Tutorial on
Mathematical models of interface dynamics and coarsening

Lecture IV, part II:
Dynamic scaling, Smoluchowski's coagulation equation and Burgers turbulence

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Smoluchowski's coagulation equations

$n(x,t)$ is the number density of size-x clusters. $K(x,y) = K(y,x)$.

Clusters of size x and y form $x+y$-clusters at a mean-field rate $K(x,y)n(x,t)n(y,t)$.

$$\boxed{x} + \boxed{y} \Longrightarrow \boxed{x+y}$$

$$\boxed{x-y} + \boxed{y} \Longrightarrow \boxed{x}$$

$$\begin{aligned} \frac{\partial n}{\partial t}(x,t) &= \frac{1}{2}\int_0^x K(x-y,y)n(x-y,t)n(y,t)\,dy \\ &\quad - \int_0^\infty K(x,y)n(x,t)n(y,t)\,dy \end{aligned}$$

Scientific applications

Smoluchowski's coagulation equations have been used to describe:

- formation of clouds and smog
- droplet sizes in spray painting, ink fog
- agglomeration of planetesimals, star clusters, galaxies
- bubble swarms
- polymerization reactions
- island size distributions in epitaxial growth
- random graph theory
- lines of descent in population genetics
- renewal processes in probability theory

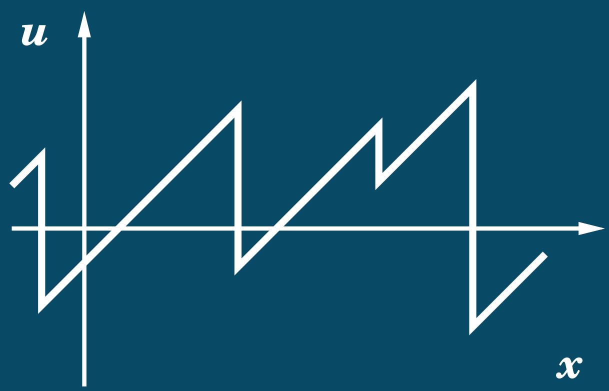
Smoluchowski's 1917 paper was one of the 58 most highly cited papers in science published before 1930. (ISI report, 1974)

Burgers' turbulence model

$$u_t + uu_x = 0, \quad u(x,0) = u_0(x)$$

Given the statistical properties of initial data u_0, what are the statistical properties of the solution?

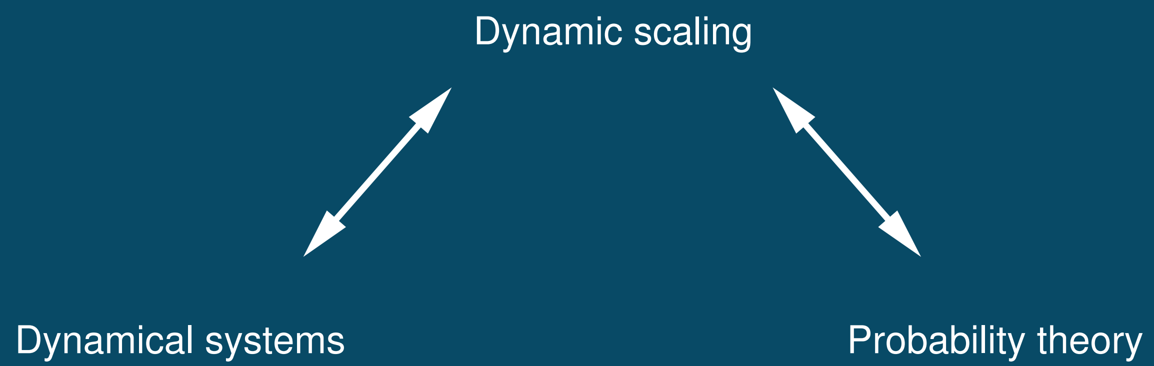
With, e.g., white noise data, the solution is like:



This is a restricted model of *ballistic aggregation.*

How do the shocks cluster? Mean-field model: $K(x,y) = x + y$

A framework for dynamic scaling analysis, inspired by dynamical systems and probability theory



- (i) What scaling solutions exist?
  (Fixed points of a renormalization group.)
- (ii) What are the domains of attraction?
  (Universality classes for scaling.)
- (iii) What other scaling limit points are possible?
  (Call the set of these the *scaling attractor*.)
- (iv) What is the ultimate dynamics on the scaling attractor?

