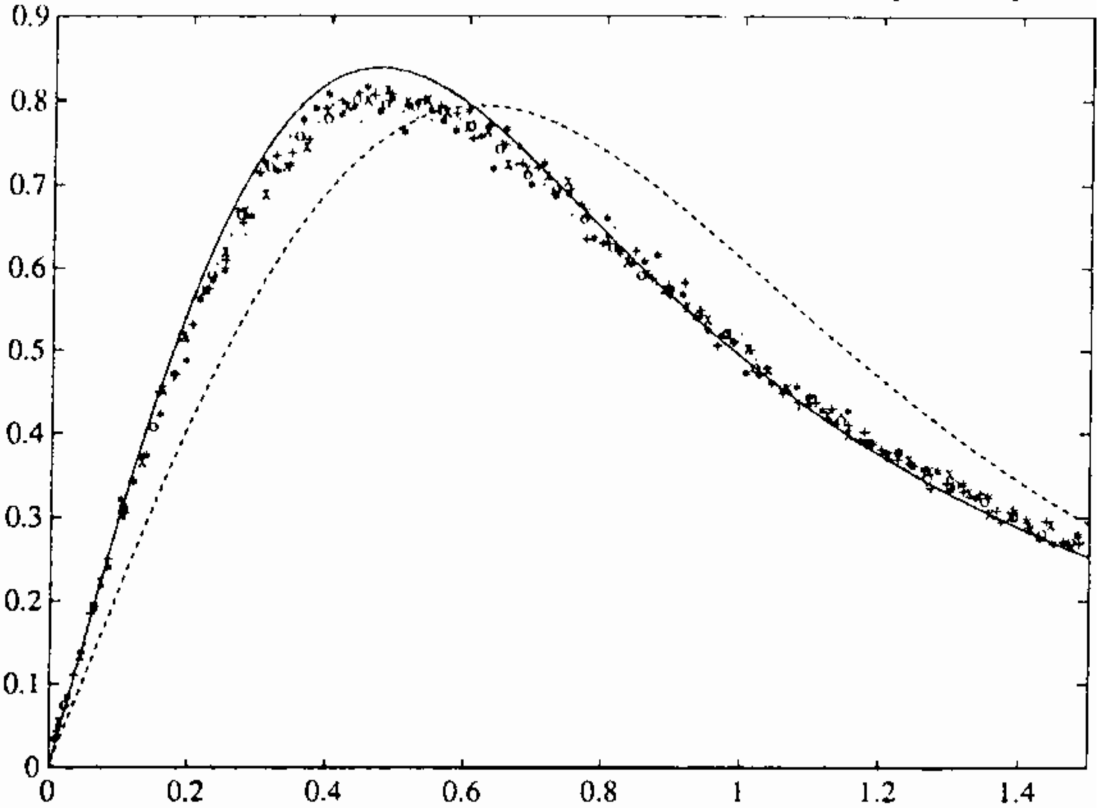

Interval density for A+A->A at step 20000
A+A → A
$g(x) \sim x e^{-x^2}$
0.035
0.03
0.025
0.02
0.015
0.01
0.005
0
0 10 20 30 40 50 60 70 80 90 100
Mean length is 25.4


Interval density for A+A->A at step 25000
0.03
0.025
0.02
0.015
0.01
0.005
0
0 10 20 30 40 50 60 70 80 90 100
Mean length is 28.29

