Conformal Invariants and Partial Differential Equations.

by

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In recent years, there have been intensive study of the relationship between the "conformally covariant operators" –that is, operators which satisfy some invariant property under conformal change of metrics on a manifold, their associated conformal invariants, and the study of the related partial differential equations. I will report on some progress in this area and some geometric applications.

A model example is that of the Laplace operator Δ_g on a compact surface M^2 with a Riemannian metric g. In this case, under conformal change of metric $g_w = e^{2w} g$, $\Delta_{g_w} = e^{-2w} \Delta_g$. Denote by K_g the Gaussian curvature of (M^2, g); then through the Gauss-Bonnet formula, $\int_M K_g dv_g$ is a topological, hence a conformal invariant; while K_{g_w} and K_g are related by the partial differential equation

$$(1) \qquad -\Delta_g w + K_g = K_{g_w} e^{2w}.$$

Equation (1) has been under intensive study in the literature. For example, to solve $K_{g_w} = c$ for some constant c is equivalent to the uniformization theorem of surfaces.

Another example is the conformal Laplacian operator $L_g = -\Delta_g + \frac{n-2}{4(n-1)} R_g$ defined on manifolds (M^n, g) of dimension $n \geq 3$; where R_g denotes the scalar curvature of the metric g. In this case, denote the conformal change of metric as $g_u = u^{\frac{4}{n-2}} g$ for some positive function u, then R_{g_u} and R_g are related by the "Yamabe equation":

$$(2) \qquad L_g u = \frac{n-2}{4(n-1)} R_{g_u} u^{\frac{n+2}{n-2}}.$$

These examples are special cases of a very general phenomena. Based on the earlier work (1985) of C.Fefferman and R. Graham on the construction of Poincare metric, there have been systematic study of the existence and construction of conformal covariants and conformal invariants of higher orders. In particular, on manifolds of dimension 4, there is a fourth order conformal covariant operator, discovered independently also by Paneitz in 1983, with leading symbol the bi-Laplace operator. The Paneitz operator applied to a conformal factor determines a fourth order curvature invariant which we will call the Q-curvature.

There are two reasons that make this Q-curvature equation attractive to study. The first consideration comes from the analytic point of view, namely that the generic singularities of the Q-curvature equation are isolated points. The second consideration comes from geometry: the Q-curvature prescribed by the Paneitz operator can be viewed as part of the integrand in the Chern-Gauss-Bonnet formula, thus the integration of Q is conformally invariant.

The Q-curvature equation is intimately related to a fully non-linear second order elliptic equation. Up to a fourth order divergence term, the Q-curvature is the second elementary symmetric function $\sigma_2(A)$ of the Weyl-Schouten tensor; $A_{ij} = R_{ij} - \frac{1}{2(n-1)} R g_{ij}$, where R_{ij} denotes the Ricci tensor and R the scalar curvature of the metric g. The positivity of $\sigma_2(A)$ implies a sign on the Ricci tensor; hence the finiteness of the fundamental group $\pi_1(M)$. Thus the $\sigma_2(A)$ equation contains geometric information that is absent in the scalar curvature equation.

In the first lecture, I will give a brief survey of the subject. I will then describe the variational approach to study the equations (1) and (2). The basic difficulty to find solutions via this approach is due to the fact that the conformal group of the standard n-sphere is not compact, thus creating a non-compact family of solutions which do not have a priori L^∞ bound and which satisfies $K_{g_w} \equiv 1$ for (1) or $R_{g_u} \equiv n(n-1)$ for (2). I will describe the blow up sequence and the analytic tools (i.e. sharp Sobolev inequality and the Moser-Trudinger inequality) to overcome the difficulty.

In the second lecture, I will discuss the study of Paneitz operator on four manifolds, the Q-curvature, connection of Q-curvature to symmetric functions of the Weyl-Schouten tensor, the associated σ_2 equations. I will also discuss some geometric applications.

In the third lecture, I will report on some recent development. One particular interesting one is the work of R.Graham and Zworski relating the existence of the conformal invariants and Q-curvature to the scattering theory of conformally compact Einstein manifold. I will discuss the concept of renormalized volume in this setting and some of the many open questions in the subject.