(ia) Self-similar solutions for "solvable" kernels

$$K = 1 + 1: \quad n = (1+t)^{-2} \exp\left(\frac{-x}{1+t}\right)$$

$$K = x + y: \quad n = \frac{1}{\sqrt{2\pi}} e^{-t} x^{-3/2} \exp(-e^{-2t} x/2)$$

$$K = xy: \quad n = \frac{1}{\sqrt{2\pi}} x^{-5/2} \exp(-(1-t)^2 x/2) \quad (t < 1)$$

These were *all* the self-similar solutions known for any K, until 2002.

Diverging moments:

$$K = x + y: \quad \int n\, dx = \infty, \quad \int xn\, dx \equiv 1.$$

$$K = xy: \quad \int n\, dx = \infty, \quad \int xn\, dx = \infty, \quad \int x^2 n\, dx \to \infty \text{ as } t \to 1^-.$$

Long-outstanding problem: *existence* for general homogenous K.

New in 2004: Fournier & Laurençot, Escobedo, Mischler & Rodriguez Ricard

(iia) Dynamic scaling: stability of SSS

Q: Is there a universal scaling limit ($\hat{x} = x/\bar{x}(t)$)

$$\bar{x}(t)^2 n(\bar{x}(t)\hat{x}, t) \to f(\hat{x}) \quad \text{as } t \to \infty?$$

A: With strong assumptions on decay & smoothness of initial density $n_0(x)\,dx$:

For $K = 1$, Kreer & Penrose (1994) get pointwise convergence of n for continuous & discrete cases.

For $K = x + y$ and xy, Deaconu & Tanré (2000) get weak convergence (i.e. in distribution).

A: For $K = 1$, Aldous (2000) gets weak convergence for any data with finite 0th and 1st moments, using classical stochastic results on *thinning of renewal processes*.

Analytic approach via Laplace transform

Introduce desingularized Laplace transforms:

$$\varphi(t,s) = \int_0^\infty (1-e^{-sx}) n(t,x)\,dx, \qquad \text{for } K = 2 \text{ or } x+y,$$

$$\psi(t,s) = \int_0^\infty (1-e^{-sx}) x n(t,x)\,dx \qquad \text{for } K = xy$$

Then

$$\partial_t \varphi = -\varphi^2 \qquad \text{for } K = 2,$$

$$\partial_t \varphi - \varphi \partial_s \varphi = -\varphi \quad \text{for } K = x+y,$$

$$\partial_t \psi - \psi \partial_s \psi = 0 \quad \text{for } K = xy,$$

Weak convergence with two finite moments

(Leyvraz 2003, Menon & P 2004)

Pointwise limits of Laplace transforms correspond to pointwise limits for the size distribution function, i.e. weak convergence of measures.

After mass-preserving rescaling, there is weak convergence to self-similar form for all initial size-distribution measures

- for $K = 2$ with finite 0th and 1st moments
- for $K = x + y$ with finite 1st and 2nd moments
- for $K = xy$ with finite 2nd and 3rd moments, as $t \uparrow T_{\text{gel}}$

Note (M & P): There is existence and uniqueness for measure-valued weak solutions with only *one* finite moment.

Uniform convergence of densities

Discrete case: one obtains uniform convergence of rescaled lattice densities with only the two finite moments described above.

Continuous case: Suppose that for the moment density

- $x\,n_0(x)$ for $K = 2$,
- $x^2 n_0(x)$ for $K = x + y$
- $x^3 n_0(x)$ for $K = xy$

the Fourier transform is integrable. Then under mass-preserving rescaling, the corresponding moment density for the solution converges to self-similar form *uniformly* for $x > 0$.

Proofs combine the uniform-convergence proof of the *central limit theorem* with study of flow along complex characteristics.

Necessary & sufficient conditions for dynamic scaling limits?

An analogy in probability theory:

The **central limit theorem** says that the Gaussian is a universal scaling limit for averages of i.i.d. random variables with *finite variance*.