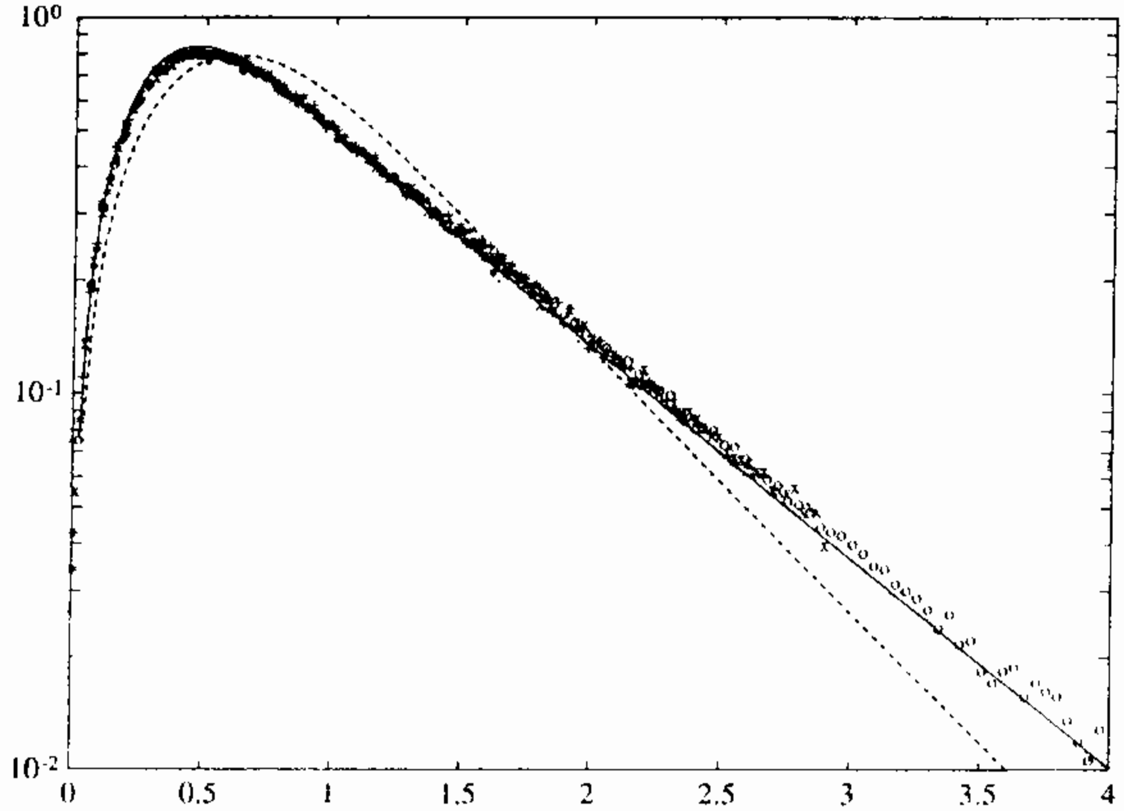

Interval densities for A+A->inert, steps 5000-25000
0.02
0.018
0.016
0.014
0.012
0.01
0.008
0.006
0.004
0.002
0
0 20 40 60 80 100 120 140 160 180 200
Density u vs x scaled to plot u/N(t) vs N(t)x, N(t)= num remaining


Interval densities for A+A->inert, steps 5000-25000
10-1
10-2
10-3
10-4
10-5
0 20 40 60 80 100 120 140 160 180 200
Density u vs x scaled to plot u/N(t) vs N(t)x, N(t)= num remaining

A+A->inert. Markers-simulation data. Dash-mean field. Solid-pair theory.



Continuum models scaled to make 0th and 1st moments = 1



Semilog plot

Nearest-neighbor closure model

Start from exact, unclosed system for

$u^{(2)}(x_1, x_2, t)$ = PDF for 2 successive interval lengths

$$u(x,t) = \int_0^\infty u^{(2)}(x,y,t)\,dy \quad , \quad u(\downarrow) := \partial_x u(0)$$

$$\partial_t u^{(2)} = \left(\partial_{11}^2 - \partial_{12}^2 + \partial_{22}^2\right) u^{(2)} + 2\, u(\downarrow)\, u^{(2)}$$

$$+ \int_0^{x_1} u^{(4)}(y, \downarrow, x_1 - y, x_2)\,dy - u^{(3)}(\downarrow, x_1, x_2)$$

$$+ \int_0^{x_2} u^{(4)}(x_1, y, \downarrow, x_2 - y)\,dy - u^{(3)}(x_1, x_2, \downarrow)$$

Closure assumption: $u(x_1 | x_2, x_3 \ldots) = u(x_1 | x_2)$

$$u^{(3)}(x_1, x_2, x_3) = u^{(2)}(x_1, x_2)\, u^{(2)}(x_2, x_3) / u(x_2)$$

$$u^{(4)}(x_1, x_2, x_3, x_4) = u^{(2)}(x_1, x_2)\, \frac{u^{(2)}(x_2, x_3)}{u(x_2)}\, \frac{u^{(2)}(x_3, x_4)}{u(x_3)}$$

Effect of noise on coarsening in 1D

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Conserved: $u_t = \partial_x^2\left(-\varepsilon^2 u_{xx} + u - u^3\right) + \sqrt{\eta}\,\partial_x \omega$

Goal: Obtain effect of stochastic ω on the restricted dynamics on the slow manifold

Context: General gradient flows $\dot{u} = -\dfrac{\delta E}{\delta u} + \omega(u,t)$

Formal result for 1D Cahn-Hilliard:

$\beta_j = \frac{1}{2}(h_j + h_{j+1}) \qquad d_j = h_{j+1} - h_j$ satisfy

$$\dot{\beta}_j = \sum_k G_{jk}\,\dot{W}_k \qquad G_{jk} = \frac{1}{\sqrt{d_j}}\left(\delta_{jk} - (-1)^{j+k}\frac{d_j}{L}\right)$$

W_k = independent Brownian motions.

2D interface growth - the KPZ model

M. Kardar, G. Parisi, Y.C. Zhang PRL 56, 1986, 889.

- Formally expand growth rate of height $z(x,y,t)$ in gradients ∇z, $\nabla\nabla z$, ... & their powers.

Retain terms formally dominant at long length scales, consistent with symmetry:

$$\partial_t z = f(x,y,t) + \frac{\lambda}{2}|\nabla z|^2 + \nu \nabla^2 z$$

Deposition beam intensity = F + f (const + fluctuation)

Invariant under symmetry:

$$z \to -z, \quad \lambda \to -\lambda, \quad f \to -f$$

- Reduction to Burgers' turbulence model: restrict to 1D, send $\nu \to 0$, $u = \partial_x z$, $f = 0$:

$$\partial_t u + \partial_x\left(\tfrac{1}{2}u^2\right) = 0$$

2D interface growth - the Lai-Das Sarma model

Z W Lai & S. DasSarma PRL 66 1991 2348
J. Villain J. Phys I 1 1991 19.

Models MBE growth. Local conservation.
Absence of desorption, gravity effects

$$\partial_t z = -\nu \nabla^4 z + \underline{\lambda \nabla^2 |\nabla z|^2} + f$$

not a gradient system. Kinetic, no chem. potential.

$z \to -z$ is not a symmetry

- C. DasGupta et al PRE 55 1997
  Chakrabarti & DasGupta 2003

Instability in discrete versions
due to growth of isolated structures