For **heavy-tailed distributions**, presuming symmetry there is a one-parameter family of possible limits, the "stable distributions". The domains of attraction of these solutions can be completely characterized (Feller, vol II) as distributions with "*almost power-law*" behavior:

$$\int_{-x}^{x} y^2 \, F\{dy\} \sim x^{2-\alpha} L(x),$$

where $0 < \alpha \leq 2$ and L is **slowly varying:** $L(\lambda x)/L(\lambda) \to 1$ as $\lambda \to \infty, \forall x$.

Heavy-tailed distributions are prevalent as statistical models, for communications networks, Web statistics, financial data, etc. (Adler, Feldman & Taqqu 1998)

Rigidity of scaling limits

and regular variation (almost power-law behavior)

Lemma (in Feller's book) Suppose that f is monotone and that there exist $\lambda_j \to \infty$ and $a_j/a_{j+1} \to 1$ such that

$$h(x) = \lim_{j\to\infty} a_j f(\lambda_j x) \text{ exists } \forall x > 0.$$

Then necessarily $h(x) = cx^p$ for some $c > 0$ and $p \in \mathbb{R}$, and furthermore, f is regularly varying at ∞, meaning

$$f(x) \sim x^p L(x) \quad \text{where } L \text{ is slowly varying.}$$

Examples $x^p \ln x$ is regularly varying at ∞, but $x^p(1 + \epsilon \sin x)$ is not.

(ib) New fat-tailed scaling limits

(Bertoin 2002, Menon & P 2004) For $K = x + y$, there is a new one-parameter family of self- similar solutions with form

$$n(x,t) = e^{-2t/\beta} f_\beta(e^{-t/\beta} x)$$

for $0 < \beta \leq \frac{1}{2}$ with mass density

$$x\, f_\beta(x) = x^{\beta-1} p(x^\beta; \alpha, 2-\alpha),$$

where p is the density of a maximally skewed α-stable Lévy distribution with

$$\alpha = \frac{1}{1-\beta} \in (1,2].$$

The mass distribution function

$$F_\alpha(x) = \int_0^x y f_\beta(y)\, dy = \sum_{k=1}^{\infty} \frac{(-1)^{k-1} x^{k\beta}}{k!} \Gamma(1+k-k\beta) \frac{\sin \pi k \beta}{\pi k \beta}.$$

(iib) Universality classes for dynamic scaling

Theorem (Menon & P) For $K = x + y$, given a weak solution corresponding to any positive measure $n_0(x)\,dx$ with finite 1st moment, let

$$F(x,t) = \int_0^x y\,n(y,t)\,dy \Big/ \int_0^\infty y\,n(y,t)\,dy.$$

(i) Suppose $\exists \lambda(t) \to \infty$ as $t \to \infty$ and a nontrivial distribution function F_* so

$$F(\lambda(t)x,t) \to F_*(x) \quad \text{as } t \to \infty \tag{1}$$

(at all points of continuity). Then for some $\alpha \in (1,2]$ and L slowly varying,

$$\int_0^x y^2\,n_0(x)\,dx \sim x^{2-\alpha}L(x). \quad \text{as } x \to \infty \tag{2}$$

(ii) Suppose for some $\alpha \in (1,2]$ and L slowly varying, (2) holds. Then (1) holds, where F_* is a trivial scaling of F_α.

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Note: Smoluchowski's equation determines a dynamical system on the space of probability measures determined by

$$F(x,t) = \int_0^x y^p\, n(y,t)\, dy \Big/ \int_0^\infty y^p\, n(y,t)\, dy,$$

under the topology of weak convergence (convergence in distribution).
Here $p = 0, 1, 2$ respectively for $K = 2,\ x+y,\ xy$.

Comparative table of answers

	Probability theory $X_1 + \cdots + X_n$	**Smoluchowski's equation** $K = x + y$
(ia) (ib)	Gaussian Lévy stable laws	$n = \dfrac{\exp(-e^{-2t}x/2)}{\sqrt{2\pi}e^t x^{3/2}}$ new SSS with $m_2 = \infty$
(iia) (iib)	CLT: $E(X^2) < \infty$ Regularly varying tails	$m_2 = \int_0^\infty y^2 \nu_0(dy) < \infty$ $\int_0^x y^2 \nu_0(dy) \sim x^{2-\alpha} L(x)$
(iii)	Infinitely divisible laws	Eternal solutions, defined for $-\infty < t < \infty$
(iv)	Doeblin's *universal laws*	Chaos on the scaling attractor

(iii) The scaling attractor

Define it to consist of all possible scaling limit points

$$F_*(x) = \lim_{n\to\infty} F_n(\lambda_n x, t_n)$$

- Points in the scaling attractor correspond 1-1 with eternal solutions (solutions defined for $-\infty < t < \infty$)
- Bertoin (2002): Eternal solutions have a Lévy-Khintchine-type representation
- The Lévy-Khintchine representation linearizes the ultimate dynamics on the scaling attractor. This dynamics is conjugate to a shift map, given simply by continuously dilating the Lévy measure.

(iv) Chaotic dynamics of scaling limits

For solutions with $K = x + y$ and total mass 1, long-time scaling behavior is sensitive to the mass distribution of the largest clusters:

- All domains of attraction are dense.
- There is a dense set of scaling-periodic solutions. (Analog to *semistable laws*.)
- There are dense orbits on the scaling attractor, and a dense set of initial data that shadow such orbits.

Relevance for Burgers turbulence

$$u_t + uu_x = 0$$

Take initial data $x \mapsto u_0(x)$ as a *continuous-time random walk, stationary with independent increments having no positive jumps* ("one-sided Lévy process with no positive jumps") This includes:

- one-sided Brownian motion
- compound Poisson processes w/no jumps up

Some Burgers turbulence literature

- White noise:

J. M. Burgers 1974 *The nonlinear diffusion equation*. Reidel.

M. Avellaneda and W. E 1995 Comm. Math. Phys., 172, 13–38.

W. E and Y. G. Sinaĭ 2000 Uspekhi Mat. Nauk, 55, 25–58

L. Frachebourg and P. A. Martin 2000 J. Fluid Mech., 417, 323–349.

P. Groeneboom 1989 Prob. Th. Rel. Fields 81, 79–109.

- Brownian motion:

Y. G. Sinaĭ 1992 Comm. Math. Phys., 148, 601–621.

Z.-S. She, E. Aurell, and U. Frisch 1992 Comm. Math. Phys., 148, 623–641.

J. Bertoin 1998 Comm. Math. Phys., 193, 397–406.

Implicit in works of J. Bertoin and Cararro & Duchon 1998 are the results that:

- For $t > 0$ the solution increments in x remain a Lévy process with no positive jumps
- *Smoluchowski mean-field theory with $K = x + y$ gives an exact description of the shock size distribution*

Lévy process initial data

Classical probability theory: The distribution of increments of a Lévy process,

$$\delta u_0(x) = u_0(x+y) - u_0(y),$$

is *infinitely divisible*. Its Lévy-Khintchine representation states that:

$$\begin{aligned} E\left(e^{iq\,\delta u_0(x))}\right) = e^{\Psi(q)x} &= e^{-\sigma^2 q^2 x/2} \cdot e^{iqbx} \cdot e^{-\Phi(q)x} \\ \delta u_0(x) &= U_{\mathrm{Br}}(x) \quad + \quad bx \quad + \quad U_{\mathrm{pj}}(x) \\ &\quad \text{Brownian} \quad \text{drift} \quad \text{pure jump} \end{aligned}$$

$$\Phi(q) = \int_{-\infty}^{\infty} (1 - e^{iqz} + iqz\mathbf{1}_{|z|<1})\,\Lambda(dz) \quad \text{where } \int_{-\infty}^{\infty}(1 \wedge z^2)\Lambda(dz) < \infty.$$

The Lévy measure Λ gives the distribution of jump sizes. Its total mass $\int_{-\infty}^{\infty} \Lambda(dz)$, if finite, gives the frequency of (Poisson-spaced) jumps.

Implicit 1-1 correspondence of Bertoin

With no positive jumps, it suffices to consider data having zero drift and $\int_1^\infty z\Lambda(dz) < \infty$, with Laplace exponent $\left(E(e^{qu_0(x)}) = e^{-x\psi(q)}\right)$

$$\psi(q) = \frac{\sigma^2 q^2}{2} - \int_0^\infty (1 - e^{-qz} - qz)\Lambda(dz).$$

- The Laplace exponent of $x \mapsto u(x,t) - u(0,t)$ has the form

$$\psi(q,t) = -\frac{1}{t}\int_0^\infty (1 - e^{-qz} - qz)\nu(dz, \ln t), \quad \int_0^\infty z\nu(dz,t) = 1,$$

  where $\nu(dz,t)$ is a measure-valued weak solution to Smoluchowski's eq!

- Initial data has ∞ total variation if and only if $\sigma^2 > 0$ or $\int_0^1 z\Lambda(dz) = \infty$. Bertoin (2002): Such Lévy pairs (σ^2, Λ) correspond 1-1 to eternal solutions of Smoluchowski's equation.

Dynamic scaling in Burgers turbulence

- **Self-similar solutions** correspond to

  Brownian motion: $\sigma^2 > 0$, $\Lambda = 0$ $(\alpha = 2)$

  Scaled α-stable laws: $\sigma^2 = 0$, $\Lambda(dz) = z^{-1-\alpha}\,dz$

  The corresponding shock size distributions are:

$$\alpha = 2: \qquad \frac{z^{-3/2}e^{-z/2t^2}}{\sqrt{2\pi}}\,dz$$

$$1 < \alpha < 2: \qquad p(z^{\beta}t^{-1};\alpha, 2-\alpha)z^{\beta-2}\,dz$$

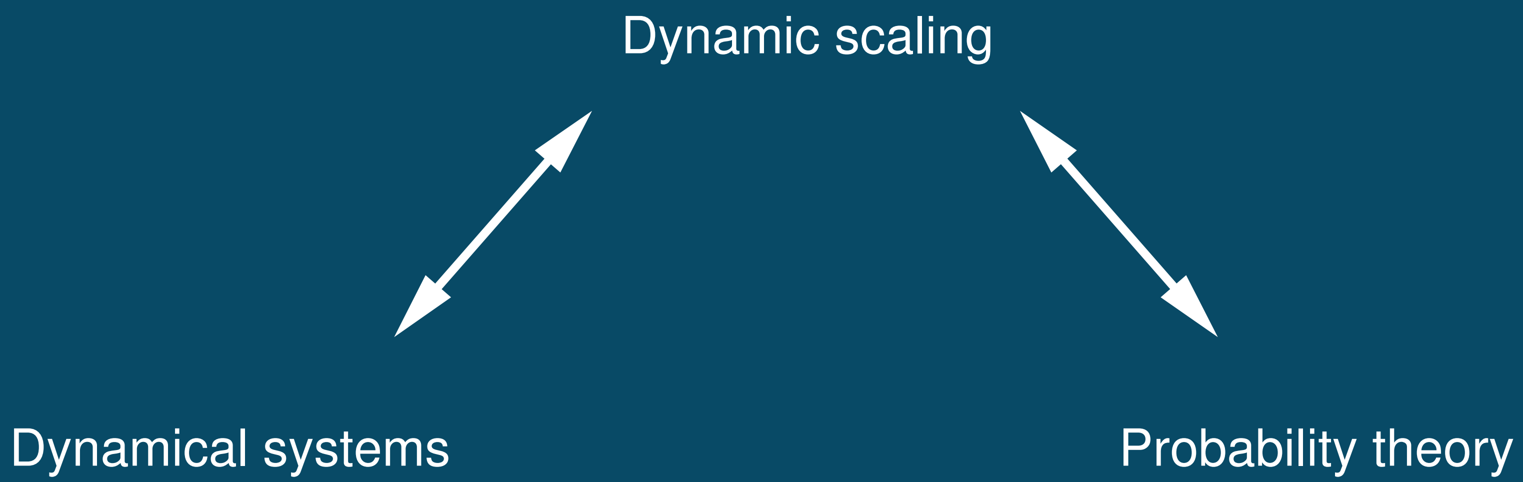
- **Domains of attraction** are determined by the rate of divergence of the 2nd moment of the Lévy jump measure:

$$\int_0^x z^2\Lambda(dz) \sim x^{2-\alpha}L(x) \quad \text{as } x \to \infty, \text{ for } 1 < \alpha \leq 2.$$

In progress:

Work to complete description of:

- all scaling limit points (scaling attractor)
- dynamics on the scaling attractor


Dynamic scaling
Dynamical systems
Probability theory

Smoluchowski's equation $K = x + y$	**Probability theory** $X_1 + \cdots + X_n$
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Eternal solutions $-\infty < t < \infty$	Infinitely divisible laws
Chaos on the S-attractor	Doeblin's *universal laws*